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Analysis and Synthesis of Discrete-Time Repetitive Controllers

Repetitive control is formulated and analyzed in the discrete-time domain. Sufficiency conditions for the asymptotic convergence of a class of repetitive controllers are given. The "plug-in" repetitive controller is introduced and applied to track-following in a disk-file actuator system. Inter-sample ripples in the tracking error were present when the "plug-in" repetitive controller was installed. The performance is enhanced, however, when the zero-holding device is followed by a low-pass filter or replaced by a delayed first-order hold.

1 Introduction

In control systems, disturbance and/or reference inputs invariably include a significant periodic component with a known period. A good example appears in the control of the read/write head of a computer memory storage device. A major function of the servo controller for a read/write head is to maintain the head accurately on a selected track of the rotating disk. This is a simple regulation problem with a constant reference input if the tracks are perfect circles on the disk; their centers agree with the center of rotation of the disk, with the head and disk not subjected to any vibrations. Unfortunately, these ideal conditions do not hold in practice; the reference input in Fig. 1 is not constant because of repeatable and nonrepeatable spindle runout. Eccentricity is a major source of the repeatable runout. This repeatable runout causes the reference input to contain a significant periodic and repetitive component and is also a source of periodic and repetitive disturbances. The period is the time for one spindle rotation. This kind of problem also arises in rotational machinery, including machine tools. Repetitive reference and disturbance inputs are also frequently encountered in robot control. When a robot manipulator has to repeat the same task a number of times, the reference input is obviously repetitive. Furthermore, if the actuator has some nonlinearities, their effects may be modelled as a periodic disturbance input.

Although tuning of general servo controllers such as PID and state feedback controllers may include considerations for periodic reference inputs and disturbances, special purpose controllers may be considered to handle these periodic signals; especially when the basic period is more or less known. The controller for this purpose is called a **repetitive** controller.

Any periodic signal with a known period, say L, can be generated by a free dynamical system which has a positive feedback around a pure time delay. This idea, combined with the internal model principle [5], has been the basis of repetitive control theory [6–10]. Repetitive control systems have been studied mainly in the continuous time domain. If the continuous time repetitive controller attempts to compensate for all high frequency components in periodic reference inputs or disturbances, an unrealistic requirement for a proper but not strictly proper controlled plant must be imposed to assure stability. This problem and a remedy were discussed in [6, 7]. Repetitive control is closely related to learning control, which has been developed primarily in robotics [1, 2, 4, 11, 14].

In this paper, the discrete time domain analysis and synthesis of repetitive control will be considered. The motivations for doing so are: 1) Digital implementation of repetitive controllers is simpler than analog and 2) The unrealistic requirement for the controlled plant does not appear since discrete time analysis naturally limits the highest frequency component. While an approximate digital implementation of repetitive controllers is discussed in [9], the discrete-time analysis and synthesis of repetitive control systems have not been considered.

Discrete time repetitive control is formulated in the next section. Section 3 presents the analysis of discrete time repetitive controllers that achieve the design objective. The digital

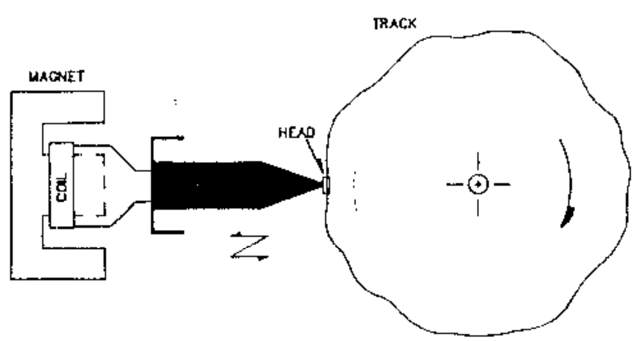

Fig. 1 Generation of repetitive signal in disk-drive system

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repetitive controller is applied to a pure inertia model for a disk file actuator in Section 4, wherein the plug-in repetitive controller concept as well as several useful ideas in the implementation of repetitive controllers are also introduced. Conclusions are in Section 5.

2 Discrete Time Repetitive Control Problem

Consider a servo system described by:

$$A(q^{-1})e(k) = q^{-d}B(q^{-1})u_r(k) + w(k) \tag{1}$$

where q^{-1} is a one-step delay operator,

$$A(q^{-1}) = 1 - a_1 q^{-1} - \ldots - a_n q^{-n}$$
$$B(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_m q^{-m} \tag{2}$$

$$b_0 \neq 0 \tag{3}$$

d represents known delay steps, $e(k)$ is the tracking error, $u_r(k)$ the control input and $w(k)$ represents the periodic signal with a known period, N.

Equation (1) is a general expression for servo systems of our interest. As an example, let the controlled plant be described by:

$$A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k) + d(k) \tag{4}$$

where $y(k)$ and $u(k)$ are the plant output and input, respectively, and $d(k)$ is the periodic disturbance with a known period of N. Define the tracking error by:

$$e(k) = y(k) - y_d(k) \tag{5}$$

where $y_d(k)$ is the periodic desired output with a known period of N. Then, equations (4) and (5) are combined into equation (1) with:

$$u_r(k) = u(k) \tag{6a}$$

and

$$w(k) = d(k) - A(q^{-1})y_d(k) \tag{6b}$$

Without loss of generality, assume equation (1) defines an asymptotically stable system: i.e., the characteristic roots of $A(z^{-1}) = 0$ are all inside the unit circle. This assumption implies that an appropriate stabilizing controller has been applied when the plant is unstable.

It is also assumed that $B(z^{-1})$ and $1 - z^{-N}$ are coprime (i.e., $B(e^{\pm i 2\pi i/N}) \neq 0$, $i = 0,1,2,\ldots,[N/2]$, where [*] denotes the largest integer $\leq$ *.). Otherwise, the characteristic roots of $B(z^{-1}) = 0$ can be either inside, on or outside the unit circle. This assumption is necessary for the asymptotic convergence of $e(k)$ to zero (see Steady State Analysis in the next section for details).

Factor $B(z^{-1})$ as:

$$B(z^{-1}) = B^{+}(z^{-1})B^{-}(z^{-1}) \tag{7}$$

where $B^{+}(z^{-1})$ and $B^{-}(z^{-1})$ are, respectively, the cancellable and uncancellable parts of $B(z^{-1})$ [12]. Thus $B^{-}(z^{-1})$ comprises roots on or outside the unit circle and undesirable roots which are in the unit circle and $B^{+}(z^{-1})$ comprises roots of $B(z^{-1})$ which are not in $B^{-}(z^{-1})$.

The design objective is to find $u_r(k)$ which assures the asymptotic stability of the overall system and asymptotic zero-error tracking, i.e.,

$$\lim_{k \to \infty} e(k) = 0 \tag{8}$$

3 Design of Discrete Time Repetitive Controllers

In discrete time, any periodic signal with a basic period of N can be generated by a delay chain with a positive feedback loop (see Fig. 2). Notice that the z-transform of a periodic signal, $w(k)$, can be expressed as:

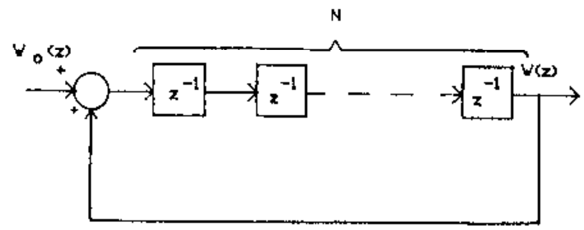

Fig. 2 Discrete-time repetitive signal generator

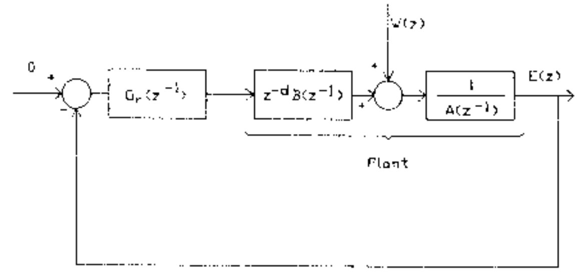

Fig. 3 Generalized discrete-time repetitive control system

$$W(z) = \frac{W_0}{1 - z^{-N}} = \frac{W_0 z^N}{z^N - 1} \tag{9}$$

where $W_0(z)$ represents the z-transform of the infinite sequence consisting of the first period of $w(k)$ and 0's, $\{w_0(k)\} = \{w(0), w(1), \ldots\ldots, w(N-1), 0, 0, 0, \ldots\}$, i.e.,

$$W_0(z) = w(0) + w(1)z^{-1} + \ldots + w(N-1)z^{-(N-1)} \tag{10}$$

Equation (9) confirms that the closed loop chain of delays in Fig. 2 is in fact the generator of repetitive signals.

For the error signal to converge to zero asymptotically, the repetitive controller must be able to generate a repetitive control signal when the error is zero. This implies that the repetitive controller must include a generator as in Fig. 2 (viz. Internal Model Principle). A general form of the internal model type repetitive controller is shown in Fig. 3. The transfer function, $G_r(z^{-1})$, in this configuration is:

$$G_r(z^{-1}) = \frac{R(z^{-1})}{S(z^{-1})(1 - z^{-N})} \tag{11}$$

With this G_r the closed loop characteristic equation is:

$$D(z^{-1}) = S(z^{-1})A(z^{-1})(1 - z^{-N}) + z^{-d}B(z^{-1})R(z^{-1}) = 0 \tag{12}$$

where $R(z^{-1})$ and $S(z^{-1})$ should be chosen such that $D(z^{-1})$ is asymptotically stable. Equation (12) is the Diophantine equation and the existence of solutions for $R(z^{-1})$ and $S(z^{-1})$ given $D(z^{-1})$ will be assured if $z^{-d}B(z^{-1})$ and $(1 - z^{-N})A(z^{-1})$ are coprime (see [3], for example). Equation (12) also characterizes all the stabilizing internal model type repetitive controllers $G_r(z^{-1})$ that result in stable $D(z^{-1})$. A special repetitive controller which belongs to this class of stabilizing controller is proposed in Section 3.2.

3.1 Steady State Analysis. The steady state performance of digital repetitive control systems is summarized in the following theorem.

Theorem 3.1 (Asymptotic Error Convergence). The error of the repetitive control system, $e(k)$, converges to zero asymptotically provided that the closed loop system remains asymptotically stable.

Proof. In the z-transform domain, $E(z) = Z\{e(k)\}$ of the closed loop system in Fig. 3 is:

$$E(z) = \frac{W(z)}{A(z^{-1}) + G_r(z^{-1})z^{-d}B(z^{-1})} \quad (13)$$

From equations (9), and (13), $e(k)$ may contain response modes from the closed loop characteristic equation $A(z^{-1}) + G_r(z^{-1})z^{-d}B(z^{-1}) = 0$ and modes from $1 - z^{-N}$, which are periodic (here, periodic modes are roots of the characteristic equation $1 - z^{-N} = 0$, which are $e^{\pm ij2\pi/N}$, $i = 0, 1, \ldots, [N/2]$. By assumption, the closed loop system is asymptotically stable; therefore it suffices to show that the associated residues of each of the periodic modes are equivalently zero: i.e.,

$$(e^{\pm ij2\pi/N} - 1)E(z)\Big|_{z = e^{\pm ij2\pi/N}} \equiv 0,$$

$$i = 0, 1, \ldots, \left[\frac{N}{2}\right] \quad (14)$$

From equations (9), (12), and (13),

$$E(z) = \frac{S(z^{-1})W_0(z)}{D(z^{-1})} \quad (15)$$

From equation (15), (14) is true since $D(z^{-1})$ is asymptotically stable by assumption.

3.2. Prototype Repetitive Controller and Its Stability Analysis. When N is large, equation (12) is not easy to solve and the order of $R(z^{-1})$ is large. This problem is more severe when equation (12) has to be solved on line, for instance, in adaptive control.

Instead, we select the following prototype repetitive controller:

$$\frac{R(z^{-1})}{R(z^{-1})} = k_r \frac{z^{-N+d+nu}A(z^{-1})(z^{-nu}B^-(z))}{B^+(z^{-1})b}$$

$$b \geq \max_{\omega \in [0,\pi]} |B^-(e^{-j\omega})|^2 \quad (16)$$

where k_r is called the **repetitive control gain**, nu is the order of $B^-(z^{-1})$, i.e., the number of uncancellable zeroes, and $B^-(z)$ is obtained by replacing every z^{-1} in $B^-(z^{-1})$ by z. For the controller in equation (16) to be realizable, we assume that $N - d - nu \geq 0$. Notice that equation (16) has a close resemblance to the zero phase error tracking controller [12]. Indeed, the asymptotic stability of the repetitive control system in equation (15) is easy to establish with the prototype repetitive controller.

For repetitive control systems with equation (16), the stability is summarized in *Theorem* 3.2.

Theorem 3.2 The repetitive system with $G_r(z^{-1})$ given by equation (11) and (16) is asymptotically stable for $0 < k_r < 2$.

Proof. The closed loop characteristic equation, equation (12), of the repetitive control system with $G_r(z^{-1})$ given by equation (11) and (16) is:

$$A(z^{-1})B^+(z^{-1})\left[z^N - 1 + \frac{k_r B^-(z)B^-(z^{-1})}{b}\right] = 0 \quad (17a)$$

Since $A(z^{-1})B^+(z^{-1})$ is stable, it suffices to show the stability of the term in the brackets, rewritten as:

$$z^{-N}\left[\frac{k_r B^-(z)B^-(z^{-1})}{b} - 1\right] + 1 = 0 \quad (17b)$$

The characteristic equation of equation (17b) can be regarded as coming from the feedback system in Fig. 4.

By construction, the frequency response gain of the open loop transfer function in the positive feedback system in Fig. 4 is less than 1 as long as $0 < k_r < 2$.

Therefore, the encirclement of the Nyquist plot around the critical point (−1) is 0. Hence, the asymptotic stability of the repetitive control system is assured. From Fig. 3, pole-zero

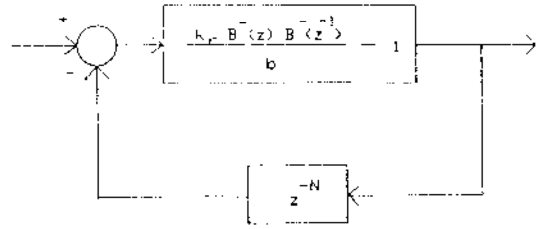

Fig. 4 Equivalent feedback system for stability analysis

cancellation occurs for every characteristic root of $A(z^{-1})$ and $B^+(z^{-1})$. These characteristic roots remain as hidden closed loop poles.

Corollary 3.3:(Finite Settling Time). Suppose $B(z^{-1})$ has no uncancellable part, i.e., $B^-(z^{-1}) = 1$. Then $k_r = b = 1$ gives the characteristic equation:

$$A(z^{-1})B^+(z^{-1})z^{-N} = 0 \quad (18)$$

where the stable hidden modes from $A(z^{-1})B^+(z^{-1})$ do not appear at the output and the N roots at the origin render finite settling time for the tracking error.

In order to establish guidelines for choosing b, Lemma 3.4 is given first

Lemma 3.4. Let $H(z^{-1}) = h_0 + h_1 z^{-1} + \ldots + h_s z^{-s}$. Then $H(e^{-j\omega})H(e^{j\omega}) = |H(e^{-j\omega})|^2$. Further

(1) $|H(e^{-j\omega})|^2 \leq (|h_0| + |h_1| + \ldots + |h_s|)^2$

(2) $0 < |H(e^{-j\omega})|^2/[H(1)]^2 \leq 1$ if all characteristic roots of $H(z^{-1})$ are in the closed left half side of the z-plane.

(3) $0 < |H(e^{-j\omega})|^2/[H(-1)]^2 \leq 1$ if all characteristic roots of $H(z^{-1})$ are in the closed right half side of the z-plane.

To achieve a fast convergent rate for the error, we would like to have $b = \max_{\omega \in [0,\pi]} |B^-(e^{-j\omega})|^2$. The following corollary gives either a tight upper bound or the exact maximum values of $|B^-(e^{-j\omega})|^2$.

Corollary 3.5:(Selection of b). The following choices of b results in asymptotically stable repetitive controller:

(1) $b = (|b_0| + |b_1| + \ldots + |b_m|)^2$

(2) $b = [B^-(1)]^2$ if all the zeros of $B^-(z^{-1})$ are in the closed left half plane.

(3) $b = [B^-(-1)]^2$ if all the zeros of $B^-(z^{-1})$ are in the closed right half plane.

The open loop transfer function with the prototype repetitive controller is $k_r z^{-N}B^-(z)B^-(z^{-1})/\{(1 - z^{-N})b\}$; which has infinite gain at the repetitive frequencies, $\pm i2\pi/N$, $i = 0, 1, 2, \ldots [N/2]$. The closed loop transfer function of $k_r z^{-N}B^-(z)B^-(z^{-1})/\{(1 - z^{-N})b + k_r z^{-N}B^-(z)B^-(z^{-1})\}$ shows that the gain is ≈ 1 when the desired signal to be regulated comprises frequencies close to the repetitive frequencies.

4 Repetitive Controller for Disk File Actuator System

Figure 1 depicts the components of a disk file servo system. In track following, the read/write head must be kept on a circular track on the spinning track. A block diagram for a simplified disk file servo system is given in Fig. 5. In the figure, the dynamics of the disk file actuator are assumed to be second order, and the controller is selected to be of PD (proportional plus derivative) type. The tracking error signal is normally the only signal directly available to the controller, i.e., the desired track and actual head positions are not separately available.

A digital repetitive controller is added to the existing servo system as a **plug-in** unit as shown in Fig. 5. A plug-in module has the advantage that the analog controller can be designed

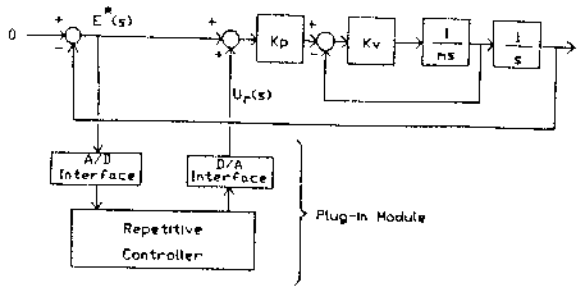

Fig. 5 Disk-drive system with repetitive controller

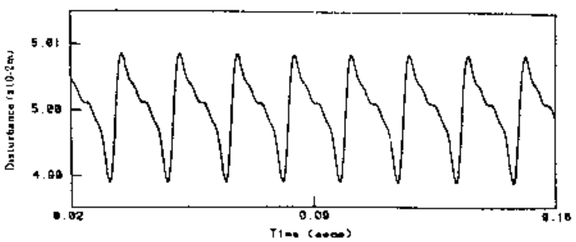

Fig. 6 Periodic disturbance used in simulations

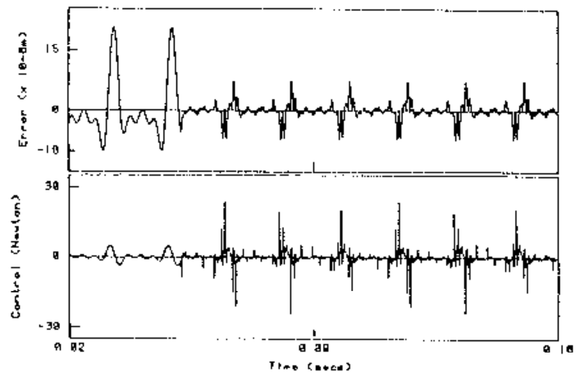

Fig. 7(a) Disk-drive simulation with ZOH ($N = 16$, $k_r = 1.0$)

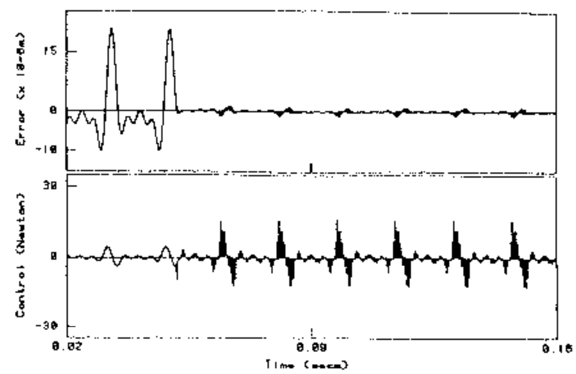

Fig. 7(b) Disk-drive simulation with ZOH ($N = 32$, $k_r = 1.0$)

without special consideration for the repetitive nature of the input/disturbance; instead the designer can concentrate his efforts on other aspects like robustness and noise rejection, for example.

It is assumed that the disk is spinning at 3600 rpm. The periodic disturbance in Fig. 6 comprises a constant signal and four out-of-phase sinusoidal signals from the first to the fourth harmonic of the spindle frequency. The analog PD loop is tuned to give a bandwidth of 1 kHz and a damping ratio of 0.7071. It is emphasized that while various techniques are available for improving the low-frequency regulation performance of the analog system, significant periodic errors will be present in tracking periodic reference inputs or when periodic disturbances are present if the analog controller is designed without a view of regulating periodic signals.

The design considerations for the repetitive controller in this problem include:

- Selection of N, which determines the sampling time, T_s, for the repetitive controller. In this case $T_s = 1/60N$. $T_s \approx 1$ millisecond when $N = 16$, for example.
- Selection of the digital to analog interface: i.e., data hold.
- Selection of the analog to digital interface: e.g., anti-alias filter.

$B(z^{-1})$ depends on the sampling time (equivalently N) of the digital controller. As long as $B(e^{\pm i2\pi/N}) \neq 0$ for $i = 0, 1, 2, \ldots [N/2]$, which is satisfied in all the examples in the ensuing discussion, perfect regulation will be achieved. The third point above is not relevant in the so-called sector servo method, since the error signal in this case is inherently discrete.

Consider first the case when a zero-order hold (ZOH) is used. In this case, for a selected N, i.e., a selected sampling time, the discrete time transfer function from $u_r(k)$ to $e(k)$ must be obtained. It can be verified from the block diagram in Fig. 5 that $E(s)$ is given by:

$$E(s) \hat{=} -E^*(s) = G_p(s)U_r(s) \tag{19}$$

where

$$G_p(s) = \frac{k_p k_v}{ms^2 + k_v s + k_p k_v} \tag{20}$$

The desired discrete time transfer function is then obtained by:

$$G_p(z) = (1 - z^{-1})Z\left(L^{-1}\left(\frac{G_p(s)}{s}\right)\right) \tag{21}$$

where L^{-1} and Z denote inverse Laplace and Z transformations, respectively.

In order to compare the tracking performance with and without the repetitive controller, the repetitive controller was turned on after the 3rd cycle (i.e., at 0.05 s). Figures 7(*a*) and 7*b*), are respectively for $(N, k_r) = (16, 1.0)$ and $(32, 1.0)$, with a zero order hold as the digital to analog interface.

From the figures, we see that the repetitive controller reduces the tracking error due to the periodic input; the *sampled* error indeed converges to zero rapidly. The inter-sample error, however, is not zero, especially when N is small. This is due to the interaction of the high bandwidth analog servo controller and the output of the digital repetitive controller with a zero-order hold, which produces a sequence of step changes. The control input, which looks like a series of finite impulses, may have undesirable effects, such as the excitation of unmodelled resonance modes in practice.

One way to reduce the inter-sample ripples in the error is to replace the zero-order hold with a first-order holding device. With a first-order hold (FOH) the desired discrete time transfer function of equation (20) is:

$$G_p(z) = (1 - 2z^{-1} + z^{-2})Z\left(L^{-1}\left(\frac{G_p(s)(1 + Ts)}{Ts^2}\right)\right) \tag{22}$$

where T is the sampling time.

From simulations which are not shown here, the repetitive controller with a first-order hold produces better inter-sample response than a zero-order hold; the ripples in inter-sample error has a lower magnitude. The output of the first-order hold is still discontinuous.

This motivates the investigation of the performance of a holding device which produces a continuous control. There are several ways to achieve this. One way is to follow the zero order hold by a low-pass filter with its break frequency at half of the sampling frequency. In this case, the analog filter

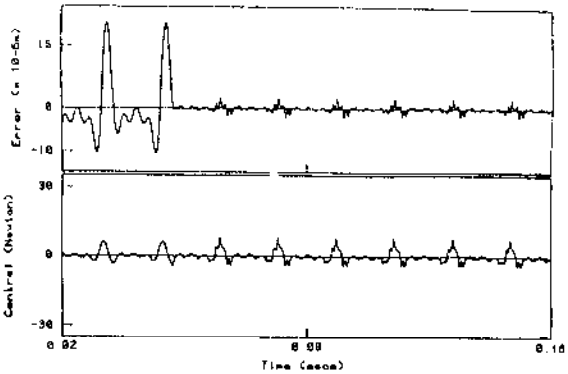

Fig. 8(*a*) Disk-drive simulation with Butterworth filter ($N = 16$, $k_r = 1.0$)

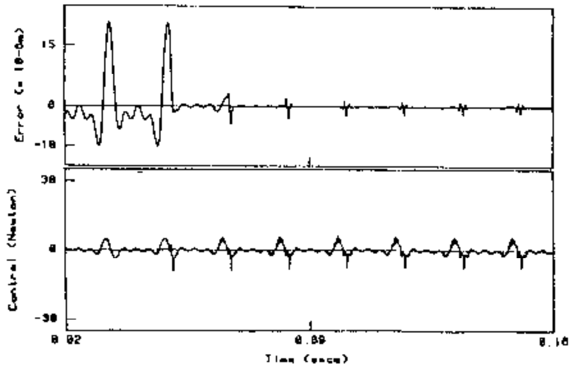

Fig. 8(*b*) Disk-drive simulation with Butterworth filter ($N = 32$, $k_r = 1.0$)

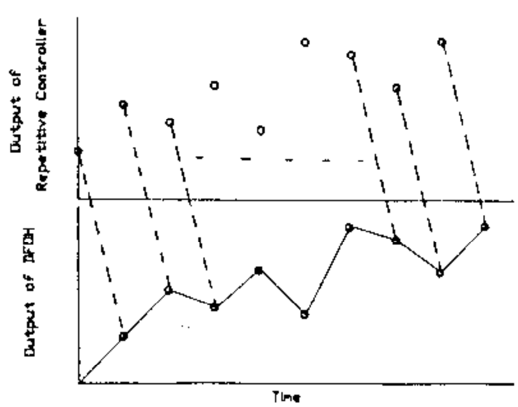

Fig. 9 Delayed first-order hold

should be taken as a part of the plant when designing the repetitive controller. Figures 8(*a*) and 8(*b*) show the results of adding a second order Butterworth filter under the same conditions for Figs. 7(*a*) and 7(*b*) respectively. Another way is to use a **delayed first-order hold** (DFOH) [13] as illustrated in Fig. (9). With a DFOH, the desired discrete time transfer function of equation (20) is:

$$G_p(z) = (1 - 2z^{-1} + z^{-2})Z\left(L^{-1}\left(\frac{G_p(s)}{Ts^2}\right)\right) \qquad (23)$$

where T is the sampling time.

Figures 10(*a*) and 10(*b*) are obtained under the same conditions as in Figs. 7(*a*) and 7(*b*), respectively, except for the replacement of the ZOH's with the DFOH's. The magnitudes of the ripples in the inter-sample error for the repetitive controller with a low-pass Butterworth filter or a DFOH are smaller than with a ZOH. Furthermore, the control input to the actuator in both cases are significantly smoother than with a ZOH.

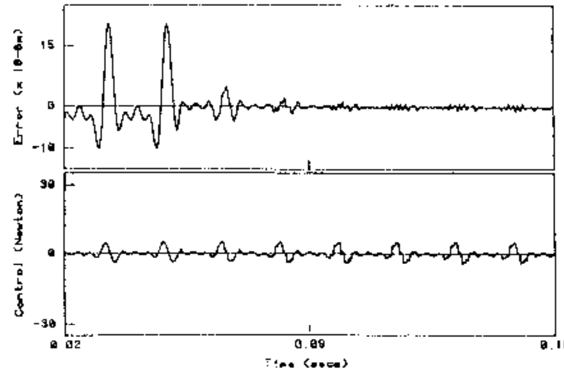

Fig. 10(*a*) Disk-drive simulation with DFOH ($N = 16$, $k_r = 1.0$)

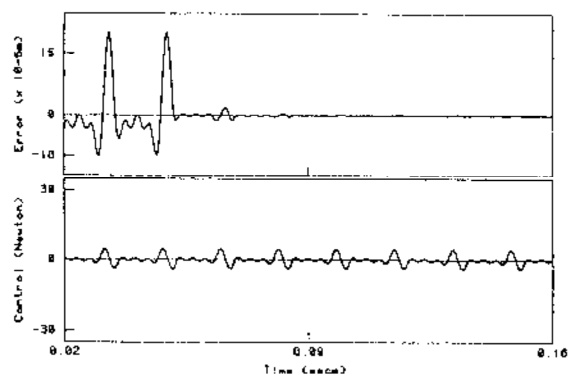

Fig. 10(*b*) Disk-drive simulation with DFOH ($N = 32$, $k_r = 1.0$)

A low-pass Butterworth filter can be easily constructed in practice using well-known analog filter design techniques. On the other hand, a DFOH may be implemented digitally by providing linear interpolation between $u_r(k-1)$ and $u_r(k)$ during the kth time instance using an interpolation time interval that is $\ll$ than the sampling time for the digital repetitive controller.

5 Conclusion

In this paper, discrete-time repetitive control was presented and the asymptotic convergence properties for a class of repetitive controllers were proven. The idea of pole-zero cancellation and/or zero-phase compensation was useful in the derivation of a stabilizing repetitive controller which has a fast convergent rate. This would save solving a Bezout-type equation and give a simpler and faster algorithm for real-time implementation. An application of discrete-time repetitive control to a simulated disk-file actuator system has shown its feasibility and benefits. The "plug-in" concept enables the designer to concentrate on other aspects of the system other than the repetitive nature of the input/disturbance. When the number of samples is low, the performance of the repetitive controller is dramatically enhanced by following the zero-order-hold with a low-pass filter or changing the holding device to a delayed first-order hold.

While the analysis in this paper is based on an accurate plant model, the plant itself may be subjected to modelling uncertainties or parameter variations. Although not specifically addressed in this current paper, the robustness of the controller can be enhanced by sacrificing regulation performance. Usually only the high frequency regulation performance need be sacrificed because a good model of a system is normally available at low frequencies. One way to attain robustness at the expense of high frequency regulation is to include a low-pass filter in the internal model.

In order for the digital repetitive controller to be viable in practice, the random noise propagation characteristics of the repetitive control system have to be analyzed. For instance, spindle speed variability (leading to variations in N for a fixed sampling time for the digital repetitive controller) for the disk-drive example in this paper may be modelled as a random perturbation to deduce its effects on the repetitive control system. These issues will be dealt with in a future paper.

Recently, it was brought to our attention by Prof. Longman, Columbia University, that the discrete-time repetitive control problem has been discussed in the following report:

Middleton, R. H., Goodwin, G. C., and Longman, R. W., "A Method for Improving the Dynamic Accuracy of a Robot Performing a Repetitive Task," Technical Report No. EE8546, Department of Electrical and Computer Engineering, University of Newcastle, Australia, Dec. 1985.

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