Lecture 11
Control applications

- optimal input design
- pole placement with low-authority control

System model

$$y(t) = h_0 u(t) + h_1 u(t-1) + h_2 u(t-2) + \cdots$$

$u(t)$ is input, $y(t)$ is output, $(h_0, h_1, \ldots)$ is impulse response

matrix description: assuming $u(t) = 0$ for $t < 0$ and $t > M$

$$\mathbf{y} = H\mathbf{u}$$

with $\mathbf{y} = (y(0), y(1), \ldots, y(N))$, $\mathbf{u} = (u(0), u(1), \ldots, u(M))$, and

$$H = \begin{bmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_M & h_{M-1} & h_{M-2} & \cdots & h_0 \\ \vdots & \vdots & \vdots & & \vdots \\ h_N & h_{N-1} & h_{N-2} & \cdots & h_{N-M} \end{bmatrix}$$

Output tracking problem

choose input sequence $u(0), \ldots, u(M)$ (with $M \leq N$) such that

- output minimizes peak deviation with desired output $y_{\rm des}(t)$,

$$\max_{t=0,\ldots,N} |y(t) - y_{\rm des}(t)|$$

- input satisfies amplitude constraints:

$$|u(t)| \leq U, \quad t = 0, \ldots, M$$

- input satisfies slew rate constraints:

$$|u(t+1) - u(t)| \leq S, \quad t = 0, \ldots, M-1$$

can include other linear constraints on inputs or outputs

Linear programming formulation

output tracking in matrix notation

$$\begin{array}{ll} \text{minimize} & \|H\mathbf{u} - \mathbf{y}_{\mathrm{des}}\|_\infty \\ \text{subject to} & \|\mathbf{u}\|_\infty \leq U \\ & \|D\mathbf{u}\|_\infty \leq S \end{array}$$

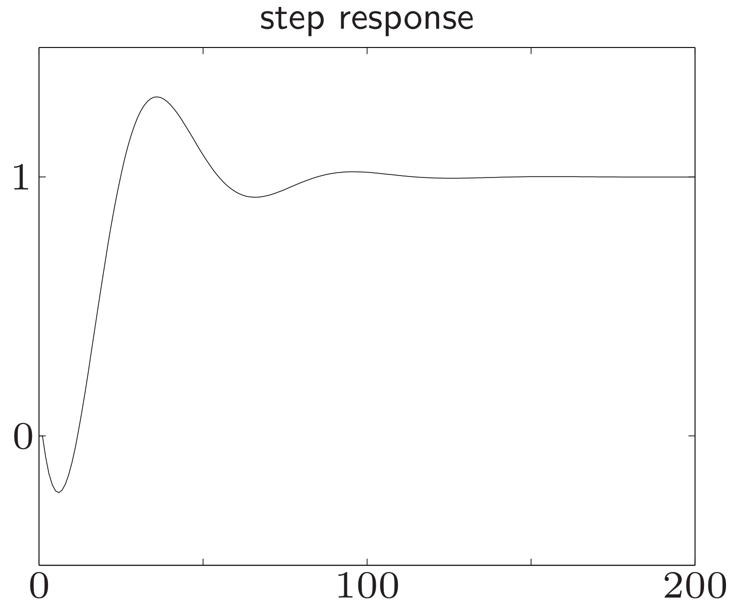
here, $\mathbf{y}_{\mathrm{des}} = (y_{\mathrm{des}}(0), \ldots, y_{\mathrm{des}}(N))$ and D is the $M \times (M+1)$ matrix

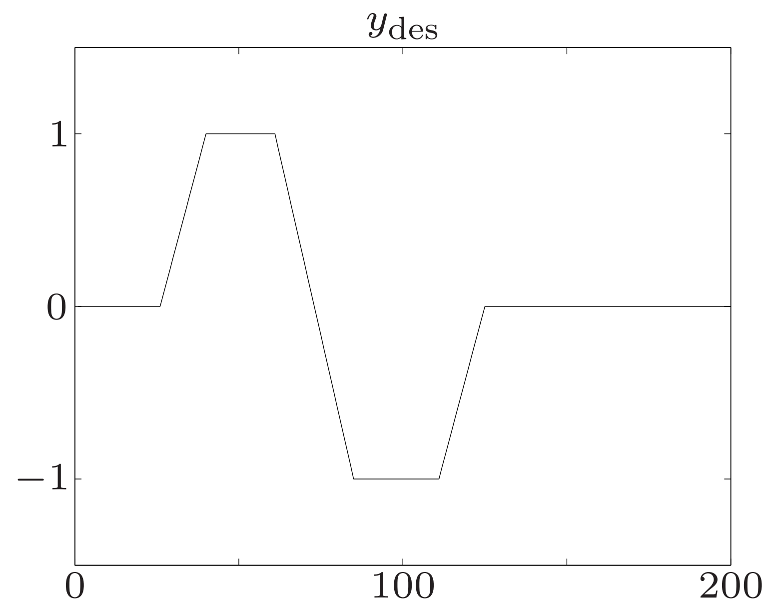
$$D = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

equivalent linear program (with variables γ, $\mathbf{u}$)

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & -\gamma\mathbf{1} \leq H\mathbf{u} - \mathbf{y}_{\mathrm{des}} \leq \gamma\mathbf{1} \\ & -U\mathbf{1} \leq \mathbf{u} \leq U\mathbf{1}, \quad -S\mathbf{1} \leq D\mathbf{u} \leq S\mathbf{1} \end{array}$$

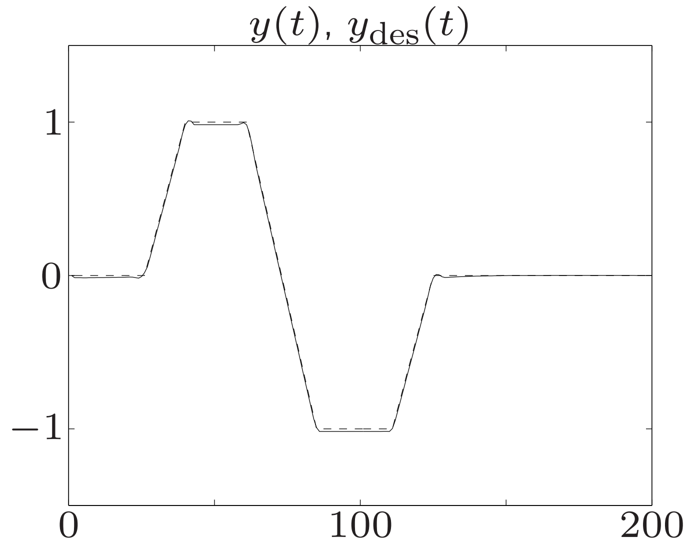
Example



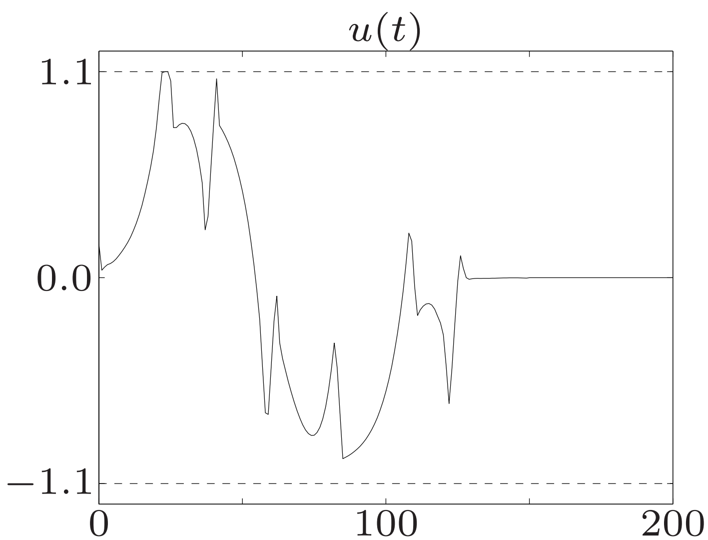


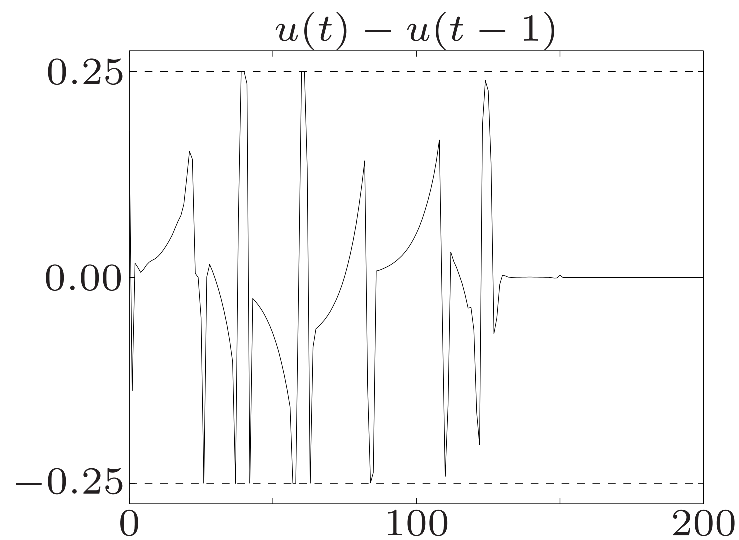
- input horizon $M = 150$
- output horizon $N = 200$
- amplitude constraint $|u(t)| \leq 1.1$
- slew rate constraint $|u(t) - u(t-1)| \leq 0.25$

output and desired output



optimal input sequence





Robust output tracking (version 1)

uncertain system model: impulse response can take two values

$$(h_{10}, h_{11}, h_{12}, \ldots), \qquad (h_{20}, h_{21}, h_{22}, \ldots)$$

robust tracking problem: minimize worst-case peak tracking error

$$\begin{array}{ll} \text{minimize} & \max\{\|H_1\mathbf{u} - \mathbf{y}_{\rm des}\|_\infty,\ \|H_2\mathbf{u} - \mathbf{y}_{\rm des}\|_\infty\} \\ \text{subject to} & \text{limits on input magnitude and slew rate} \end{array}$$

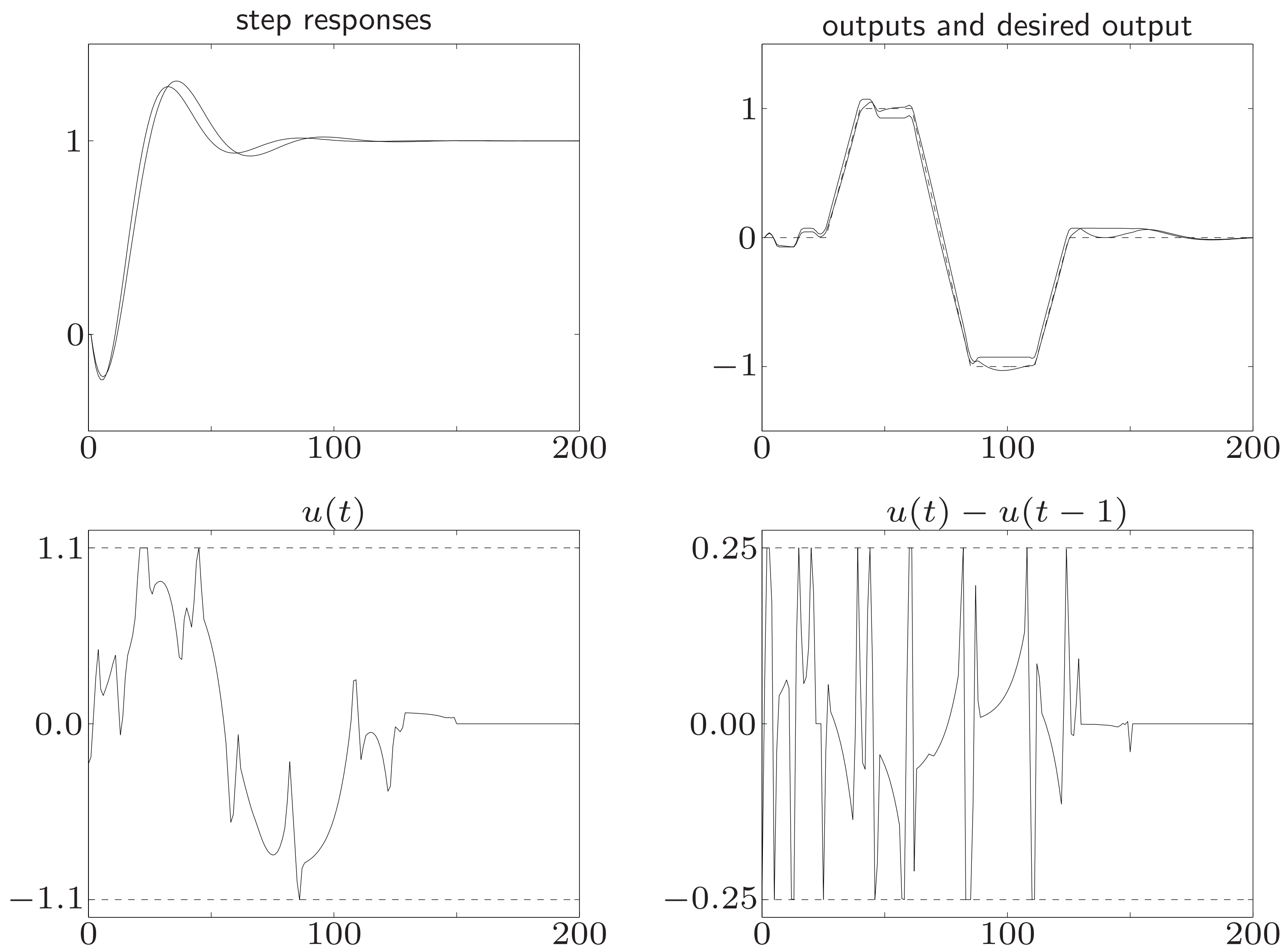
H_1, H_2 are Toeplitz matrices constructed from two impulse responses

equivalent linear program (variables γ, $\mathbf{u}$)

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & -\gamma\mathbf{1} \leq H_1\mathbf{u} - \mathbf{y}_{\rm des} \leq \gamma\mathbf{1} \\ & -\gamma\mathbf{1} \leq H_2\mathbf{u} - \mathbf{y}_{\rm des} \leq \gamma\mathbf{1} \\ & A\mathbf{u} \leq b \end{array}$$

(inequalities $A\mathbf{u} \leq b$ include $-U\mathbf{1} \leq \mathbf{u} \leq U\mathbf{1}$ and $-S\mathbf{1} \leq D\mathbf{u} \leq S\mathbf{1}$)

Example


step responses
outputs and desired output
$u(t)$
$u(t) - u(t-1)$
1
0
−1
1.1
0.0
−1.1
0.25
0.00
−0.25
100
200

Robust output tracking (version 2)

uncertain system model

$$\begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_N \end{bmatrix} = \begin{bmatrix} \bar{h}_0 \\ \bar{h}_1 \\ \vdots \\ \bar{h}_N \end{bmatrix} + \begin{bmatrix} v_{01} & v_{02} & \cdots & v_{0K} \\ v_{11} & v_{12} & \cdots & v_{1K} \\ \vdots & \vdots & & \vdots \\ v_{N1} & v_{N2} & \cdots & v_{NK} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_K \end{bmatrix}$$

- $(\bar{h}_0, \bar{h}_1, \ldots)$ is known (nominal) impulse response
- s_1, . . . , s_K are unknown parameters in $[-1, 1]$

robust tracking problem: minimize worst-case tracking error

$$\begin{array}{ll} \text{minimize} & \max_{\|s\|_\infty \leq 1} \|(\bar{H} + \sum_{k=1}^{K} s_k V_k)\mathbf{u} - \mathbf{y}_{\text{des}}\|_\infty \\ \text{subject to} & A\mathbf{u} \leq b \end{array}$$

$\bar{H}$, V_k are Toeplitz matrices constructed from $(\bar{h}_0, \bar{h}_1, \ldots)$, $(v_{0k}, v_{1k}, \ldots)$

Naive linear programming formulation

enumerate 2^K corners of model set and apply the method of p. 11–7

LP formulation (variables γ, $\mathbf{u}$)

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & -\gamma\mathbf{1} \leq (\bar{H} + \sum_{k=1}^{K} s_k V_k)\mathbf{u} - \mathbf{y}_{\mathrm{des}} \leq \gamma\mathbf{1} \quad \text{for all } s \in \{-1,+1\}^K \\ & A\mathbf{u} \leq b \end{array}$$

a very large set of inequalities when K is not small

Compact linear programming formulation

worst-case tracking error (as an explicit function of $\mathbf{u}$):

$$
\begin{aligned}
& \max_{\|s\|_\infty \le 1} \|(\bar{H} + \sum_{k=1}^{K} s_k V_k)\mathbf{u} - \mathbf{y}_{\text{des}}\|_\infty \\
&= \max_{i=0,\ldots,N} \max_{\|s\|_\infty \le 1} |(\bar{H}\mathbf{u} + \sum_{k=1}^{K} s_k V_k \mathbf{u} - \mathbf{y}_{\text{des}})_i| \\
&= \max_{i=0,\ldots,N} \left(|(\bar{H}\mathbf{u} - \mathbf{y}_{\text{des}})_i| + \sum_{k=1}^{K} |(V_k \mathbf{u})_i| \right)
\end{aligned}
$$

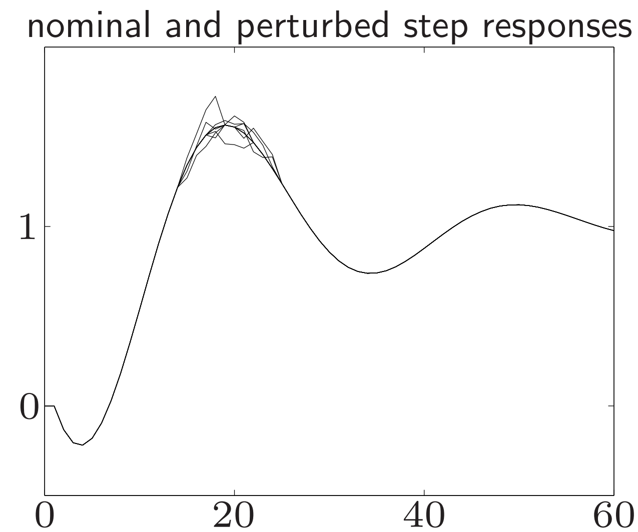
equivalent linear program (variables γ, $\mathbf{u}$, $\mathbf{w}_i$, $i = 1, \ldots, K$)

$$
\begin{array}{ll}
\text{minimize} & \gamma \\
\text{subject to} & -\mathbf{w}_i \le V_i \mathbf{u} \le \mathbf{w}_i, \quad i = 1, \ldots, K \\
& -\gamma \mathbf{1} + \sum_{i=1}^{K} \mathbf{w}_i \le \bar{H}\mathbf{u} - \mathbf{y}_{\text{des}} \le \gamma \mathbf{1} - \sum_{i=1}^{K} \mathbf{w}_i \\
& A\mathbf{u} \le b
\end{array}
$$

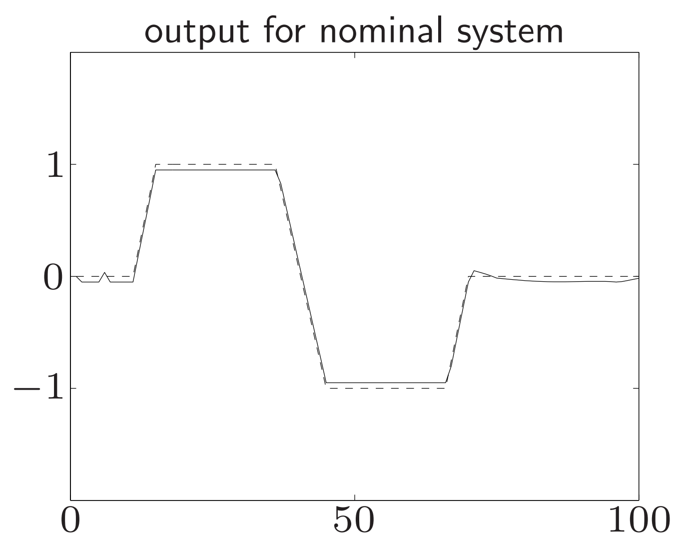
Example

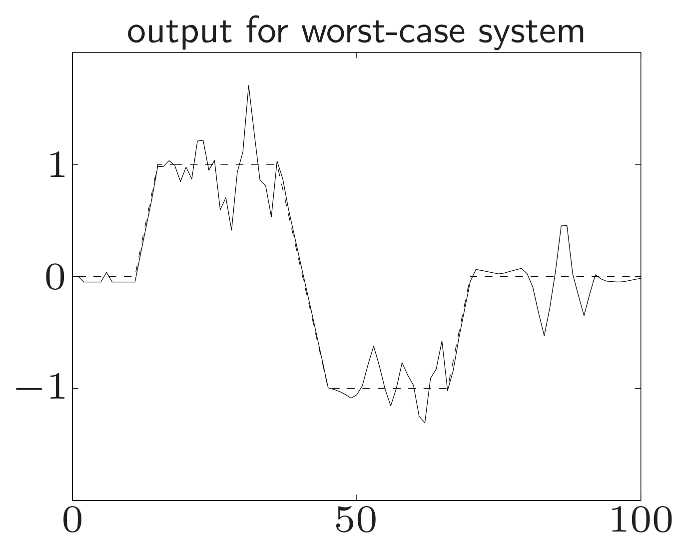
system models $(K = 6)$

figure shows a few step responses from model set

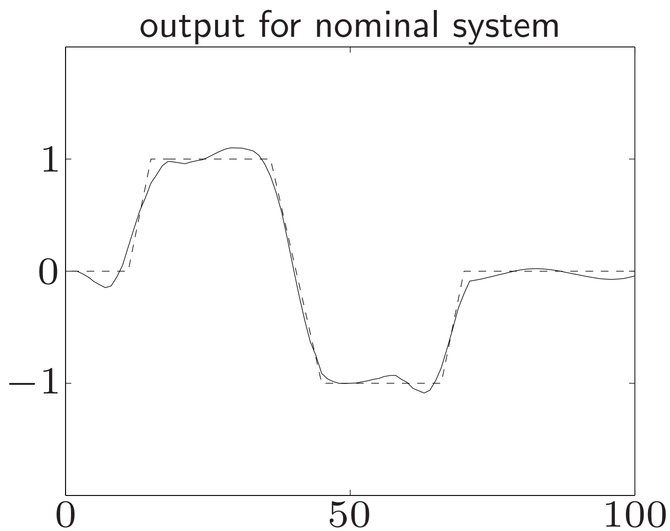


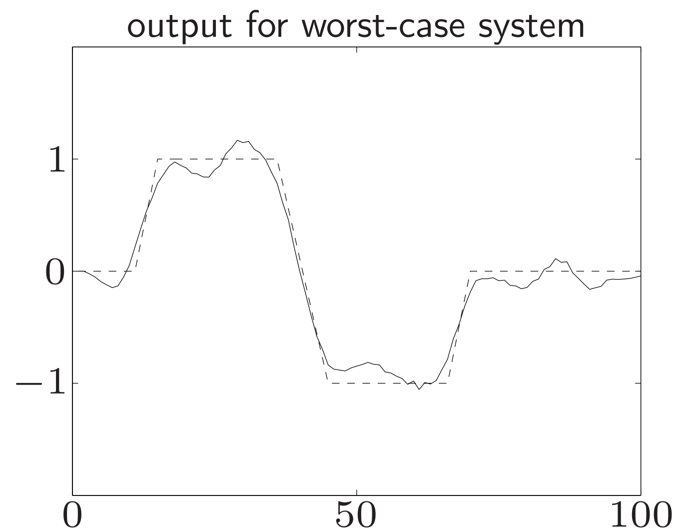
design for nominal system





robust design





- on nominal system, robust design does worse than non-robust design
- however, performance does not degrade much over the model set

Outline

- optimal input design
- **pole placement with low-authority control**

Pole placement

autonomous linear system

$$\dot{z}(t) = A(x)z(t), \qquad z(0) = z_0$$

where $A(x) = A_0 + x_1 A_1 + \cdots + x_p A_p \in \mathbf{R}^{n \times n}$

- solutions have the form

$$z_i(t) = \sum_k \beta_{ik} e^{\sigma_k t} \cos(\omega_k t - \phi_{ik})$$

  $\lambda_k = \sigma_k \pm \mathrm{j}\omega_k$ are the eigenvalues of $A(x)$

- $x \in \mathbf{R}^p$ is design parameter

goal: place eigenvalues of $A(x)$ in a desired region by choosing x

Low-authority control

eigenvalues of $A(x)$ are very complicated functions of x

first-order perturbation: if $\lambda_i(A_0)$ is a *simple* eigenvalue,

$$\lambda_i(A(x)) = \lambda_i(A_0) + \sum_{k=1}^{p} \frac{w_i^* A_k v_i}{w_i^* v_i} x_k + o(\|x\|)$$

where w_i, v_i are the left and right eigenvectors:

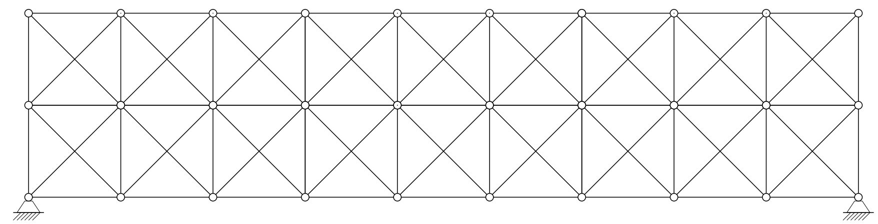
$$w_i^* A_0 = \lambda_i(A_0) w_i^*, \qquad A_0 v_i = \lambda_i(A_0) v_i$$

'low-authority' control

- use linear first-order approximations for λ_i
- can place λ_i in a polyhedral region by imposing linear inequalities on x
- we expect this to work only for small shifts in eigenvalues

Truss example

30 nodes, 83 bars



$$M\ddot{d}(t) + D\dot{d}(t) + Kd(t) = 0$$

- $d(t)$: vector of horizontal and vertical node displacements
- M is mass matrix, D damping matrix, K stiffness matrix
- to increase damping, we attach external dampers to bars:

$$D(x) = D_0 + x_1 D_1 + \cdots + x_p D_p$$

  $x_i > 0$: amount of external damping at bar i

- can be written as $\dot{z}(t) = A(x)z(t)$ with

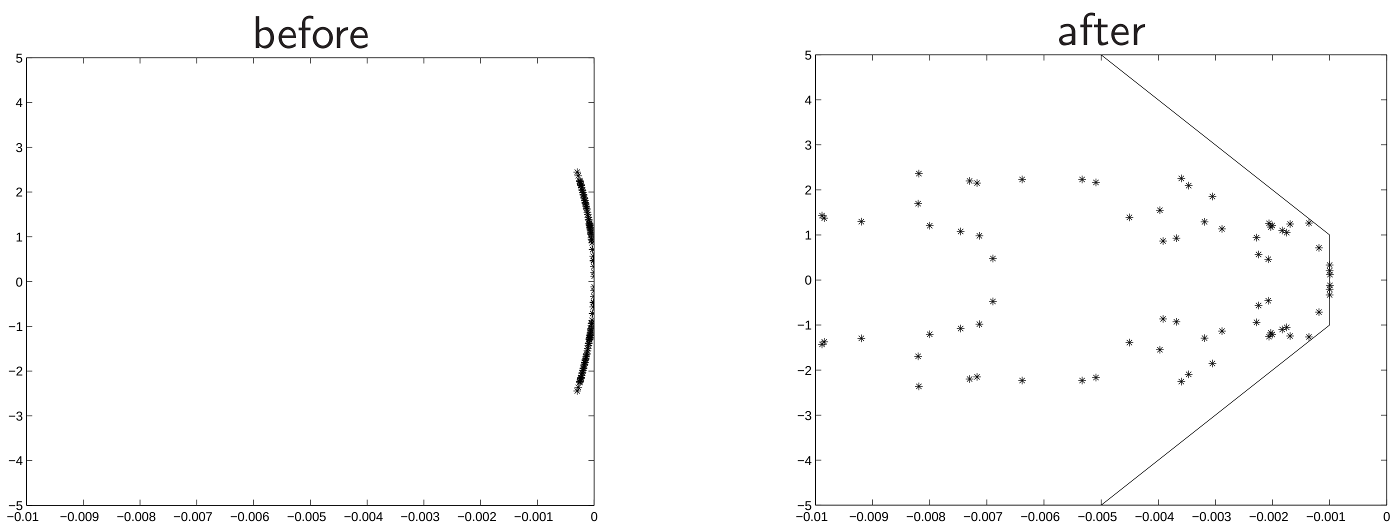
$$z(t) = \begin{bmatrix} d(t) \\ \dot{d}(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D(x) \end{bmatrix}$$

approximate eigenvalue placement with least external damping:

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{p} x_i \\
\text{subject to} & \lambda_i(M, D(x), K) \in \mathcal{C}, \quad i = 1, \ldots, n \\
& x \geq 0
\end{array}
$$

an LP if $\mathcal{C}$ is polyhedral and we use the first-order approximation for λ_i

eigenvalues



location of dampers