Systems & Control Letters 22 (1994) 223–225
North-Holland

Simple stability criteria for nonlinear time-varying discrete systems

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Received 26 February 1993

Abstract: We present two sufficient conditions for asymptotic stability of nonlinear time-varying discrete systems. Our main results generalize Mori's result for linear discrete time-invariant systems to nonlinear time-varying systems.

Keywords: Stability criteria; nonlinear time-varying systems; delay independent stability; discrete systems.

Introduction

Mori et al. [1] gave a delay independent stability condition for linear discrete systems. The system

$$x(n+1) = Ax(n) + Bx(n-h),$$

where $x \in \mathbb{R}^n$, A and $B \in \mathbb{R}^{n \times n}$, is asymptotically stable if

$$\|A\| + \|B\| < 1.$$

This result provides a simple way of stability check without solving characteristic roots. For nonlinear systems, a similar condition can be derived.

Main theorem. *Consider a nonlinear discrete time-varying system of the following form:*

$$x(n+1) = \sum_{i=1}^{p} A_i(\cdot,\cdot)\, x(n-h_i), \tag{1}$$

where $x \in \mathbb{R}^n$, $A_i(\cdot,\cdot) \in \mathbb{R}^{n \times n}$ is a function of time indices $n, n-1, n-2, \ldots$, etc., and $x(n), x(n-1), x(n-2), \ldots$, etc., and $0 \le h_1 < h_2 < h_3 \cdots < h_p$. Suppose there exists operator norm $\|\cdot\|$ and constants $C_i > 0$, $1 \le i \le p$ such that

$$\|A_i(\cdot,\cdot)\| \le c_i.$$

Then system (1) *is asymptotically stable if*

$$\sum_{i=1}^{p} c_i < 1.$$

Lemma 1. *Consider the following scalar difference equation:*

$$y(n+1) = a_1 y(n-h_1) + a_2 y(n-h_2) + \cdots + a_p y(n-h_p), \tag{2}$$

where $y(n) \ge 0$ and $a_i \ge 0$.

Then

$$\sum_{i=1}^{p} a_i < 1 \iff y(n) \to 0 \quad \text{as } n \to \infty.$$

Proof. ($\Rightarrow$) Suppose $\sum_{i=1}^{p} a_i < 1$. By applying z-transform to equation (2), the characteristic polynomial is

$$H(z) = z^{h_p+1} - a_1 z^{(h_p-h_1)} - a_2^{(h_p-h_2)} - \cdots - a_p.$$

Let

$$H_1(z) = z^{h_p+1}, \quad \text{and}$$

$$H_2(z) = -(a_1 z^{(h_p-h_1)} + \cdots + a_p).$$

Then $\forall z$, $|z| = 1$, we have

$$|H_1(z)| = 1 \quad \text{and}$$

$$|H_2(z)| \le a_1 + a_2 + \cdots + a_p < 1.$$

Hence,

$$|H_1(z)| > |H_2(z)|.$$

By Rouché Theorem [2] H_1 and $H_1 + H_2$ have same number of roots (counting multiplicity) inside

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SSDI 0167-6911(93)E0070-W

the unit circle. Therefore, all roots of $H(z) = H_1(z) + H_2(z)$ are inside the unit circle and the solution of equation (2) asymptotically converges to zero.

($\Leftarrow$) Suppose $\sum_{i=1}^{p} a_i \geq 1$.

(i) $\sum a_i = 1 \Rightarrow H(1) = 0 \Rightarrow$ equation (2) is not asymptotically stable.

(ii) $\sum a_i > 1$. Then $H(1) = 1 - \sum_{i=1}^{p} a_i < 0$. Choose a number b sufficiently large such that $H(b) > 0$; then $H(1)H(b) < 0$, which implies that there exists e, $1 < e < b$, such that $H(e) = 0$. Therefore, the system is unstable.

Lemma 2. *Consider the following scalar difference inequality:*

$$w(n+1) \leq a_1 w(n-h_1) + a_2 w(n-h_2) + \cdots + a_p w(n-h_p), \tag{3}$$

where $w(n) \geq 0$, $a_i \geq 0$.

If (2) and (3) have same initial condition, then $w(n) \leq y(n)$ for $n = 1, 2, \ldots$

Proof. Suppose there exists a positive integer m such that

$$w(k) \leq y(k), \quad k \leq m \text{ and } w(m+1) > y(m+1).$$

Since

$$w(m+1) \leq \sum_{i=1}^{p} a_i w(m-h_i),$$

$$y(m+1) = \sum_{i=1}^{p} a_i y(m-h_i),$$

imply $w(m+1) \leq y(m+1)$, it contradicts (3). Hence the proof is complete. □

Therefore, by Lemmas 1 and 2, we can conclude that

$$w(n) \to 0 \quad \text{as } n \to \infty$$

if

$$y(n) \to 0 \quad \text{as } n \to \infty.$$

Proof of main theorem. Since equation (1) can be expressed as

$$x(n+1) = \sum_{i=1}^{p} A_i(\cdot,\cdot)\, x(n-h_i) \tag{4}$$

and $\|A_i(\cdot,\cdot)\| \leq c_i$, taking norm on both sides of equation (4) yields

$$\|x(n+1)\| = \left\| \sum_{i=1}^{p} A_i(\cdot,\cdot)\, x(n-h_i) \right\| \leq \sum_{i=1}^{p} \|A_i(\cdot,\cdot)\|\, \|x(n-h_i)\| \leq \sum_{i=1}^{p} c_i \|x(n-h_i)\|.$$

By Lemmas 1 and 2, $\|x(n)\| \to 0$ as $n \to \infty$.

Corollary. *Suppose*

$$x(n+1) = \sum_{i=1}^{p} f_i(\cdot, x(n-h_i)), \tag{5}$$

where $x \in \mathbb{R}^n$, $0 \leq h_1 < h_2 < h_3 \cdots < h_p$, and $f_i : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 function of x, which may vary with time and $f(\cdot, 0) = 0$.

Suppose there is an operator norm such that for every $x \in \mathbb{R}^n$

$$\|Jf_i(\cdot, x)\| \leq c_i$$

($Jf_i(\cdot, x)$ is the Jacobian matrix of f_i at x)

then equation (5) is asymptotically stable if $\sum_{i=1}^{p} c_i < 1$.

Proof. $\forall x \in \mathbb{R}^n$,

$$f_i(\cdot, x) = f_i(\cdot, x) - f_i(\cdot, 0) = \int_0^1 \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} f_i(\cdot, \lambda x) \right] \mathrm{d}\lambda = \int_0^1 [Jf_i(\cdot, \lambda x)]\, x\, \mathrm{d}\lambda = A_i(\cdot, x)x,$$

where

$$A_i(\cdot, x) = \int_0^1 Jf_i(\cdot, \lambda x)\, \mathrm{d}\lambda.$$

Further,

$$\|A_i(\cdot, x)\| = \int_0^1 \|Jf_i(\cdot, \lambda x)\|\, \mathrm{d}\lambda \leq c_i.$$

The proof is completed by applying the main theorem.

The result of Mori et al. [1] is a special case of the main theorem or of the Corollary, supposing $f_i(x) = A_i x$.

Example 1. Consider

$$x(n+1) = \begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 0.5\cos(n)\sin(x_1(n-5))\,x_2(n) \\ 0.4\cos(x_1(n))\,x_2(n-10) \end{bmatrix}.$$

Then

$$A_1(\cdot,\cdot)\,x(n) = \begin{bmatrix} 0 & 0.5\cos(n)\sin(x_1(n-5)) \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix},$$

$$A_2(\cdot,\cdot)\,x(n-10) = \begin{bmatrix} 0 & 0 \\ 0 & 0.4\cos(x_1(n)) \end{bmatrix} \times \begin{bmatrix} x_1(n-10) \\ x_2(n-10) \end{bmatrix}.$$

Then $\forall x \in \mathbb{R}^n$,

$$\|A_1(\cdot,\cdot)\|_\infty \le c_1 = 0.5, \quad \|A_2(\cdot,\cdot)\|_\infty \le c_2 = 0.4.$$

Since $c_1 + c_2 = 0.9 < 1$, by the main theorem, $x(n)$ converges to 0 asymptotically.

Example 2. Consider

$$x(n+1) = \sin(n)[0.5\,x(n) - 0.4\sin(x(n-10))].$$

Since $f_1(\cdot,0) = f_2(\cdot,0) = 0$, $Jf_1(\cdot,x) = 0.5\sin(n)$ and $Jf_2(\cdot,x) = 0.4\sin(n)\cos(x(n-10))$ by the Corollary, $x(n)$ converges to 0 asymptotically.

Concluding remark

Simple delay-independent sufficient conditions for asymptotic stability of nonlinear time-varying discrete systems have been presented.

References

[1] T. Mori, N. Fukuma and M. Kuwahara, Delay-independent stability criteria for discrete-delay systems, *IEEE Trans. Automat. Control* **27** (1982) 964–966.

[2] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1974).