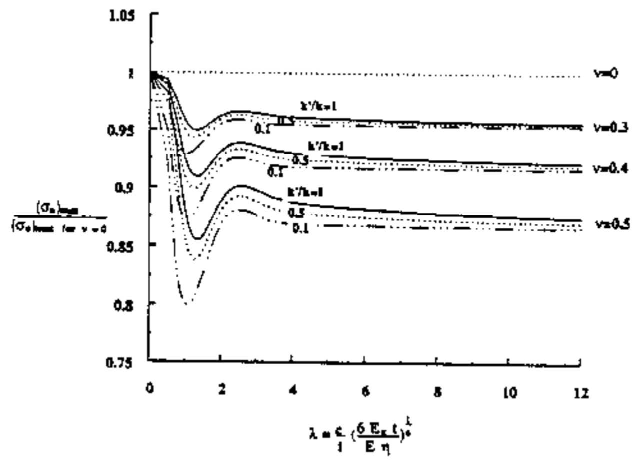

Fig. 6 The sensitivity of the maximum peel stress, normalized by $(\sigma_o)_{max}$ for $\nu=0$, to ν, k'/k and λ

In order to understand the sensitivity of the solution to ν, $(\sigma_o)_{max}$ Eq. (2) is normalized by $(\sigma_o)_{max\ for\ \nu=0}$:

$$\frac{(\sigma_o)_{max}}{(\sigma_o)_{max\ for\ \nu=0}} = \frac{\lambda_o^2 \frac{1}{2}\frac{\sinh 2\lambda_o - \sinh 2\lambda_o}{\sinh 2\lambda_o + \sin 2\lambda_o} + \lambda_o \frac{k'}{k}\frac{\cosh 2\lambda_o + \cos 2\lambda_o}{\sinh 2\lambda_o + \sin 2\lambda_o}}{\lambda^2 \frac{1}{2}\frac{\sinh 2\lambda - \sin 2\lambda}{\sinh 2\lambda + \sin 2\lambda} + \lambda \frac{k'}{k}\frac{\cosh 2\lambda + \cos 2\lambda}{\sinh 2\lambda + \sin 2\lambda}} \qquad (15)$$

where $\lambda_o = \lambda\,(1-\nu^2)^{1/4}$. Equation (15) is then plotted against λ in Fig. 6. It is shown that increasing ν leads to reduced $(\sigma_o)_{max}$. That is, for $\lambda > 3$ and $\nu = 0.3 \sim 0.4$, Eq. (3) without the Poisson effect ($\nu = 0$) gives five ~ seven percent more for $(\sigma_o)_{max}$ than Eq. (7).

Conclusions

It has been shown, through inspection of the original derivation, equilibrium considerations, and numerical analyses, that the original Goland and Reissner solutions for adhesive peel stress, which has been misused and misquoted by several researchers, is correct, even though there were some errors in the derivation. However, there are still some minor discrepancies in the peel stress equations, which are related to neglecting the Poisson effect, and accurately determining edge shear force, V_o. It has been demonstrated that Poisson effect should be included so that Eq. (3) should be replaced by Eq. (7). It is also suggested that the edge shear V_o expression in Eq. (8) be used in Eq. (4).

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A Sufficient Stability Condition for Linear Conservative Gyroscopic Systems

Jinn-Wen Wu[5] and Tsu-Chin Tsao[5,6]

A sufficient stability condition for linear conservative gyroscopic systems with negative definite stiffness matrices is given. The condition for the stability is stated in terms of the coefficients of system matrices without solving the spectrum of the entire system. An example is given for comparison with existing results.

1 Introduction

One interesting phenomenon for gyroscopic dynamic systems is that unstable conservative systems can be stabilized by increasing the gyroscopic forces. Criteria to determine reasonably small gyroscopic forces for stabilization are important for practical problems, for example, in the spacecraft attitude control problem, the artillery shell moving problem, and the flexible rotating shaft problem (Chetayev, 1961; Hughes, 1986). Recent results (Inman, 1988; Walker, 1991) have stimulated new interest in providing simple stability conditions for such systems without resorting to eigenvalue calculation.

Linear conservative gyroscopic systems can be described by a set of ordinary differential equations satisfying

$$\ddot{x}(t) + G\dot{x}(t) + Kx(t) = 0, \qquad (1)$$

where $G^T = -G$, $K^T = K$. The special case of interest here is when K is negative definite and G is nonsingular. The stability used here is in the sense of Lyapunov (see, for instance, Bellman, 1953).

It is well known that the system described by Eq. (1) is stable if and only if the latent values associated with (1) are purely imaginary and nonzero (see, for instance, Huseyin, 1978) since if λ is a latent value so is $-\lambda$. It has also been known that dissipative damping can easily violate gyroscopic stabilization. Hence in a practical sense, since dissipative forces almost always exist, the gyroscopic stability is only temporary.

2 Previous Results

It has been known that when the determinant of G is sufficiently large, Eq. (1) is stable (Müller, 1985; Roseau, 1987). However, so far only a few criteria to determining the strength of the gyroscopic force for stabilizing the system are available. Summaries of results to date are given in Inman (1988) and Walker (1991).

For a two-degree-of-freedom system, Teschner (1977) showed that the stability for negative definite K results if $4K - G^2$ is positive definite. Inman and Saggio (1985) showed that the stability results if the trace of $4K - G^2$ and the determinant of K are both positive and applied the result to the design of a dynamically tuned gyroscope. In fact, for a two-

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degree-of-freedom system, eigenvalues can be evaluated directly. Thus the following necessary and sufficient condition not appeared in the literature previously can be derived:

$$[\det(G)]^2 + \mathrm{tr}(K) > 2\sqrt{\det(K)} > 0. \quad (2)$$

Our main interest here is to provide simple stability criteria for n-degree-of-freedom systems. Huseyin (1983) obtained the following result: Suppose $GK = KG$. Then Eq. (1) is stable if and only if

$$4K - G^2 > 0. \quad (3)$$

When G and K does not commute, the construction of stability conditions becomes much more difficult. Inman (1988) proposed that the system described by Eq. (1) is stable if

$$4K - G^2 - \epsilon I > 0, \text{ where } 2\epsilon = \lambda_{max}(-G^2). \quad (4)$$

However, a rigorous proof was not given as pointed out by Walker (1991). Based on the Lyapunov approach, Walker (1991) derived that the system described by Eq. (1) is stable if there exists a real number α such that

$$K(K - \alpha I) > 0 \text{ and } K - \alpha I - \alpha G^T(K - \alpha I)^{-1}G > 0. \quad (5)$$

The evaluation of this stability condition requires searches of the α values. Our main result of this paper is a simple sufficient stability condition with a rigorous proof.

3 Main Results

Lemma 1: (Huseyin, 1978): Consider the system described by Eq. (1). The system is stable if and only if for every latent vector q the following is true:

$$4\langle Kq, q\rangle - \langle Gq, q\rangle^2 > 0 \text{ and } \langle Kq, q\rangle \neq 0.$$

Proof: The solution of Eq. (1) is stable if and only if every latent value λ is purely imaginary. Let C^n be the n-dimensional complex Euclidean space and for all $z, w \in C^n$ define the inner product $\langle z, w\rangle = \Sigma_{i=1}^n z_i \overline{w_i}$ and norm $\|z\|^2 = \langle z, z\rangle$. It is not difficult to verify that $\forall\, x \in C^n$, $\langle Gx, x\rangle$ is pure imaginary and $\langle Kx, x\rangle$ is real. Suppose λ is a latent value of Eq. (1) and q is the corresponding latent vector, where $\lambda \in C$, $q \in C^n$. Then

$$x(t) = e^{\lambda t} q \quad (6)$$

is a fundamental solution of Eq. (1). Substitute Eq. (6) into Eq. (1). It follows that

$$\lambda^2 q + \lambda G q + K q = 0. \quad (7)$$

Take the inner product to the both sides of Eq. (7) with q:

$$\lambda^2\langle q, q\rangle + \lambda\langle Gq, q\rangle + \langle Kq, q\rangle = 0. \quad (8)$$

Assume that the latent vector q has unit length, i.e., $\|q\| = 1$. Then

$$\lambda^2 + \lambda\langle Gq, q\rangle + \langle Kq, q\rangle = 0. \quad (9)$$

Let $\lambda = i\omega$, $k = -\langle Kq, q\rangle$ and $\langle Gq, q\rangle = i\beta$. Then Eq. (9) becomes

$$\omega^2 + \beta\omega + k = 0. \quad (10)$$

To show that λ is purely imaginary and nonzero is equivalent to showing that ω is real and non-zero, which is true if and only if

$$\beta^2 - 4k = -\langle Gq, q\rangle^2 + 4\langle Kq, q\rangle > 0 \text{ and } k \neq 0. \quad \text{QED.}$$

Although Lemma 1 gives the necessary and sufficient condition for stability, it requires the knowledge of a system latent vector and hence it is not useful for the stability investigation when the system dimension becomes large. A sufficient stability condition which does not require the calculation of system latent vectors or values is given next.

Theorem 1: The system described by Eq. (1), with $K < 0$ and det. $G \neq 0$, is stable if

$$3K + \frac{1}{\delta}K^2 - G^2 > 0. \quad (11)$$

where $\delta < 0$ is the maximum eigenvalue of K.

Proof: Suppose there exists a latent value $\lambda = i\omega$ and the corresponding latent vector q, satisfying $\beta^2 - 4k = -\langle Gq, q\rangle^2 + 4\langle Kq, q\rangle \leq 0$. Then the roots ω of Eq. (10) are complex conjugates and $|\omega|^2 = k > 0$. A contradiction is verified in the following.

By Eq. (9), $\langle Gq, q\rangle = -1/\lambda\,(\langle Kq, q\rangle + \lambda^2)$. Then

$$-\langle Gq, q\rangle^2 = |\langle Gq, q\rangle|^2$$

$$= \left(\frac{|k-\lambda^2|}{|\lambda|}\right)^2 = \frac{(k-\lambda^2)(k-\overline{\lambda^2})}{|\lambda|^2}$$

$$= \frac{k^2 - 2\mathrm{Re}(\lambda^2)k + |\lambda|^4}{|\lambda|^2}$$

$$= \frac{1}{|\omega|^2}\{k^2 + 2\mathrm{Re}(\omega^2)k + |\omega|^4\}. \quad (12)$$

By Eq. (7), it follows $Gq = -1/\lambda\,(Kq + \lambda^2 q)$. Then

$$\langle -G^2 q, q\rangle = \langle Gq, Gq\rangle$$

$$= \left\langle \frac{1}{\lambda}(Kq + \lambda^2 q), \frac{1}{\lambda}(Kq + \lambda^2 q)\right\rangle$$

$$= \frac{1}{|\lambda|^2}\langle Kq + \lambda^2 q, Kq + \lambda^2 q\rangle$$

$$= \frac{1}{|\lambda|^2}\{\|Kq\|^2 + \overline{\lambda^2}\langle Kq, q\rangle + \lambda^2\langle Kq, q\rangle + \langle \lambda^2 q, \lambda^2 q\rangle\}$$

$$= \frac{1}{|\lambda|^2}\{\|Kq\|^2 - 2\mathrm{Re}(\lambda^2)k + |\lambda|^4\}$$

$$= \frac{1}{|\omega|^2}\{\|Kq\|^2 + 2\mathrm{Re}(\omega^2)k + |\omega|^4\}. \quad (13)$$

Since $3K + 1/\delta\, K^2 - G^2 > 0$,

$$-3\langle Kq, q\rangle + \frac{1}{\delta}\langle K^2 q, q\rangle - \langle -G^2 q, q\rangle > 0$$

$$\rightarrow -3k|\omega|^2 + \frac{|\omega|^2}{\delta}\|Kq\|^2 + \|Kq\|^2 + 2\mathrm{Re}(\omega^2)k + |\omega|^4 > 0. \quad (14)$$

Noting that $\frac{|\omega|^2}{\delta} = \frac{k}{\delta} = \frac{-\langle Kq, q\rangle}{\underset{x \in C^n}{\mathrm{Max}}\langle Kx, x\rangle} \leq -1$,

$$\rightarrow -3k^2 - \|Kq\|^2 + \|Kq\|^2 + 2\mathrm{Re}(\omega^2)k + k^2 > 0.$$

$$\rightarrow \mathrm{Re}(\omega^2)k > k^2. \quad (15)$$

Then by Eqs. (12) and (15) we have

$$-\langle Gq, q\rangle^2 + 4\langle Kq, q\rangle$$

$$= \frac{2\mathrm{Re}(\omega^2)k - 2k^2}{k}$$

$$> 0.$$

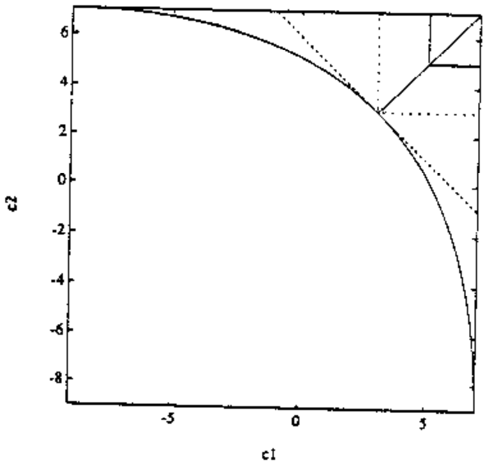

Fig. 1 Stability region predicted by various stability criteria

A contradiction is concluded. Therefore, every latent vector must satisfy $4\langle Kq, q\rangle - \langle Gq, q\rangle^2 > 0$ and the system is stable by The Lemma. QED.

Since for the case that $GK = KG$, the stability condition is $4K - G^2 > 0$, it is natural to ask whether the premises could be relaxed. The following stability criterion does not require K and G be commutative.

Theorem 2: Consider Eq. (1) in the following form:

$$G=\begin{bmatrix}0 & B\\ -B^T & 0\end{bmatrix},\; K=-\begin{bmatrix}k_1I & 0\\ 0 & k_2I\end{bmatrix} \tag{16}$$

where B is an arbitrary nonsingular $n \times n$ matrix, k_1, k_2, are nonzero positive real numbers, and I is the $n \times n$ identity matrix. If $4K - G^2 > 0$, then the system is stable.

Proof: Let

$$\phi=\det(\lambda^2 I_{2n}+\lambda G+K). \tag{17}$$

It follows that

$$\phi=\det((\lambda^2-k_1)I)\det((\lambda^2-k_2)I+\lambda^2B^TB(\lambda^2-k_1)^{-1}I)$$

$$=\lambda^{2n}\det\left(\frac{(\lambda^2-k_1)(\lambda^2-k_2)}{\lambda^2}I+B^TB\right) \tag{18}$$

since

$$-G^2+4K>0 \text{ implies } BB^T-4k_1I>0 \text{ and } B^TB-4k_2I>0 \text{ and} \tag{19}$$

since BB^T and B^TB have the same eigenvalues,

$$\mu>\lambda_{\min}(B^TB)>\mathrm{Max}\{4k_1,\, 4k_2\}, \tag{20}$$

where μ is any eigenvalue of B^TB. From Eq. (20), every root of $\phi(\lambda) = 0$ must satisfy

$$\frac{(\lambda^2-k_1)(\lambda^2-k_2)}{\lambda^2}+\mu=0. \tag{21}$$

Let $\omega = \lambda^2$. Then Eq. (21) is equivalent to

$$\omega^2+(\mu-(k_1+k_2))\omega+k_1k_2=0. \tag{22}$$

Then the roots of Eq. (22) can be expressed as

$$\omega_\pm=\frac{[(k_1+k_2)-\mu]\pm\sqrt{(\mu-(k_1+k_2))^2-4k_1k_2}}{2}.$$

By (20), it follows that

$$k_1+k_2-\mu<0, \text{ and } (\mu-(k_1+k_2))^2-4k_1k_2>\mu^2 -2\mu(k_1+k_2)>\mu^2-4\mu(\mathrm{Max}(k_1,\, k_2))>0. \tag{23}$$

Therefore $\omega_\pm$ are negative real. Therefore, all roots are purely imaginary and hence the system is stable. QED.

4 An Example

Consider a simplified model of a mass mounted on a noncircular weightless rotating shaft (Inman, 1991). The equation of motion in the rotating reference frame is (Huseyin, 1976)

$$\ddot{x}+2\xi\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\dot{x}+\begin{bmatrix}c_1-\xi^2-\eta & 0\\ 0 & c_2-\xi^2-\eta\end{bmatrix}x=0, \tag{24}$$

where x represents the displacements of the mass in the rotating reference coordinate frame, ξ is the shaft angular velocity, η is the axial compression force, and $c_1 \geq c_2 \geq 0$ are elastic rigidities in the two principle directions. Using the same parameter values as those in Inman's example (1988), $\xi^2 = 4$, $\eta = 3$, Fig. 1 shows the stability regions for c_1 and c_2 obtained from various stability criteria. The wing shape enclosed by the solid curve and lines $c_1 = 7$ and $c_2 = 7$ is the stability region predicted by the necessary and sufficient condition for two-degree-of-freedom systems per Eq. (2). The triangular shape enclosed by the dashed line is the stability region predicted by Inman and Saggio's (1985) sufficient condition for two-degree-of-freedom systems. The square enclosed by the dash-dotted lines is the stability region predicted by Teschner's (1977) sufficient condition for two-degree-of-freedom systems and also by the sufficient condition for n-degree-of-freedom systems per Theorem 2. The football shaped dotted curves enclose the stability region predicted by the sufficient condition for n-degree-of-freedom systems per Theorem 1. The 45-deg straight solid line represents Huseyin's (1983) necessary and sufficient condition for the case $GK = KG$. The smaller square enclosed by the solid line is the stability region predicted by Inman's (1988) sufficient condition for n-degree-of-freedom systems per Eq. (4).

Since the simple stability conditions developed herein are meant for systems of large degree-of-freedom, where direct eigenvalue evaluation becomes cumbersome, this two-degree-of-freedom example is only for the purpose of graphical illustration of various stability conditions.

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