Simple stability criteria
for nonlinear time-varying discrete systems

Tsu-Chin Tsao

Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, 1206 West Green Street, Urbana, IL 61801, USA

Received 26 February 1993

Abstract: We present two sufficient conditions for asymptotic stability of nonlinear time-varying discrete systems. Our main results generalize Mori's result for linear discrete time-invariant systems to nonlinear time-varying systems.

Keywords: Stability criteria; nonlinear time-varying systems; delay independent stability; discrete systems.

Introduction

Mori et al. [1] gave a delay independent stability condition for linear discrete systems. The system

\[ x(n + 1) = Ax(n) + Bx(n - h), \]

where \( x \in \mathbb{R}^n \), \( A \) and \( B \in \mathbb{R}^{n \times n} \), is asymptotically stable if

\[ \| A \| + \| B \| < 1. \]

This result provides a simple way of stability check without solving characteristic roots. For nonlinear systems, a similar condition can be derived.

Main theorem. Consider a nonlinear discrete time-varying system of the following form:

\[ x(n + 1) = \sum_{i=1}^{p} A_i(\cdot, \cdot) x(n - h_i), \tag{1} \]

where \( x \in \mathbb{R}^n \), \( A_i(\cdot, \cdot) \in \mathbb{R}^{n \times n} \) is a function of time indices \( n, n - 1, n - 2, \ldots, \) etc., and \( x(n), x(n - 1), x(n - 2), \ldots, \) etc., and \( 0 \leq h_1 < h_2 < h_3 \cdots < h_p \).

Suppose there exists operator norm \( \| \cdot \| \) and constants \( C_i > 0, 1 \leq i \leq p \) such that

\[ \| A_i(\cdot, \cdot) \| \leq C_i. \]

Then system (1) is asymptotically stable if

\[ \sum_{i=1}^{p} a_i < 1. \]

Lemma 1. Consider the following scalar difference equation:

\[ y(n + 1) = a_1 y(n - h_1) + a_2 y(n - h_2) + \cdots + a_p y(n - h_p), \tag{2} \]

where \( y(n) \geq 0 \) and \( a_i \geq 0 \).

Then

\[ \sum_{i=1}^{p} a_i < 1 \iff y(n) \to 0 \text{ as } n \to \infty. \]

Proof. \( (\Rightarrow) \) Suppose \( \sum_{i=1}^{p} a_i < 1 \). By applying z-transform to equation (2), the characteristic polynomial is

\[ H(z) = z^{h_p - h_1} a_1 + z^{h_p - h_2} a_2 + \cdots + a_p. \]

Then \( \forall z, |z| = 1 \), we have

\[ |H_1(z)| = 1 \quad \text{and} \]

\[ |H_2(z)| \leq a_1 + a_2 + \cdots + a_p < 1. \]

Hence.

\[ |H_1(z)| > |H_2(z)|. \]

By Rouche Theorem [2] \( H_1 \) and \( H_1 + H_2 \) have same number of roots (counting multiplicity) inside
the unit circle. Therefore, all roots of
\[ H(z) = H_1(z) + H_2(z) \]
are inside the unit circle and the solution of equation (2) asymptotically converges to zero.

(\(\Rightarrow\)) Suppose \(\sum_{i=1}^{p} a_i \geq 1\).

(i) \(\sum a_i = 1 \Rightarrow H(1) = 0 \Rightarrow \) equation (2) is not asymptotically stable.

(ii) \(\sum a_i > 1\). Then \(H(1) = 1 - \sum_{i=1}^{p} a_i < 0\).

Choose a number \(b\) sufficiently large such that \(H(b) > 0\); then \(H(1)H(b) < 0\), which implies that there exists \(e, 1 < e < b\), such that \(H(e) = 0\). Therefore, the system is unstable.

Lemma 2. Consider the following scalar difference inequality:
\[ w(n + 1) \leq a_1 w(n - h_1) + a_2 w(n - h_2) + \cdots + a_p w(n - h_p) \]
where \(w(n) \geq 0, a_i \geq 0\).

If (2) and (3) have same initial condition, then \(w(n) \leq y(n)\) for \(n = 1, 2, \ldots\)

Proof. Suppose there exists a positive integer \(m\) such that
\[ w(k) \leq y(k), \quad k \leq m \quad \text{and} \quad w(m + 1) > y(m + 1). \]
Since
\[ w(m + 1) \leq \sum_{i=1}^{p} a_i w(m - h_i), \]
\[ y(m + 1) = \sum_{i=1}^{p} a_i y(m - h_i), \]
imply \(w(m + 1) \leq y(m + 1)\), it contradicts (3). Hence the proof is complete. \(\Box\)

Therefore, by Lemmas 1 and 2, we can conclude that
\[ w(n) \to 0 \quad \text{as} \quad n \to \infty \]
if \(y(n) \to 0 \quad \text{as} \quad n \to \infty\).

Proof of main theorem. Since equation (1) can be expressed as
\[ x(n + 1) = \sum_{i=1}^{p} A_i(\cdot, \cdot) x(n - h_i) \]
and \(\|A_i(\cdot, \cdot)\| \leq c_i\), taking norm on both sides of equation (4) yields
\[ \|x(n + 1)\| = \left\| \sum_{i=1}^{p} A_i(\cdot, \cdot) x(n - h_i) \right\| \leq \sum_{i=1}^{p} \|A_i(\cdot, \cdot)\| \|x(n - h_i)\| \leq \sum_{i=1}^{p} c_i \|x(n - h_i)\|. \]
By Lemmas 1 and 2, \(\|x(n)\| \to 0\) as \(n \to \infty\).

Corollary. Suppose
\[ x(n + 1) = \sum_{i=1}^{p} f_i(\cdot, x(n - h_i)), \]
where \(x \in \mathbb{R}^n, 0 \leq h_1 < h_2 < h_3 \cdots < h_p\), and \(f_i: \mathbb{R}^n \to \mathbb{R}^n\) is \(C^1\) function of \(x\), which may vary with time and \(f(\cdot, 0) = 0\).

Suppose there is an operator norm such that for every \(x \in \mathbb{R}^n\),
\[ \|f_i(\cdot, \cdot)\| \leq c_i \]
then equation (5) is asymptotically stable if \(\sum_{i=1}^{p} c_i < 1\).

Proof. \(\forall x \in \mathbb{R}^n\),
\[ f_i(\cdot, x) = f_i(\cdot, x) - f_i(\cdot, 0) = \int_{0}^{1} \left[ J f_i(\cdot, \lambda x) \right] d\lambda \]
where
\[ A_j(\cdot, x) = \int_{0}^{1} J f_i(\cdot, \lambda x) d\lambda. \]
Further,
\[ \|A_i(\cdot, x)\| = \int_{0}^{1} \|J f_i(\cdot, \lambda x)\| d\lambda \leq c_i. \]
The proof is completed by applying the main theorem.

The result of Mori et al. [1] is a special case of the main theorem or of the Corollary, supposing \(f(\cdot) = A_i(\cdot)\).
Example 1. Consider

\[
x(n + 1) = \begin{bmatrix} x_1(n + 1) \\ x_2(n + 1) \end{bmatrix} = \begin{bmatrix} 0.5 \cos(n) \sin(x_1(n - 5)) x_2(n) \\ 0.4 \cos(x_1(n)) x_2(n - 10) \end{bmatrix}
\]

Then

\[
A_1(\cdot, \cdot) x(n) = \begin{bmatrix} 0 & 0.5 \cos(n) \sin(x_1(n - 5)) \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}
\]

\[
(5)
\]

Since \(c_1 + c_2 = 0.9 < 1\), by the main theorem, \(x(n)\) converges to 0 asymptotically.

Example 2. Consider

\[
x(n + 1) = \sin(n) [0.5 x(n) - 0.4 \sin(x(n - 10))].
\]

Since \(f_1(\cdot, 0) = f_2(\cdot, 0) = 0\), \(Jf_1(\cdot, x) = 0.5 \sin(n)\) and \(Jf_2(\cdot, x) = 0.4 \sin(n) \cos(x(n - 10))\) by the Corollary, \(x(n)\) converges to 0 asymptotically.

Concluding remark

Simple delay-independent sufficient conditions for asymptotic stability of nonlinear time-varying discrete systems have been presented.

References
