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Active and Passive Damping of Euler-Bernoulli Beams and Their Interactions

Active and passive damping of Euler-Bernoulli beams and their interactions have been studied using the beam's exact transfer function model without mode truncation or finite element or finite difference approximation. The combination of viscous and Voigt damping is shown to map the open-loop poles and zeros from the imaginary axis in the undamped case into a circle in the left half plane and into the negative real axis. While active PD collocated control using sky-hooked actuators is known to stabilize the beam, it is shown that the derivative action using proof-mass (reaction-mass) actuators can destabilize the beam.

1 Introduction

There is often a strong temptation to rely on computationally intensive methods like the finite element method when studying the dynamics of distributed systems neglecting those methods that may be more analytically based. The superiority of the former over the latter is unquestionable if the geometry of the structural system is complicated. However, this may not be true for "simple" systems like uniform Euler-Bernoulli or Timoshenko beams, where the analytical approach yields exact solutions and the finite element results are approximate. In analyzing large space structures, where beams are abundant, analytical methods may be more appealing since they can provide accurate, and even exact, solutions with a minimum of additional effort. Much of what will be reported herein involves the use of Green's functions, which can be transformed to the s-domain to give an exact transfer function for the distributed parameter system. Such a distributed transfer function will be desirable from the standpoint of control system design and stability analysis.

The use of Green's functions in structural dynamics has been explored by Bergman and coworkers (Marek and Bergman, 1985; McFarland and Bergman, 1985, 1986). This previous work has been done in the time domain, thereby restricting the scope for structural control. This paper extends the analysis to the frequency domain, as did Butkovskiy (1969, 1983), and more recently, Yang and Mote (1991) and Yang (1992), in order to study the problem of discrete control of a continuous Euler-Bernoulli beam and the effect of distributed damped on system poles. The use of proof-mass actuators for structural control is also investigated. Unlike similar work by Inman (1989, 1990), Politansky and Pilkey (1989), and Miller and Crawley (1988), where a lumped beam model is used, this paper analyzes the structural control problem utilizing a distributed model without mode-truncation. Utilization of a closed-form distributed model for the structure allows an accurate analysis, particularly of the higher modes and how they are affected by control effort targeted at the lower ones.

The effect of distributed damping on the system's open-loop poles and zeroes and its interaction with controllers has received little attention in the recent literature, even though it can significantly affect the dynamics of the system (Vidyasagar and Morris, 1990; Yang, 1991). Two types of distributed damping will be considered here: distributed viscous damping and distributed Voigt damping. The effect of both types will be analyzed in conjunction with the proportional and derivative controllers.

The derivation of the Green's function for the beam and its transformation to a distributed transfer function are discussed in Section 2. The open-loop poles and zeros of the damped beam are related to those of the undamped beam in Section 3, and the effect of PD control in the "sky-hooked" configuration is briefly mentioned. The distributed transfer function model of the beam with a proof-mass actuator is derived and used for PD control root-loci analysis in Section 4.

2 Transfer Function of Fixed-Free Beams

Green's function formulations in the time domain inevitably run into the problem of "non-self-adjointness" of operators if distributed damping is included, as the damping terms give rise to first derivatives in time in the governing equations of the distributed system. The condition of self-adjointness is desirable when formulating the discrete-continuous problem using Green's function methods. To avoid this, previous work with Green's functions in the time domain has been restricted to the case where the unforced distributed system is undamped. The nondimensionalized governing equation in free vibration is thus confined to the category

$$w^{IV}(x, t) + \ddot{w}(x, t) = 0$$

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When then system is damped, the computation of the system eigensolution is usually quite involved and necessarily approximate in the time domain. Though neglect of distributed damping is often justifiable based on relative magnitudes, it would be interesting to study the effect it has on the system's overall dynamic properties. This insight will provide a better understanding of the relationship between damping and system stability and also facilitate a more intelligent selection of structural properties to produce certain damping characteristics.

However, where the time domain proves barren, greener pastures can be found in the frequency domain. Since damping invariably produces complex eigenvalues, the frequency domain is better suited for handling damped systems. Futhermore, self-adjointness previously not available to operators with time derivatives is now assured, since the time derivative is simply replaced by the Laplace variable, s, which is a parameter in space.

With this in mind, the Euler-Bernoulli beam can be more accurately modeled by

$$w^{IV}(x, t) + a_4\dot{w}^{IV}(x, t) + a_0\dot{w}(x, t) + \dot{w}(x, t) = f(x, t) \quad (1)$$

Here, a_0 is the distributed viscous damping factor, and a_4 is the Voigt damping factor.

In the time domain, the Green's function $G(x, \zeta; \lambda)$ is the response of the system at "x" due to a harmonically varying point load applied at "ζ." This response is the superposition of the responses of the infinity of modes of the distributed system excited by that single input. When the Laplace transform of the Green's function is taken, the result is essentially an infinite order transfer function, $G(x, \zeta; s)$.

There are many ways for obtaining the transfer function, $G(x, \zeta; s)$. The method of Butkovskiy (1983) yields a transfer function comprised of an infinite series in the modes of the distributed structure. Yang (1992) adopted a method that casts the distributed problem into a set of spatial state space equations. However, since the closed-form Green's functions for Euler-Bernoulli beams for various boundary conditions are available, it would be desirable to find a way of transforming these results in the time domain into closed-form transfer functions in the frequency domain.

For an undamped Euler-Bernoulli beam subjected to a distributed load, the governing equation is given by

$$w^{IV}(x, t) + \ddot{w}(x, t) = f(x, t) \quad (2)$$

Letting $f(x, t)$ be a point harmonic force of frequency λ^2 at ζ, and separating variables, $w(x, t) = W(x)\, e^{i\lambda^2 t}$, the spatial part of (2) yields

$$W^{IV}(x) - \lambda^4 W(x) = \delta(x - \zeta) \quad (3)$$

where λ is the system eigenparameter and δ is the Dirac delta. Applying the appropriate boundary conditions and using the method of initial parameters (McFarland and Bergman, 1985; Shah 1967), the Green's function for the clamped-free Euler-Bernoulli beam is found to be:

$$G(x, \zeta; \lambda) = \frac{1}{D(\lambda)} \begin{cases} \lambda^{-3} Z_1(x, \zeta; \lambda) & x \le \zeta \\ \lambda^{-3} Z_2(x, \zeta; \lambda) & x \ge \zeta \end{cases} \quad (4)$$

$D(\lambda) = 4(1 + \cosh\lambda\, \cos\lambda)$

$Z_1(x, \zeta; \lambda) =$

$[\psi_1(\lambda(1-\zeta))\psi_2(\lambda) - \psi_2(\lambda(1-\zeta))\psi_1(\lambda)]\,\psi_3(\lambda x) +$

$[\psi_2(\lambda(1-\zeta))\psi_4(\lambda) - \psi_1(\lambda(1-\zeta))\psi_1(\lambda)]\,\psi_4(\lambda x)$

$Z_2(x, \zeta; \lambda) =$

$[\psi_3(\lambda\zeta)\psi_2(\lambda) - \psi_4(\lambda\zeta)\psi_1(\lambda)]\,\psi_1(\lambda(1-x)) +$

$[\psi_4(\lambda\zeta)\psi_4(\lambda) - \psi_3(\lambda\zeta)\psi_1(\lambda)\psi_2(\lambda(1-x))]$

$\psi_1(\alpha) = \cosh\alpha + \cos\alpha$

$\psi_2(\alpha) = \sinh\alpha + \sin\alpha$

$\psi_3(\alpha) = \cosh\alpha - \cos\alpha$

$\psi_4(\alpha) = \sinh\alpha - \sin\alpha$

The same procedure is followed to derive the transfer function except Eq. (2) is now replaced by its Laplace transform

$$w^{IV}(x, s) + s^2 w(x, s) = f(x, s) \quad (5)$$

The resulting transfer function for the clamped-free Euler-Bernoulli beam has the same exact form as Eq. (4), with the λ replaced by "$(-s^2)^{1/4}$."

Thus the distributed damping are drawn into the inertia and stiffness operators and the system operator becomes self-adjoint once the Laplace transform is taken, giving

$$w^{IV}(x, s) + a_4 s w^{IV}(x, s) + a_0 s w(x, s) + s^2 w(x, s) = f(x, s) \quad (6)$$

Inclusion of Voigt damping modifies the boundary conditions at the free end of the beam (Inman and Banks, 1989),

$$w^{II}(1, s) + a_4 s w^{II}(1, s) = -J_f s^2 w^{I}(1, s) \quad (7a)$$

$$\frac{\partial}{\partial x}[w^{II}(x, s) + a_4 s w^{II}(x, s)] = M_f s_2 w(x, s),\ x = 1. \quad (7b)$$

where J_f is the rotational moment of inertia of the point mass, M_f, at the free end. It is assumed that the point moment at the free end is negligible. The boundary condition of Eq. (7b) can also be simplified by treating the contribution of the point mass as an external, pointwise forcing function (with acceleration feedback) in Eq. (6). With these treatments, the boundary conditions of Eq. (7a) and (7b) can be reduced to the usual form,

$$w^{II}(1, s) = 0 \quad (7c)$$

$$w^{III}(1, s) = 0 \quad (7d)$$

Equation (6) can be written in a form similar to Eq. (3)

$$w^{IV}(x, s) + \left(\frac{s^2 + a_0 s}{a_4 s + 1}\right) w(x, s) = \frac{f(x, s)}{a_4 s + 1} \quad (8)$$

Just like s^2 in the undamped case, the factor $(s^2 + a_0 s/a_4 s + 1)$ is a parameter in the spatial domain of x and is analogous to $-\lambda^4$ of Eq. (3). Applying the method of initial parameters as before, the transfer function can be found to be of the same form as Eq. (4) with λ replaced by

$$\lambda = \left(-\frac{s^2 + a_0 s}{a_4 s + 1}\right)^{1/4} \quad (9)$$

The function thus obtained is the distributed transfer function from $(f(x, s)/a_4 s + 1)$ to $w(x, s)$.

While this transfer function can be used to relate distributed output to the distributed input, it also characterizes the response at the point x due to point force at ζ. Thus, if we consider a collocated actuator/sensor pair placed at the beam free end, the transfer function is

$$G_b(1, 1; s) = \frac{\frac{1}{\lambda^3}(\sin\lambda\cosh\lambda - \cos\lambda\, \sinh\lambda)}{(1 + \cos\lambda\cosh\lambda)(a_4 s + 1)} \quad (10)$$

It can be shown, using L'Hopital's rule, that the term, $1/\lambda^3$, is not responsible for any poles at the origin.

3 Effect of Distributed Damping on Open-Loop Poles and Zeros

It is well known that the open-loop poles and zeros of the undamped beam ($a_0 = a_4 = 0$) lie on the imaginary axis, where the poles are interlaced with the zeros. In this case the open-loop poles and zeros, denoted as "s_u" are related to the roots of the denominator and numerator of Eq. (10), denoted as λ, by

$$s_u^2 = -\lambda^4 \quad (11)$$

With distributed viscous damping, λ is replaced by

$$\lambda^4 = -s_u^2 = -(s^2 + a_0 s), \text{ or} \quad (12)$$

$$s = -\frac{a_0}{2} \pm \frac{\sqrt{a_0^2 + 4s_u^2}}{2} \quad (13)$$

It can be seen that viscous damping shifts the poles and zeros on the imaginary axis into the left half plane by $a_0/2$. It also lowers the undamped natural frequencies, ω_{ud}, to ω_{vd} according to

$$\omega_{vd} = \sqrt{\omega_{ud}^2 - \frac{a_0^2}{4}} \quad (14)$$

The influence of this form of distributed damping is uniform for all modes. A similar observation was made by Vidyasagar and Morris (1990).

With Voigt damping, the λ term in the characteristic equation is changed to

$$\lambda^4 = -s_u^2 = \frac{-s^2}{a_4 s + 1} \quad (15)$$

The damped poles and zeros may be expressed as:

$$s = \frac{a_4 s_u^2}{2} \pm \frac{\sqrt{s_u^2(a_4^2 s_u^2 + 4)}}{2} \quad (16)$$

From Eq. (16), it can be seen that the effect of Voigt damping is not uniform for all modes, unlike viscous damping. The higher modes, identified by larger λ values, are more highly damped than the lower ones. It fact, beyond a certain λ value, the modes become overdamped. Vidyasagar and Morris (1990) mentioned that the open-loop poles of the infinite-order modes modes are at negative infinity. Analysis of Eq. (16) indicates that there are actually two branches of the "open-loop locus." The locus returns to the real axis and breaks up into two branches, one approaching negative infinity, the other approaching $-1/a_4$ on the real axis. Voigt damping also imposes an upper bound on the imaginary part of the open-loop system at $1/a_4$. In fact, it can be easily verified that the mapping forms a circle centered at $(-1/a_4, 0)$ with radius $1/a_4$.

With both viscous and Voigt damping, it can be verified that the open-loop poles and zeros lie on a circle and the interval $(-\infty, -1/a_4)$ on the negative real axis. The circle is centered at $(-1/a_4, 0)$ with radius $\sqrt{\frac{1-a_0 a_4}{a_4}}$, if $1 - a_0 a_4 > 0$.

This open loop pole-zero mapping result may be useful for the identification of the distributed damping factors, in which a set of open-loop poles and zeros obtained from experimental test results is used to fit to a circle. The mapping can also be used for root-loci analysis. For example, consider P control using sky-hooked actuators. For an undamped beam, open-loop poles and zeros are interlaced and are located on the imaginary axis. The closed-loop poles with P control will still lie on the imaginary axis with increase in natural frequencies. Therefore for a beam with distributed damping, the closed-loop poles with P control must lie on a circle determined by the mapping. Since sky-hooked P action and D action are equivalent to adding a discrete spring and damper to the beam, instability cannot occur. This is not necessarily true if we use proof-mass actuators, discussed next.

4 Active Damping Using Proof-Mass Actuators

An infinitely large reaction mass is not always available for the controller to act on as in the sky-hooked configuration. Such is the case for control of tall structures or spaceborne structures. In such situations, the controller will have to carry its own reaction mass, also known as a reaction-mass. The controlling force comes from the reaction in accelerating the reaction-mass. Actuators that are based on this principle of action-reaction are termed proof-mass (reaction-mass) actuators; hence, PMA. Figure 1 shows the combined system. In the diagram, M_a, C_a, K_a are the actuator parameters, with C_a and K_a providing the restoring force for the subsystem. The lumped mass, M_f, is due to the dead mass of the PMA, i.e., mass due to the stationary electromagnetic coils and frame. Considering the resultant of the control force F, the M_f inertial force and the Ca, Ka restoring forces as the external force to beam, the transfer function for the open loop combined system can be obtained as

$$G_{PMA}(s) = \frac{W(1,s)}{F(s)} \triangleq \frac{B(s)}{A(s)}$$

$$= \frac{N(s)M_a s^2}{[D(s) + N(s)M_f s^2][M_a s^2 + C_a s + K_a] + N(s)M_a s^2(C_a s + K_a)} \quad (17)$$

where $N(s)$ and $D(s)$ are, respectively, the numerator and denominator of $G(1,1;s)$.

Consider a proportional control scenario. Figure 2 shows the root loci plot of four modes of system vibration: the first three natural modes of the beam and the mode caused by interaction with the PMA dynamics. The "mode" caused by the PMA will, hence, be called the PMA mode.

The above plot, obtained by a Newton-Raphson numerical iteration procedure for solving the transcendental equation, assumes that distributed viscous and Voigt damping factors

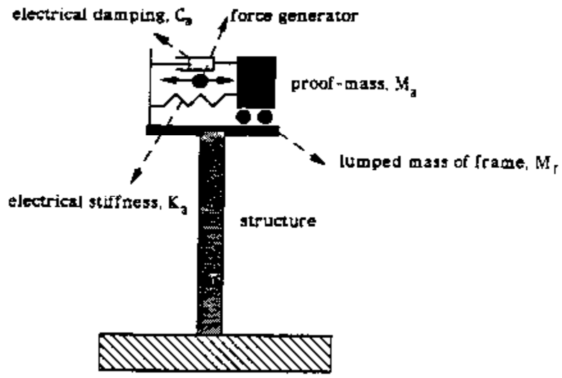

Fig. 1 Beam and proof-mass actuator combined system

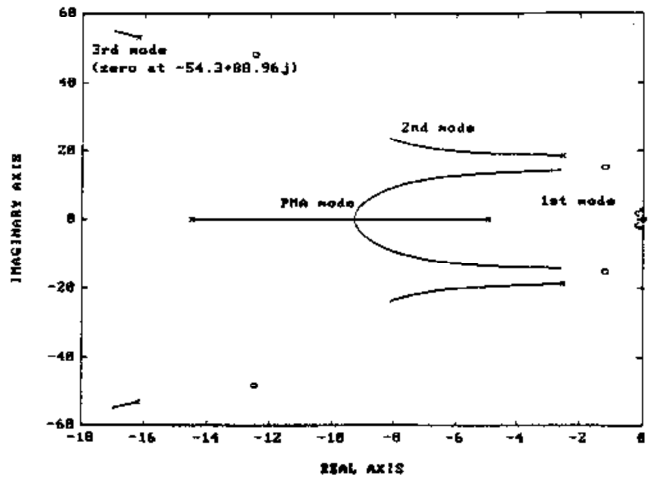

Fig. 2 Root-loci of the combined system under P control

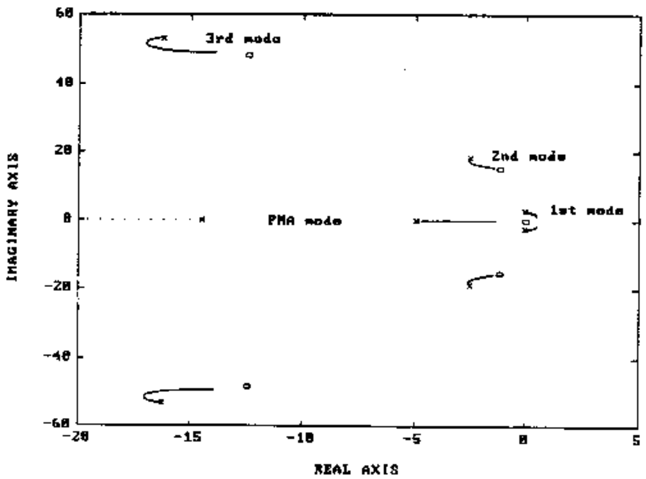

Fig. 3 Root-loci of the combined system under D control

are both 0.01 The frame mass and the reaction-mass are assumed to be ten percent of the beam mass, and the PMA is tuned critically damped.

Coupling the critically damped subsystem to the beam adds damping to it. The open-loop poles of all beam modes are shifted down and slightly to the left. The PMA-coupling exerts a stronger influence on the lower modes than on the higher ones. After the first two modes, the amount of displacement in open loop poles caused by the PMA-coupling is gradually reduced. The locations of the beam's zeroes are unchanged, as can be seen from the open loop transfer function of the combined system. The root-loci plot indicates that the PMA mode of vibration imparted to the system by the motion of the proof mass is initially overdamped, both open-loop poles being on the real axis. The actuator dynamics add two origin zeroes to the combined system. These two zeroes draw to it the loci of the first mode. The pair of zeroes that the first beam mode loci approached in the ideal case are now approached by the loci of the PMA mode. As the proportional gain is increased, the PMA mode starts to become underdamped. The higher beam modes become more stable with large K_p due to the more negative real parts of the closed-loop poles. The natural frequencies of these modes are also raised at the same time. Unlike these higher modes, there is a drop in frequency of the first mode as the gain is increased. With a PMA attached, the scope of controller design for the first beam mode is very limited since its loci do not venture far into the left half plane.

Use of D control on the PMA-beam system runs the risk of instability, though this might seem a little counter-intuitive. Inman (1990) has shown this to be true using a lumped model of the beam. By showing that the symmetric part of the damping matrix is negative definite if K_d is greater than a certain value, he has shown that the stability of the PMA-beam system, under a derivative control law, is not guaranteed. This condition of positive definiteness, however, is only necessary and not sufficient. If the condition is not satisfied, no conclusion about the system's stability can be drawn. A more conclusive study of the system's stability can be made by root-loci analysis, as shown in Fig. 3.

The above plots indicate that the instability associated with use of derivative control is solely due to the fundamental beam mode. The loci of the higher beam modes are all situated safety in the left half plane. The fact that their locations are successively further from the imaginary axis is attributed to the presence of Voigt damping. However, even if the beam is undamped, these modes will still be stable under a D control since the departure angles of their loci point to the left. The PMA mode is also stable. It is overdamped due to the presence of the third zero at the origin, which draws to the origin the right branch of the PMA loci.

On the other hand, the departure angles of the first mode's loci point to the right. Any increase in D gain from zero immediately reduces the stability of the mode and eventually destabilizes it and the system therewith. For a combined system that has no damping, a locus departing to the right immediately implies instability. If the system is damped, the open-loop poles will be shifted to the left. Here, a rightward departing locus does not immediately imply instability; however, instability still sets in once the gain is large enough to drive the first-mode loci beyond the imaginary axis.

Since the departure angles of the first-mode loci point to the left for P control and to the right for D control, there is a critical ratio of derivative to proportional gain below which the first-mode loci will remain in the left half plane.

By checking the definiteness of the damping matrix, Inman (1990) has defined a set of α's that will make the lumped parameter system asymptotically stable. The similar approach for finding α_c for a distributed parameter system will be difficult. For the continuous system, α_c can be found by expressing the departure angle of the first mode as a function of α and then setting it to 90 deg or 270 deg. With the PD control expressed as

$$G_c(s) = K(1 + \alpha s) \tag{18}$$

The departure angle is given by

$$\frac{\partial s}{\partial K} = \frac{-(1+\alpha s)B(s)}{\dfrac{dA(s)}{ds}} \tag{19}$$

Evaluating Eq. (19) at the open-loop pole location of the first mode and then solving for α by setting the real part to zero (to get a stationary point for the real part with respect to changes in K) gives

$$\alpha_c = 0.318 \tag{20}$$

The impact of α on system stability is shown in Fig. 4. Below the critical α, the system is always stable for all gains but beyond the critical α, there exist high gains which destabilize the system.

The first mode becomes less damped with larger values of α. This might prompt one to ask if a D action is at all desirable in a PMA system. The answer to this will not be complete if the higher modes are not taken into account. Though the derivative feedback has a destabilizing effect on the first mode, its effect on the higher modes is exactly the opposite. This is reflected in Table 1, which shows how the pole locations of the first four beam modes of a system without distributed damping shift under the P ($k = 10$) and PD control ($k = 10$, $\alpha = 0.2$).

The implication of this study is that when the PD control is used, damping for the high frequency vibrations of the closed-loop system is higher than when the P control is used (since the poles of the higher modes have more negative real parts when the former is used). While the P control is more effective than the PD control in damping out low frequency vibrations caused primarily by the first mode, it is not as effective as PD control in damping out high frequency vibrations of the closed loop system. This is illustrated in the impulse response plots in Fig. 5.

5 Conclusion

By working with a distributed plant model, accurate conclusions can be drawn about how the different modes respond to a certain control effort. This is best described by the various root-loci plots presented. The s-plane analysis of distributed damping has provided a detailed picture of how each of the two forms of damping affects the individual modes. Viscous

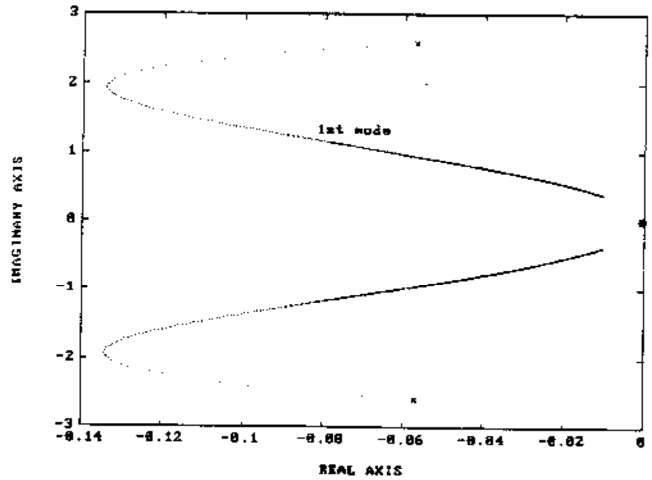

Fig. 4(*a*)

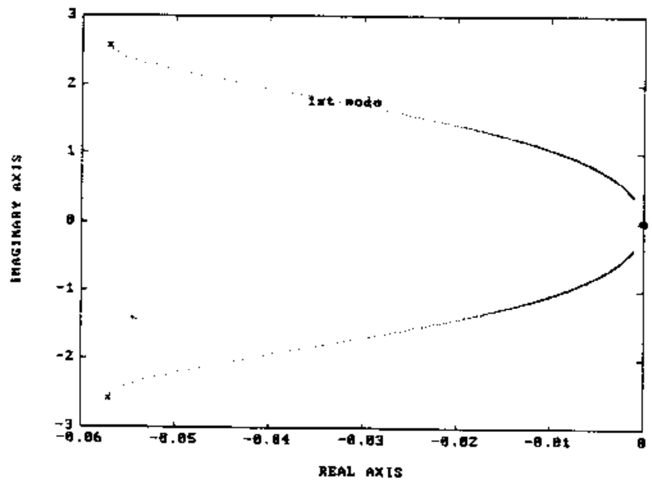

Fig. 4(*b*)

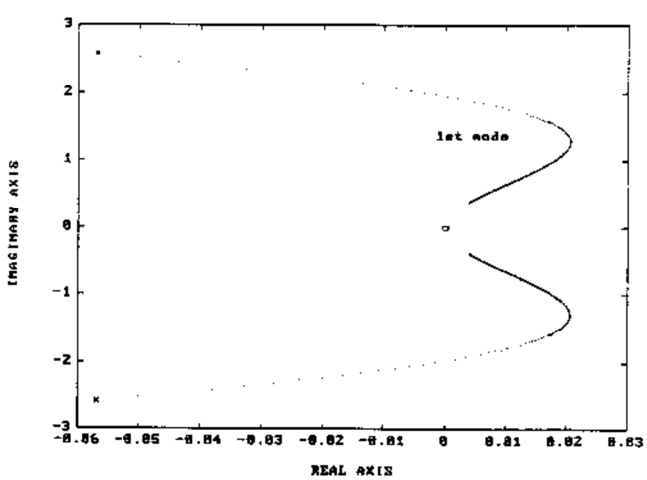

Fig. 4(*c*)

Fig. 4 Root-loci for the first beam mode of the combined system under PD control. The ratios of D gain to P gain are (*a*) $\alpha = 0.2$, (*b*) $\alpha = 0.318$, and (*c*) $\alpha = 0.35$, respectively.

damping maps the undamped, open loop poles and zeroes from the imaginary axis onto a vertical line in the left-half plane, thereby imposing a lower bound on the system damping. Voigt damping, on the other hand, maps the open loop poles and zeroes onto a circle in the left half plane, thereby making the higher modes more stable than the lower ones, and at the same time, imposing an upper bound on the system's natural fre-

Table 1 Closed-loop pole locations for the first four modes under P and PD control

	1st mode	2nd mode	3rd mode	4th mode
P Control	-0.2083+2.1998j	-1.0246+18.880j	-0.7443+55.379j	-0.5169+110.65j
P-D Control	-0.0690+2.1570j	-1.1848+17.829j	-1.525+54.7820j	-1.1790+110.39j

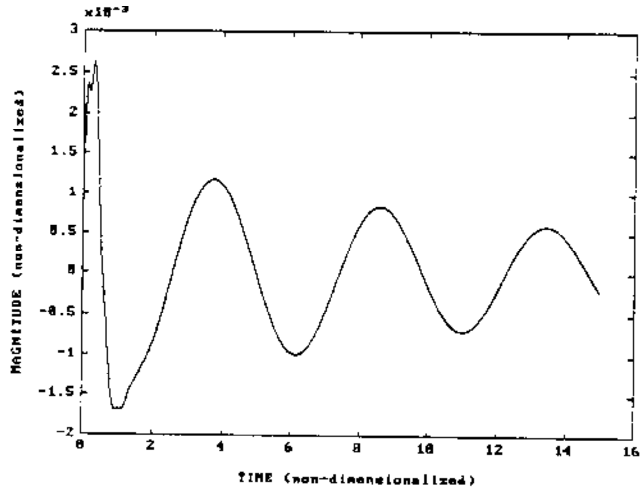

Fig. 5(*a*)

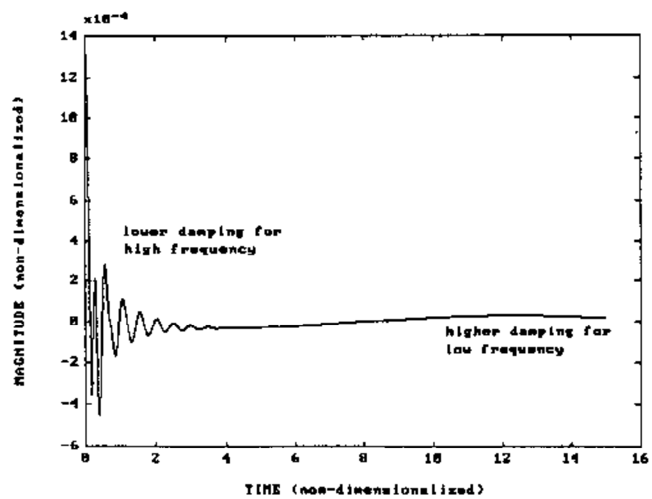

Fig. 5(*b*)

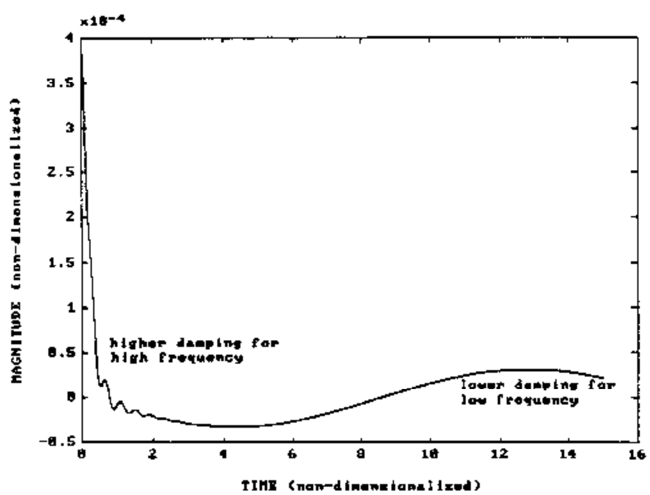

Fig. 5(*c*)

Fig. 5 Impulse responses of (*a*) open-loop system, (*b*) P control with $K = 10$, $\alpha = 0$, (*c*) PD control with $K = 10$, $\alpha = 0.2$

quencies. With Voigt damping, the stability of the system can be improved with just a P controller alone since the various root-loci move leftwards along the circular trajectory. It is obvious, therefore, that these forms of distributed damping provide a certain level of safeguard against spill-over instability in structural control, especially Voigt damping. Also, controllers of the types discussed will never run the risk of instability as long as positive gains (negative feedback) are used. The addition of a critically damped PMA shifts all the open loop poles and zeroes down and to the left slightly, with the lower modes affected more than the higher ones. The risk of instability in a PMA system using derivative feedback has been addressed, and is attributed to the first mode only. Derivative feedback has to be used in conjunction with a proportional feedback, with due regards for the critical gain ratio, in order to avoid instability. In controlling high frequency vibrations, a PD control is more effective than a P control, and the reverse is true in controlling low frequency vibrations.

The work reported can be extended to the problem of structural control with noncollocated actuator-sensor pairs, since it is only a matter of changing the "x" and "ζ" variables of the distributed transfer function. The control in the noncollocated case is more complicated due to the system nonminimum-phase zeros. The exact transfer function model used in this analysis can also be used as a basis for comparison of the various model reduction techniques commonly used in structural dynamics (Pang et al. 1991).

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