David M. Alter
Graduate Student.

Tsu-Chin Tsao
Assistant Professor.

Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801

Two-Dimensional Exact Model Matching With Application to Repetitive Control

This paper concerns matching a system with time-delayed feedback to a rational transfer function model and its application to repetitive controller design. Necessary and sufficient conditions on the order of the plant, reference model, and controller are obtained for the existence of causal and stabilizing exact model matching solutions. The results are applied to robust repetitive controller design, in which a delayed feedback is introduced in the repetitive controller for rejecting periodic disturbances while simultaneously achieving input-output model matching. Furthermore, the 2-D model matching method also renders computationally efficient solutions. Also addressed are some subtle points on the selection of a low-pass filter required for robust stability. Finally, the approach is experimentally applied for turning noncircular shapes.

1 Introduction

Exact model matching control is attractive as a means of performance design. One chooses a reference model that meets specified transient and steady-state specifications, and then determines a controller which matches the system closed loop transfer function to that of the reference model. When the plant, reference model, and controller are all represented by linear rational transfer functions, the problem is called that of 1-D model matching, and is currently well understood (Kucera, 1979). Model matching design when the plant is time-delayed does not conform to the 1-D theory as the plant is now of infinite order. However, it is often still desirable to choose a rational reference model so that lumped linear system theory may be applied in performance evaluation. Such a problem falls under the category of 2-D model matching.

As it pertains to delay-differential systems, the most general form of the 2-D model matching problem allows the plant, controller, and reference model to all be represented by rational mappings in two operators: the Laplace differential operator s, and the Laplace delay operator $e^{-\tau s}$. Eising and Emre (1979) investigated this problem in the multivariable case and presented some possible solutions utilizing a so called generalized dynamic cover. Sebek (1983, 1985) determined necessary and sufficient conditions for solution existence under the restriction that the plant be strictly causal. Johansson (1986) formulated a direct adaptive control scheme for the 2-D problem, but did not further address solution existence properties. This paper is concerned with a special case of the 2-D model matching control problem where the reference model is restricted to being a rational transfer function, and the plant belongs to the class of single-input single-output systems represented by

$$\frac{y(s)}{u(s)}=\frac{B(s)}{A_1(s)+A_2(s)e^{-\tau s}} \tag{1}$$

The plant class (1) is directly pertinent to machining systems with regenerative feedback (Merritt, 1965). Such a plant class may also be obtained when designing repetitive controllers. In this case, one approach is to form an augmented open loop system as the series connection of the plant and a repetitive signal generator. The basic transfer function of said generator is of form $1/(1-e^{-\tau s})$, and thus the resulting augmented system is of form (1). Hara et al. (1988) presented sufficient conditions for the stability of repetitive control systems in the continuous-time domain. It will be shown in this paper that both input-output exact model matching and repetitive disturbance rejection can be simultaneously met by solving a corresponding 2-D model matching problem, with the resulting controller being computationally efficient. It will also be shown that for certain cases, the repetitive controller does not offer better repetitive disturbance rejection performance than integral control.

The ability to obtain a rational closed loop system starting from a distributed open-loop system motivates the present use of the 2-D model matching control. From a design point of view, it is a standard matter to choose a rational reference model to meet a wide range of performance requirements by applying lumped linear system theory. Additionally, not all performance specifications need be met by the model matching controller alone; an outer control loop is easily designed using conventional techniques once the closed-loop system is of lumped rational form. In this paper, the existence of causal stabilizing controllers for proper and strictly proper plants will be given by use of the polynomial approach. Since the existence conditions are described in the system polynomial orders, their evaluation is more straightforward than previous works.

The remainder of this paper is organized as follows. Section

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2 gives the polynomial formulation for the 2-D model matching problem. Section 3 presents conditions under which there exists a causal and stabilizing controller. Application to repetitive control is shown in Section 4, with continuous-time domain examples studied in Section 5, and discrete-time domain examples studied in Section 6. Section 7 presents experimental results for the turning of noncircular shapes using the presented 2-D controller for tool motion control. Finally, conclusions are presented in Section 8.

2 Problem Formulation

Consider a single-input single-output plant with transfer function given by (1). The objective is to design a distributed feedback and feed-forward controller of proposed form

$$R(s)u(s) = -[S_1(s) + S_2(s)e^{-\tau s}]y(s) + T(s)u_c(s) \quad (2)$$

where $R(s)$, $S_1(s)$, $S_2(s)$, and $T(s)$ are polynomials, such that the resulting closed-loop system matches some desired rational transfer function

$$\frac{y_m(s)}{u_c(s)} = \frac{B_m(s)}{A_m(s)} \frac{A_0(s)}{A_0(s)} \quad (3)$$

The cancelable ratio of A_0 polynomials represent the fact that a non-minimal reference model may be required if a model matching solution is to exist. Henceforth, the Laplace notation (s) will be dropped. Substituting (2) into (1), the closed-loop system becomes

$$[(A_1 + A_2 e^{-\tau s})R + B(S_1 + S_2 e^{-\tau s})]y = BTu_c \quad (4)$$

Factor $B = B' B^-$, where B' contains all the stable (cancelable) roots of B, and B^- contains all the unstable (uncancelable) roots of B. Note that it may not be desirable to cancel roots with small negative real parts or low damping ratios, and thus technically stable roots may not necessarily be cancelable from a practical viewpoint. Now, B' is to be canceled in (4), so it is required that B' be a factor of R. Furthermore, it may be desired to incorporate asymptotic disturbance rejection into the controller design via the internal model principle. Let the disturbance signal have a rational Laplace transform of form $X(s)/D(s)$ where $X(s)$ and $D(s)$ are both constant coefficient polynomials in s. It is then required that $D(s)$ also be a factor of $R(s)$. Thus:

$$R = B' D R' \quad (5)$$

which, after the stable pole-zero cancellation of B', reduces the closed-loop system (4) to

$$[(A_1 + A_2 e^{-\tau s})R' D + B^- (S_1 + S_2 e^{-\tau s})]y = B^- T u_c \quad (6)$$

The reference model (3) may be rewritten as

$$A_m A_0 y_m = B_m A_0 u_c \quad (7)$$

and then the corresponding sides of (6) and (7) equated. Noting that the delay terms in (6) must additively cancel, the following three equations are obtained:

$$A_1 DR' + B^- S_1 = A_m A_0 \quad (8)$$

$$A_2 DR' + B^- S_2 = 0 \quad (9)$$

$$B^- T = B_m A_0 \quad (10)$$

Equations (8), (9), and (10) will henceforth be considered the general formulation of the 2-D model matching design being investigated here. Note that one can always choose $D = 1$, which would not design for any disturbance rejection properties. A causal and stabilizing solution to these three equations thus solves the proposed model matching problem. If these equations are solvable, then solutions to (9) and (10) may be found via simple polynomial division. One solution technique for the Diophantine Eq. (8) is polynomial coefficient matching, which may be posed as a linear set of algebraic equations by use of the Sylvester matrix (Astrom, 1990).

The 2-D model matching controller consists of feedback and feedforward compensators. The feedback part is required to move any unstable open-loop plant poles into the stable region. Feedback can also be used to move already stable open-loop poles to more desirable locations. The controller obtained by solving the above formulation of the 2-D model matching problem is such that the feedback compensator places all open-loop plant poles to exactly match the reference model poles. The feedforward compensator adjusts only the stable plant zeros. An infinite number of possibilities actually exist for controller designs that meet the model matching criteria. It is also possible, for example, to design a feedback controller to arbitrarily place the open loop plant poles into the stable s-plane region, and then use feedforward compensation to locate these now stable poles to the desired reference model matching locations using pole-zero cancellation. Such controllers are easily obtained by modifying the reference model poles to match such arbitrary stable locations, and then adjusting the feedforward compensator to achieve the necessary pole-zero cancellation.

3 Stabilizing and Causal Solutions

The existence of the solution to the proposed model matching problem is based upon the solvability of the polynomial Diophantine equation. Two well-known preliminary results are needed, here given without proof:

Lemma 3.1: (Kucera, 1979) The Diophantine equation $AX + BY = C$, where A, B, and C are known polynomials with real coefficients has a solution (X, Y) if and only if the greatest common factor of A and B divides C.

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Lemma 3.2: (Kucera, 1979) If a solution exists to the Diophantine equation $AX + BY = C$, then there exists a solution such that $\deg(X) < \deg(B)$, and also a (possibly different) solution such that $\deg(Y) < \deg(A)$.

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Since the resulting closed-loop system is to exactly match (7) after a stable pole-zero cancellation of the factor B', it is clearly necessary and sufficient for closed-loop stability that A_m and A_0 be chosen as stable polynomials. A stabilizing solution thus refers to a solution $[R(s), S_1(s), S_2(s), T(s)]$ of the form (2) which solves (8), (9), and (10) under the constraint that A_m and A_0 be stable polynomials.

Theorem 3.3: A stabilizing solution to (8) exists if and only if B^- and $A_1 D$ are factor coprime.

Proof: Immediately follows from Lemma (3.1).

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Theorem 3.4: If a stabilizing to (8) exists, then a stabilizing solution to (9) exists if and only if B^- is a factor of A_2.

Proof: Equation (9) gives $S_2 = -(A_2 DR)/B^-$. Since DR' and B^- are factor coprime by Theorem 3.3, B^- must be a factor of A_2.

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Theorem 3.5: A stabilizing solution to (10) exists if and only if B^- is a factor of B_m.

Proof: Equation (10) gives $T = B_m A_0 / B^-$. Since A_0 is stable, B^- must be a factor of B_m.

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Necessary and sufficient conditions for the existence of a stabilizing solution to the model matching problem have now been given. However, such a solution (controller) need not be causal. Causality is a necessary property of a controller from a practical point of view. In the continuous-time domain, implementation of a noncausal controller requires signal derivatives. In discrete-time, noncausal implementation requires signal preview. In either case, such information is likely unavailable. Conditions for the existence of a causal (implementable) solution to the 2-D model matching problem will be developed next.

Fact 3.6: (Sebek, 1983) A plant of the form (1), with $\tau \geq$

0, is causal if and only if $\deg(A_1) \geq \deg(B)$ and $\deg(A_1) \geq \deg(A_2)$, and strictly causal if $\deg(A_1) > \deg(B)$ and $\deg(A_1) \geq \deg(A_2)$.

Fact 3.7: (Sebek, 1983) A controller of the form (2), with $\tau \geq 0$, is causal if and only if $\deg(R) \geq \deg(S_1)$, $\deg(R) \geq \deg(S_2)$, and $\deg(R) \geq \deg(T)$.

Theorem 3.8: Given a causal plant, if a solution exists to (8) and (10), then a causal solution exists such that $\deg(R) \geq \deg(S_1)$, and $\deg(R) \geq \deg(T)$ if

1. $\deg(A_0) \geq 2\deg(A_1) - \deg(A_m) - \deg(B') + \deg(D) - \delta$ where δ equals 1 for a strictly causal plant, and 0 for a proper plant
2. $\deg(A_m) - \deg(B_m) \geq \deg(A_1) - \deg(B)$

Proof: Equation (8) gives $A_1DR' + B^-S_1 = A_mA_0$. Equating the degree of each side, one obtains

$$\deg(A_1DR' + B^-S_1) = \deg(A_m) + \deg(A_0) \tag{11}$$

Also, direct expansion of $A_1DR' + B^-S_1$ gives

$$\deg(A_1DR' + B^-S_1) = \max\{[\deg(A_1) + \deg(D) + \deg(R')], [\deg(B^-) + \deg(S_1)]\} \tag{12}$$

Lemma 3.2 allows the choice $\deg(S_1) = \deg(A_1) + \deg(D) - 1$. Thus

$$\deg(A_1DR' + B^-S_1) = \deg(A_1) + \deg(D) + \max\{\deg(R'), \deg(B^-) - 1\} \tag{13}$$

Now, if condition 1 holds, define $j \geq 0$ and let

$$\deg(A_0) = 2\deg(A_1) - \deg(A_m) - \deg(B') + \deg(D) - \delta + j \tag{14}$$

Combining (11), (13), and (14) yields

$$\max\{\deg(R'), \deg(B^-) - 1\} = \deg(A_1) - \deg(B') - \delta + j \tag{15}$$

Now suppose $\deg(B^-) - 1 > \deg(R')$. Then (15) yields $\deg(A_1) - \deg(B) - \delta + j + 1 = 0$. This can never be true with δ as defined in condition 1. This supposition $\deg(B^-) - 1 > \deg(R')$ must therefore be incorrect, and it must be true that $\deg(R') \geq \deg(B^-) - 1$. Thus, (15) gives:

$$\deg(R') + \deg(D) + \deg(B') = \deg(A_1) - \delta + j + \deg(D)$$

Substituting yields

$$\deg(R) = \deg(S_1) - \delta + j + 1$$

Hence, $\deg(R) \geq \deg(S_1)$. It remains to show $\deg(R) \geq \deg(T)$ Equation (10) gives

$$\deg(T) = \deg(A_0) + \deg(B_m) - \deg(B^-) \tag{16}$$

For a causal plant, and using the result $\deg(R) \geq \deg(S_1)$, one obtains $\deg(A_1R + BS_1) = \deg(A_1R)$. Equation (11) may now be rearranged to give

$$\deg(R) = \deg(B') + \deg(A_m) + \deg(A_0) - \deg(A_1) \tag{17}$$

If condition 2 holds, then $\deg(A_m) - \deg(B_m) \geq \deg(A_1) - \deg(B') - \deg(B^-)$, from which it follows that

$$\deg(A_0) + \deg(A_m) + \deg(B') - \deg(A_1) \geq \deg(A_0) + \deg(B_m) - \deg(B^-)$$

Substituting in Eqs. (16) and (17) gives $\deg(R) \geq \deg(T)$.

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Theorem 3.9: If a solution exists to (9), then a causal solution exists such that $\deg(R) \geq \deg(S_2)$ if and only if $\deg(B) \geq \deg(A_2)$.

Proof: Equation (9) gives $A_2R + BS_2 = 0$, and thus $\deg(R) = \deg(S_2) + \deg(B) - \deg(A_2)$. If $\deg(B) \geq \deg(A_2)$, then $\deg(B) - \deg(A_2) \geq 0$. It follows that $\deg(R) \geq \deg(S_2)$. To prove the reverse, suppose $\deg(R) \geq \deg(S_2)$ and thus $\deg(R) - \deg(S_2) \geq 0$. Again, (9) gives $\deg(B) = \deg(A_2) + \deg(R) - \deg(S_2)$. Hence, it follows that $\deg(B) \geq \deg(A_2)$.

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Combining Theorems 3.3–3.5, 3.8, and 3.9 results in the following summary theorem:

Theorem 3.10: 2-D Model Matching Causal and Stabilizing Solution Existence. Given a causal plant and reference model of the forms:

$$\frac{y(s)}{u(s)} = \frac{B(s)}{A_1(s) + A_2(s)e^{-\tau s}} \qquad \frac{y_m(s)}{u_c(s)} = \frac{B_m(s)}{A_m(s)}\frac{A_0(s)}{A_0(s)}$$

there exists a causal and stabilizing model matching controller of the form

$$R(s)U(s) = -[S_1(s) + S_2(s)e^{-\tau s}]Y(s) + T(s)U_c(s)$$

if all six of the following conditions hold:

1. B^- and DA_1 are factor coprime
2. B^- is a factor of A_2
3. B^- is a factor of B_m
4. $\deg(B) \geq \deg(A_2)$
5. $\deg(A_0) \geq 2\deg(A_1) - \deg(A_m) - \deg(B') + \deg(D) - \delta$ where δ equals 1 for a strictly causal plant, and 0 for a proper plant
6. $\deg(A_m) - \deg(B_m) \geq \deg(A_1) - \deg(B)$

In addition, said controller cannot exist if any of conditions 1 through 4 do not hold.

Proof: Immediately follows from Theorems 3.3–3.5, 3.8, and 3.9.

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Thus, given a minimal reference model B_m/A_m, one may find a solution to the 2-D model matching problem, if one exists, by choosing:

$$\deg(A_0) = 2\deg(A_1) - \deg(A_m) - \deg(B') + \deg(D) - \delta$$
$$\deg(R') = \deg(A_m) + \deg(A_0) - \deg(A_1) - \deg(D)$$
$$\deg(S_1) = \deg(A_1) + \deg(D) - 1$$

The degrees of S_2 and T follow immediately from Eqs. (9) and (10).

Implementation of the controller may be achieved by employing an arbitrary stable polynomial $E(s)$ with $\deg(E) = \deg(R)$, as shown in Fig. 1. If it happens that $R(s)$ is itself stable, one may make the choice $E(s) = R(s)$, considerably reducing computational expense.

Extension of the 2-D model matching technique to the discrete-time domain is straightforward, simply requiring a reexamination of the causality definitions. The discrete-time domain plant, controller, and reference model are of the forms (1), (2), and (3) respectively, but with the Laplace variable s replaced with the Z-transform variable z, and also with the Laplace delay operator $e^{-\tau s}$ replaced with z^{-N}:

$$\frac{y(z)}{u(z)} = \frac{B(z)}{A_1(z) + A_2(z)z^{-N}} \tag{18}$$

$$R(z)u(z) = -[S_1(z) + S_2(z)z^{-N}]y(z) + T(z)u_c(z) \tag{19}$$

$$\frac{y_m(z)}{u_c(z)} = \frac{B_m(z)}{A_m(z)} \cdot \frac{A_0(z)}{A_0(z)} \tag{20}$$

Since z^{-N} is not an infinite dimensional operator, 1-D model matching can of course be applied as well as 2-D model matching. However, the 2-D approach has an advantage over the 1-D approach. When the delay N is large, the derivation and implementation of the 1-D solution, the order of which is in general comparable to N, becomes rather difficult. In contrast, the 2-D solutions are low-order compensation along with the easy to implement pure N-step delay. In other words, the 2-

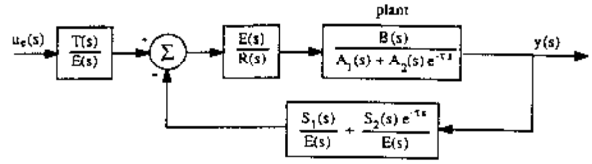

Fig. 1 Implementation of the 2-D model matching controller

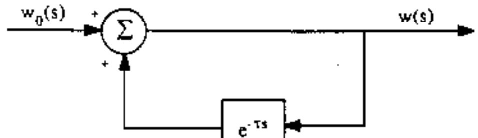

Fig. 2 Continuous-time ideal repetitive signal generator

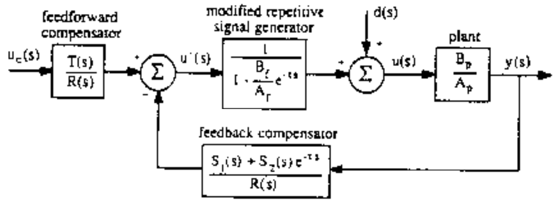

Fig. 3 Model matching repetitive control scheme

D solutions force to zero most coefficients of the approximately Nth order controller.

It will be assumed that $N \geq \deg(A_2)$, and therefore the plant is causal if and only if $\deg(A_1) \geq \deg(B)$. For controller causality, it is first needed that $\deg(R) \geq \deg(S_1)$, and second that $\deg(R) \geq \deg(S_2) - N$. Following a similar proof to that of Theorem 3.9, it is easy to verify that the second requirement is always satisfied. Condition 4 of Theorem 3.10 has thus been relieved. The remaining conditions of Theorem 3.10 apply directly in the discrete-time domain, as can be easily verified analogous to the continuous-time formulation.

4 Application to Continuous-Time Domain Repetitive Control

Any periodic signal with a known period, say τ, can be thought of as generated by a dynamical system with positive feedback around a pure time delay, as shown in Fig. 2. The Laplace transform of the periodic signal $w(s)$ may thus be expressed as

$$w(s) = \frac{1}{1 - e^{-\tau s}} w_0(s) \tag{21}$$

where $w_0(s)$ is the Laplace transform of a signal composed of the first period of $w(t)$, and zero for $t > \tau$. If the control system is to reject the periodic disturbance, it must be capable of generating a periodic signal itself. A repetitive signal generator (RSG) must therefore be included in the control loop, as stated by the internal model principle (Francis and Wonham, 1975). However, it has been shown that strictly proper plants cannot be exponentially stabilized when the ideal periodic signal generator (21) is used in the control loop (Hara, 1988). This is due to the unrealistic high-gain demand on the controller of handling the infinitely-high frequency modes of the periodic disturbance. Hara showed that this difficulty could be relaxed by including a suitable low-pass filter in the RSG to reduce the feedback gains at high frequencies. Even though asymptotic stability can be obtained when using the RSG in (21) with a proper plant, the system will still suffer from poor robustness against unmodeled dynamics, obviously due to infinitely-high gain feedback. The low-pass filter also improves high-frequency robustness. The proposed control scheme with the modified repetitive signal generator is depicted in Fig. 3. The cascade of modified RSG and plant combine to form a new open loop system of form (1):

$$\frac{y(s)}{u'(s)} = \frac{B(s)}{A_1(s) + A_2(s)e^{-\tau s}} = \frac{A_f(s)B_p(s)}{A_f(s)A_p(s) - B_f(s)A_p(s)e^{-\tau s}} \tag{22}$$

Equations (22), (2), and (3) combine to form a 2-D model matching problem. The low-pass filter $B_f(s)/A_f(s)$ must be chosen so that the new plant (22) conforms to the solution requirements of Theorem 3.10. The following corollary results:

Corollary 4.1: Given a causal plant $B_p(s)/A_p(s)$, define a 2-D model matching problem (with repetitive disturbance rejection) by Eqs. (22) and (3), and factor $B_p = B_p' B_p^-$ where B_p' and B_p^- contain the cancelable and uncancelable roots of B_p, respectively. There exists a causal and stabilizing solution of form (2) if all six of the following conditions hold:

1. (*a*) A_f is stable
   (*b*) B_p' and A_pD are factor coprime
2. B_p' is a factor of B_f
3. B_p' is a factor of B_m
4. $\deg(A_f) - \deg(B_f) \geq \deg(A_p) - \deg(B_p)$
5. $\deg(A_0) \geq \deg(A_f) + 2\deg(A_p) - \deg(A_m) - \deg(B_p') + \deg(D) - \delta$ where δ equals 1 for a strictly causal plant, and 0 for a proper plant
6. $\deg(A_m) - \deg(B_m) \geq \deg(A_p) - \deg(B_p)$

In addition, said controller cannot exist if any of conditions 1 through 4 do not hold.

Proof: Each condition follows from direct application of the identically numbered condition in Theorem 3.10.

Condition 1a reflects the intuitive expectation that the low-pass filter be chosen stable, while condition 2 requires that the filter zeros contain all unstable plant zeros. Condition 4 shows that the relative degree of the filter must be at least as great as that of the plant. Based upon these necessary conditions, the low-pass filter may be appropriately designed with a bandwidth encompassing a suitable number of Fourier frequencies of the periodic disturbance. Now, the following closed loop transfer functions result from the design:

$$\frac{y(s)}{u_c(s)} = \frac{B_m}{A_m} \cdot \frac{A_0 B_p' A_f}{A_0 B_p' A_f} \qquad \frac{y(s)}{d(s)} = \frac{B_p^- R}{A_m A_0}\left(1 - \frac{B_f}{A_f} e^{-\tau s}\right) \tag{23}$$

As expected, the mapping y/u_c depends only on the choice of reference model (after stable pole-zero cancellations), while the mapping y/d contains the modified repetitive signal generator.

5 Continuous-Time Domain Repetitive Control Design Example

Consider the following simple illustrative example:

$$\frac{B_p}{A_p} = \frac{1}{s} \qquad \frac{B_m}{A_m} = \frac{2}{s+2} \tag{24}$$

For comparison, model matching will be achieved via three different designs: 1-D model matching with integral action only, and 2-D model matching with either a first or second order low-pass repetitive generator filter.

Design 1: 1-D Model Matching With Integral Action: This design does not include the RSG (i.e., $B_f = 0$), but rather provides integral action by choosing $D(s) = s$ in Eq. (5). The polynomial choice $A_0 = s + 20$ has been used in the design, resulting in the following controller:

$$T = a(s + v_1) \quad R = s \quad S_1 = (a + v_1)s + av_1 \quad S_2 = 0$$

$$\text{parameters:} \quad a = 2 \quad v_1 = 20$$

$$\frac{y(s)}{d(s)} = \frac{s}{(s+a)(s+v_1)} \tag{25}$$

Design 2: 2-D Model Matching With First Order Low-Pass Repetitive Filter:

$$B_f = b_1 \quad A_f = s + b_1 \quad A_0 = s + v_1$$

$$T = a(s + v_1) \quad R = s + b_1 \quad S_1 = (a + v_1 - b_1)s + av_1 \quad S_2 = b_1 s$$

$$\text{parameters:} \quad a = 2 \quad v_1 = 20 \quad b_1 = 20$$

$$\frac{y(s)}{d(s)} = \frac{(s + b_1) - b_1 e^{-\tau s}}{(s+a)(s+v_1)} \tag{26}$$

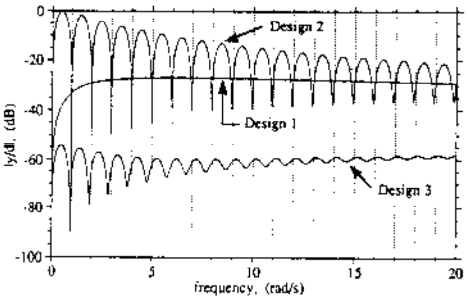

Fig. 4 Disturbance rejection magnitude responses of continuous-time domain examples

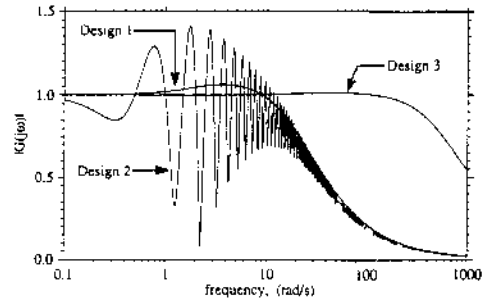

Fig. 5 Return difference function magnitudes of continuous-time domain examples

Design 3: 2-D Model Matching With 2nd Order Low-Pass Repetitive Filter:

$$B_f = b_1 b_2 \qquad A_f = (s+b_1)(s+b_2) \qquad A_0 = (s+v_2)(s+v_2)$$

$$T = a(s+v_1)(s+v_2) \qquad R = (s+b_1)(s+b_2) \qquad S_2 = (b_1 b_2)s$$

$$S_1 = (a+v_1+v_1-b_1-b_2)s^2 + [a(v_1+v_2)s + v_1 v_2 - b_1 b_2]s + a v_1 v_2$$

$$\text{parameters: } a_2 \quad v_1 = 20 \quad v_2 = 625 \quad b_1 = 5 \quad b_2 = 5$$

$$\frac{y(s)}{d(s)} = \frac{[s^2 + (b_1+b_2)s + b_1 b_2] - b_1 b_2 e^{-\tau s}}{(s+a)(s+v_1)(s+v_2)} \tag{27}$$

The periodic disturbance $d(s)$ has a discrete frequency spectrum containing only those Fourier series harmonics given by $\omega_i = 2i\pi/\tau$, $i = 0, 1, 2, \ldots$. The frequency response $|y/d|$ at these distinct frequencies provides a measure of the repetitive disturbance rejection capabilities of the controller designs. The frequency response of (26) reduces to that of (25) at the frequencies $\omega = \omega_i$ regardless of the choice of the RSG low-pass filter pole. For this example then, the use of the first order repetitive generator filter cannot improve steady-state repetitive disturbance rejection performance over that of integral action.

The above observation on the disturbance rejection similarities between integral action and the first order filter repetitive control displayed in the example may in fact be concluded for a specific class of plants. Equation (23) may be rewritten as

$$\frac{y(s)}{d(s)} = \frac{B_p D}{A_m A_0} \cdot R' [A_f - B_f e^{-\tau s}] \tag{28}$$

For a given reference model and with specified disturbance rejection properties in $D(s)$, the terms B_p, D, A_m, and A_0 will be common to all model matching designs. Only the terms R', B_f, and A_f vary among designs. For a first-order low-pass filter ($B_f = b$, $A_f = s + b$), the frequency response of the repetitive signal generator becomes independent of the filter parameter b at the Fourier harmonics:

$$|A_f - B_f e^{-\tau j\omega_i}| = \omega_i \tag{29}$$

If the plant is such that $\deg(A_p) - \deg(B_p') = 1$ (i.e., a minimum phase relative degree 1 plant, or a relative degree zero plant with one unstable zero), a solution may be found which gives $\deg(R') = 0$. Examination of Eq. (8) reveals that this scalar R' is completely independent of the low-pass filter choice, regardless of the filter degree. Therefore, for plants conforming to the above relative degree condition, a first order filter repetitive control design provides exactly the same magnitude characteristics at the Fourier harmonic frequencies as an integral action design. The response relations at other frequencies, however, cannot be conclusively predicted.

Returning to the example, and with some abuse of notation, Design 3 (27) reduces at the harmonic frequencies to

$$\left.\frac{y(s)}{d(s)}\right|_{s=j\omega_i} = \frac{s}{(s+a)(s+v_1)} \cdot \left.\frac{s+(b_1+b_2)}{s+v_2}\right|_{s=j\omega_i} \tag{30}$$

Equation (30) is seen to contain (25), and also an extra pole-zero combination. This pole-zero combination can be independently designed to attenuate the response $|y/d|$ through some suitable bandwidth. Therefore, the example 2nd order repetitive filter design offers superior repetitive disturbance rejection capabilities over the two simpler designs.

Figure 4 shows the frequency response $|y/d|$ for the three parametrized designs with $\tau = 2\pi$. Note that y/d is equal to the system sensitivity function multiplied by $B_p(s)/A_p(s)$. This plot confirms that $|y/d|$ of Designs 1 and 2 collocate at $\omega = \omega_i = 1, 2, 3, \ldots$. It is also seen that the response magnitude of Design 2 is greater than that of Design 1 at a majority of other frequencies, and therefore the "Energy" of response (i.e., the integral area) of Design 2 exceeds that of Design 1. On the other hand, the placement of the extra pole-zero in the RSG low-pass filter of Design 3 has produced significant response reductions at not only the Fourier frequencies, but also at all other frequencies. This demonstrates the potential advantages of using a higher than minimal order RSG low-pass filter, as well as the possible performance improvements of repetitive control over conventional integral action.

Any phase lag introduced by the RSG low-pass filter shifts the magnitude response of the entire modified RSG so that the minimums do not occur at the desired harmonics. The effect of this on the closed loop design is evident in Fig. 4, where the periodic depressions in $|y/d|$ of Designs 2 and 3 have similarly been shifted away from the Fourier harmonics. Since the response valleys are extremely narrow band, even a slight shift can result in a significant increase in system response at a Fourier frequency. Generally, the effects of this frequency shift become more pronounced as frequency increases. It will be shown in Section 7 how this difficulty may be alleviated.

The return difference function for the three designs are shown in Fig. 5 (i.e., the "Robustness" to a stable multiplicative plant perturbation $\Delta(s)$). For robust stability, it is necessary that $|G(j\omega)| < |1/\Delta|$. An expected result is that Design 3 has sacrificed some high frequency robustness in order to achieve its better disturbance rejection properties. The figure also shows Design 2 with less robustness than Design 1 at most frequencies.

Simulation results are shown in Figs. 6 and 7, with excitation provided by a square wave disturbance of ± 1 amplitude and period 2π, beginning at $t = 10$ seconds. Examining the transient plant response in Figs. 6*a*) and 6(*b*), one sees those of Designs 2 and 3 as respectively being on the order of 10 times greater, and 40 times smaller than that of Design 1. The steady-state response of Design 2 is essentially identical to Design 1, which is to be expected since they have identical disturbance to output closed loop transfer functions. The steady-state response of Design 3 is again seen on the order of 40 times less than Design 1. The spiked appearance of all three design curves is due to the inability of the controllers to respond to the higher frequency components of the square wave disturbance. Finally, Figs. 7(*a*) and 7(*b*) show little difference between designs in control input demands, and thus similar actuating hardware would be required in all three cases.

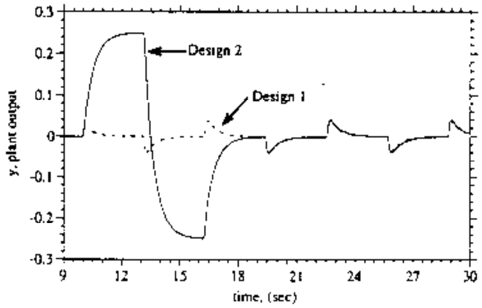

Fig. 6(*a*) Simulated plant output with unit amplitude square wave disturbance, $\tau = 2\pi$

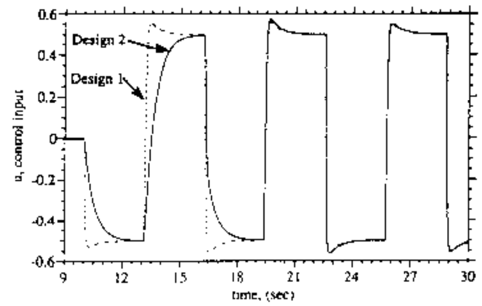

Fig. 7(*a*) Simulated control input with unit amplitude square wave disturbance, $\tau = 2\pi$

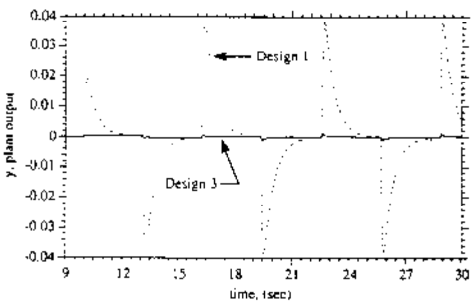

Fig. 6(*b*) Simulated plant output with unit amplitude square wave disturbance, $\tau = 2\pi$

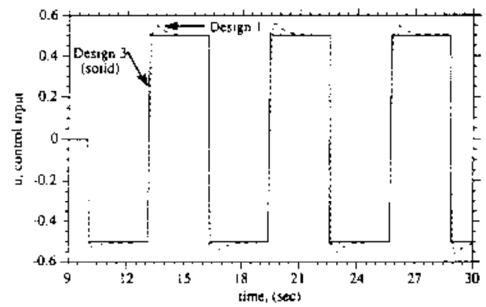

Fig. 7(*b*) Simulated control input with unit amplitude square wave disturbance, $\tau = 2\pi$

6 Application to Discrete-time Domain Repetitive-Control

The discrete-time domain modified repetitive signal generator is given by

$$w(z) = \frac{1}{1 - \dfrac{B_f(z)}{A_f(z)} z^{-N}} w_0(z) \tag{31}$$

Unlike in the continuous-time case, the RSG low-pass filter is not needed in discrete-time for achieving exponential stability. Tomizuka et al. (1989) developed a method for synthesizing repetitive controllers without using the low-pass filter, and under the assumption of a stable plant. In implementation, the low-pass filter is typically included in order to improve high frequency robustness (Tsao and Tomizuka, 1988). The 2-D model matching-repetitive control problem does not assume a stable open loop plant, and is formulated parallel to the continuous-time case previously presented in Section 5. Since condition 4 of Theorem 3.10 need not hold in the discrete-time case, it follows that condition 4 of Corollary 4.1 need not hold either. The remainder of Corollary 4.1 may be easily shown applicable in its entirety.

Design in the discrete-time domain offers the advantage of choosing $B_f(z)/A_f(z)$ with a zero phase frequency response at all frequencies. In view of (23), the output to disturbance magnitude ratio, at those Fourier frequencies of the periodic signal, is proportional to

$$\left| 1 - \frac{B_f(z)}{A_f(z)} \right|_{z = e^{j\omega_i}} \tag{32}$$

Among low-pass filters which have similar magnitude ratio responses, the one with zero phase response will render a minimum in (32). In addition, the periodic disturbance rejection response minimums will coincide exactly with its Fourier harmonic frequencies. Therefore, a zero-phase low-pass filter offers better periodic disturbance rejection. This may be accomplished by choosing (Tsao, 1988)

$$\frac{B_f(z)}{A_f(z)} = \frac{z^m P(z) P(z^{-1})}{z^m} \tag{33}$$

where $m = \deg(P)$, and $P(z^-)$ is $P(z)$ with z replaced by z^{-1}. Although (33) is noncausal, the complete modified RSG (31) is causal. Also, the discrete 2-D model matching assumption that $N \ge \deg(A_2)$ requires that $m \le [N - \deg(A_p)]/2$. Finally, note that z^m is a common factor of B and A_1, and can be absorbed into z^{-N}. Therefore, controller complexity may be reduced by multiplying the resultant new open loop system (22) by z^{-m}/z^{-m}. The delay operator for 2-D model matching design then becomes $z^{-(N+m)}$.

The example plant and reference model (24) has been sampled with a zero-order hold, yielding:

$$\frac{B_p}{A_p} = \frac{h}{z-1} \qquad \frac{B_m}{A_m} = \frac{1 - e^{-2h}}{z - e^{-2h}} \tag{34}$$

where h is the sampling period. Two example designs will now be given, each utilizing a different RSG low-pass filter.

Design 4: Tustin Approximation of $b/(s+b)$ Low-Pass Filter:

$$b_1 := bh/(bh+2) \qquad b_2 := (bh-2)/(bh+2)$$
$$B_f = b_1(z+1) \qquad A_f = z + b_2 \qquad A_0 = z + v_1$$
$$T = (1 - e^{-ah})(z + v_1) \qquad R = h(z + b_2) \qquad S_2 = b_1(z^2 - 1)$$
$$S_1 = (v_1 - e^{-ah} - b_2 + 1)z + (b_2 - v_1 e^{-ah})$$
$$\text{parameters:} \quad a = 2 \qquad v_1 = e^{-20h} \qquad b = 100 \qquad h = \pi/300$$

Design 5: Zero-Phase Discrete Low-Pass Filter:

$$\frac{B_f}{A_f} = \frac{\dfrac{1}{(1+b)^2} z(z+b)(z^{-1}+b)}{z} \qquad A_0 = z + v_1$$
$$T = (1 - e^{-ah})(z + v_1) \quad R = hz \quad S_2 = (bz^3 + z^2 - z - b)/(1+b)^2$$
$$S_1 = (v_1 - e^{-ah} - 1)z - v_2 e^{-ah}$$
$$\text{parameters:} \quad a = 2 \quad v_1 = e^{-20h} \quad b = 1 \quad h = \pi/300$$

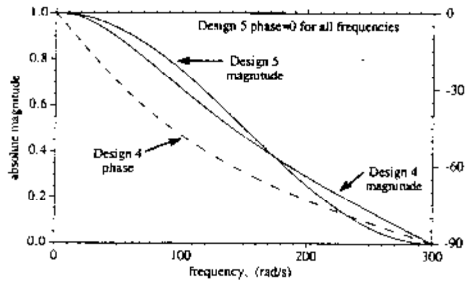

Fig. 8 Frequency responses of discrete-time domain low-pass filters

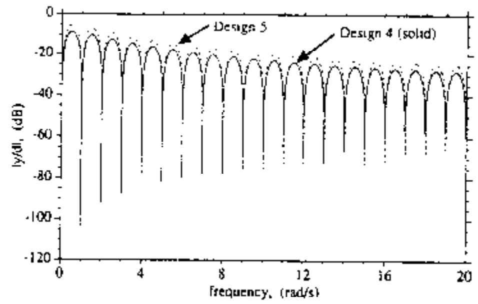

Fig. 9 Disturbance rejection magnitude responses for discrete-time domain examples

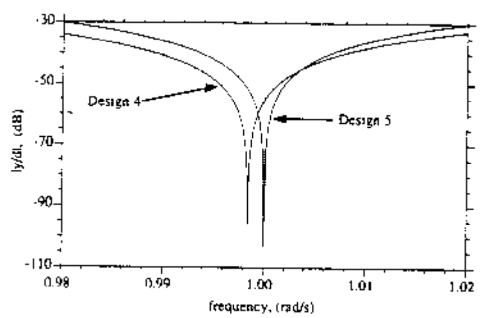

Fig. 10 Repetitive control frequency response showing first Fourier harmonic depression error for discrete-time domain examples

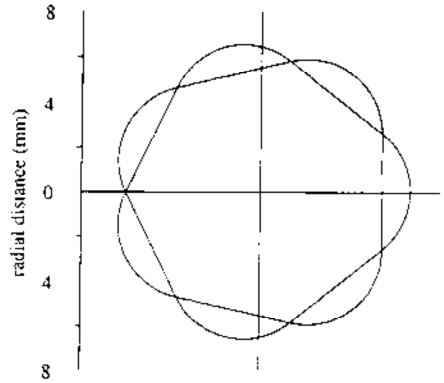

Fig. 11 Synthesizing profile of heptagon in the X-Y plane

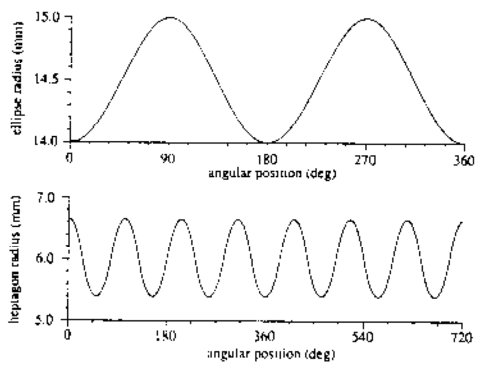

Fig. 12 Synthesizing profiles of ellipse and heptagon in the r-θ plane

Figure 8 shows the frequency responses of the RSG low-pass filters used in the above two examples. Although the magnitude responses are quite similar, Design 5 shows zero phase for all frequencies while Design 4 displays increasing phase lag with frequency. Figure 9 shows Bode magnitudes of the resultant closed loop transfer functions y/d. The overall responses look similar, being of the same order and wave form. However, closer examination at the Fourier harmonics reveals that the periodic valleys of Design 5 occur exactly at the harmonics, while those of Design 4 occur at a frequency prior to each harmonic. This, for example, increases the actual magnitude response of Design 4 at the first harmonic of $w = 1$ rad/s by 45 dB over its locally minimum response at approximately 0.9985 rad/s, as shown in Fig. 10.

7 Application to Noncircular Turning: Experiments

The utility of 2-D model matching repetitive control is illustrated in this section by the lathe turning of two different noncircular shapes: a heptagon (7-sided polygon), and an ellipse. To achieve this, the 2-D model matching design must be adjusted to result in asymptotic repetitive tracking. This may be accomplished by a straight forward open-loop system manipulation. The input-output equation of the open-loop system is represented by

$$A_p\, y = B_p\, u \tag{35}$$

The control goal is to asymptotically track a desired repetitive reference signal y_d. The error system is formed by defining $e = y - y_d$. The system may now be manipulated as follows:

$$(A_p y - A_p y_d) + A_p y_d = B_p u \tag{36}$$

$$A_p e = B_p u - A_p y_d \tag{37}$$

The term $A_p y_d$ may be treated as a periodic disturbance to the error system. With a stable closed-loop system design, asymptotic rejection of this disturbance is equivalent to asymptotic tracking (i.e., $e \rightarrow 0$). In this tracking design, the feedforward part of the 2-D model matching controller is not utilized; only the feedback part is needed. Thus, the 2-D controller becomes a pole-placement tracking algorithm driven from the error signal:

$$R(s)u(s) = -[S_1(s) + S_2(s)e^{-\tau s}]e(s) \tag{38}$$

A profile resolution of 144 points per workpiece revolution was chosen to generate a digital repetitive reference signal. Any N-sided polygon may be synthesized by some continuously differentiable profile in a finite number of revolutions (typically only 2 or 3 revolutions are needed) (Tsao, 1990). The heptagon synthesizing profile in the X-Y plane is shown in Fig. 11, requiring 2 workpiece revolutions to complete the shape for a total of 288 points. The ellipse is itself a continuously differentiable profile, and is thus machinable in a single revolution. Major and minor axis diameters of 30 and 28 mm, respectively, were chosen. Synthesizing profiles in the r-θ plane are shown in Fig. 12 for both noncircular shapes.

A direct drive linear motor was used to actuate the tool position, as shown in Fig. 13. Tool position was obtained using an optical encoder of 2 μm resolution mounted to the motor slide. A Texas Instruments TMS320C30 processor, accessed through an IBM-AT host, performed controller implementation. Since synchronization of tool motion with workpiece angular position is desired, control time-step advancement is triggered based upon workpiece angular position. This information was obtained from a 1440 pulse per revolution rotary encoder attached to the lathe spindle. Spindle speeds for the

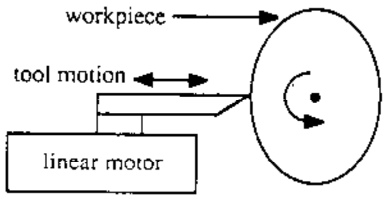

Fig. 13 Noncircular turning setup

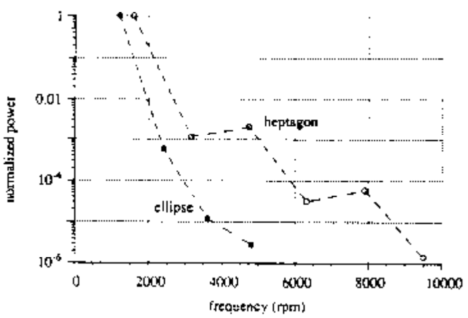

Fig. 14 Normalized discrete power spectral densities of synthesizing profiles

heptagon and ellipse were chosen as 450 and 600 rpm respectively, with corresponding effective sampling rates of 64800 and 86400 samples per minute.

At the above spindle speeds, the synthesizing profiles display the power spectral densities shown in Fig. 14. Note that the DC component is not shown. The ellipse contains harmonics at integer multiples of twice the spindle speed, whereas the heptagon contains harmonics at integer multiples of 3.5 times the spindle speed (i.e., half of 7 sides). The power of the harmonics at higher frequencies (not shown on graph) rapidly diminishes for both profiles. For robustness, a 14th order zero-phase RSG filter was chosen of the form:

$$\frac{B_f(z)}{A_f(z)}=\frac{z^7(z+1)^7(z^{-1}+1)^7}{2^{14}z^7} \tag{39}$$

With the above spindle speeds, the bandwidth of this filter is 4563 and 6084 rpm (i.e., samples per minute) for the heptagon and ellipse, respectively. Workpiece material was 6061-T6 aluminum.

Tracking errors for a typical workpiece revolution are shown in Figs. 15(*a*) and 15(*b*), and are seen similar under cutting and non-cutting conditions. The heptagon shows maximum errors of about 4 percent reference peak to peak, whereas the ellipse shows only one percent maximum error.

8 Conclusions

Solution existence requirements have been developed for exact 2-D model matching control of a class of linear distributed plants. The attractiveness of this controller stems from its ability to form a rational closed-loop input-output mapping starting from the distributed open loop plant. It has been shown how 2-D model matching may be applied to design a repetitive controller for a rational plant, such that both reference model matching and asymptotic rejection of a periodic disturbance of specified period may be simultaneously achieved. This approach does not require the plant to be first stabilized. Further, the resulting controller is computationally efficient, being a low-order filter with time delay. The discrete-time domain design offers the advantage of selecting an RSG low-pass filter with a zero phase frequency response at all frequencies. Chosen as such, the disturbance rejection response minimums will coincide exactly with the Fourier frequencies of the periodic disturbance. Finally, experimental results have been presented for the turning of two noncircular shapes, using the 2-D model matching approach to achieve asymptotic tracking.

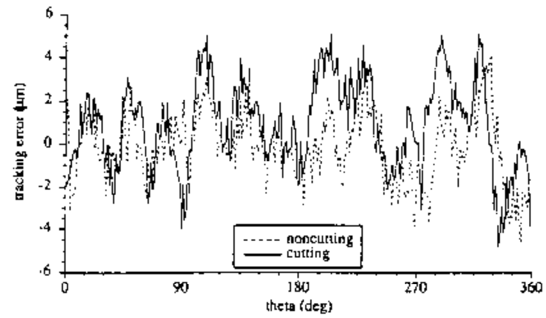

Fig. 15(*a*) Experimental result of ellipse tracking error

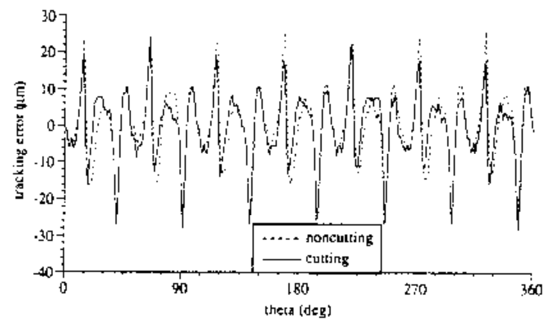

Fig. 15(*b*) Experimental result of heptagon tracking error

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References

Astrom, K., and Wittenmark, B., 1990, *Computer Controlled Systems*, Prentice Hall, Englewood Cliffs, N.J., 2nd ed., pp. 283–299.

Chen, B., Wang, S., and Lu, H., 1989, "Minimal Sensitivity Perfect Model Matching Control," *IEEE Transactions on Automatic Control*, Vol. 34, No. 12, pp. 1279–1283.

Eising, R., and Emre, E., 1979, "Exact Model Matching of 2-D Systems," *IEEE Transactions on Automatic Control*, Vol. 24, No. 1, pp. 134–135.

Francis, B. A., and Wonham, W. M., 1975, "The Internal Model Principle for Linear Multivariable Regulators," *Appl. Math. Opt.*, Vol. 2, pp. 170–194.

Hara, S., Yamamoto, Y., Omata, T., and Nakano, M., 1988, "Repetitive Control System: A New Type Servo System for Periodic Exogenous Signals," *IEEE Transactions on Automatic Control*, Vol. 33, No. 7, pp. 659–667.

Johansson, R., 1986, "Estimation and Direct Adaptive Control of Delay-Differential Systems," *Automatica*, Vol. 22, No. 5, pp. 555–560.

Kucera, V., 1979, *Discrete Linear Control*, Wiley, New York, NY, pp. 41–59.

Merritt, H. E., 1965, "Theory of Self-Excited Machine-Tool Chatter," ASME *Journal of Engineering for Industry*, Vol. 87, No. 4, pp. 447–454.

Sebek, M., 1985, "On 2-D Pole Placement," *IEEE Transactions on Automatic Control*, Vol. AC-30, No. 8, pp. 819–822.

Sebek, M., 1983, "2-D Exact Model Matching Control," *IEEE Transactions on Automatic Control*, Vol. AC-28, No. 2, pp. 215–217.

Tomizuka, M., Tsao, T.-C., and Chew, K., 1989, "Analysis and Synthesis of Discrete-Time Repetitive Controllers," ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 111, pp. 353–358.

Tsao, T.-C., Tomizuka, M., 1988, "Adaptive and Repetitive Digital Control Algorithms for Noncircular Machining," *Proceedings 1988 American Control Conference*, Atlanta GA., pp. 115–120.

Tsao, T.-C., 1988, "Digital Tracking Control and Its Application to Noncircular Machining," Ph.D. dissertation, Univ. of California-Berkeley.

Tsao, T.-C., Chen, Y. Y., and Tomizuka, M., 1990, "Noncircular Turning of Workpieces with Sharp Corners," ASME *Journal of Engineering for Industry*, Vol. 112, No. 2, pp. 181–183.