# Lecture 7
# Duality II

- sensitivity analysis
- two-person zero-sum games
- circuit interpretation

# Sensitivity analysis

**purpose:** extract from the solution of an LP information about the sensitivity of the solution with respect to changes in problem data

**this lecture:**

- sensitivity w.r.t. to changes in the right-hand side of the constraints
- we define $p^\star(u)$ as the optimal value of the modified LP (variables $x$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b + u \end{array}$$

- we are interested in obtaining information about $p^\star(u)$ from primal, dual optimal solutions $x^\star$, $z^\star$ at $u = 0$

# Global inequality

**dual of modified LP**

$$\begin{array}{ll} \text{maximize} & -(b+u)^T z \\ \text{subject to} & A^T z + c = 0 \\ & z \geq 0 \end{array}$$

**global lower bound:** if $z^\star$ is (any) dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T z^\star \\ &= p^\star(0) - u^T z^\star \end{aligned}$$

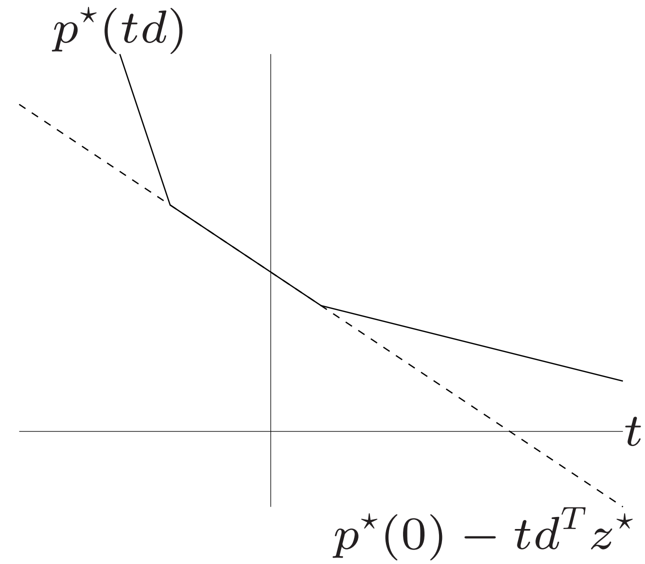
- follows from weak duality and feasibility of $z^\star$
- inequality holds for all $u$ (not necessarily small)

# Example (one varying parameter)

take $u = td$ with $d$ fixed:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b + td \end{array}$$

$p^\star(td)$ is optimal value as a function of $t$




sensitivity information from lower bound (assuming $d^T z^\star > 0$):

- if $t < 0$ the optimal value increases (by a large amount of $|t|$ is large)
- if $t > 0$ optimal value may increase or decrease
- if $t$ is positive and small, optimal value certainly does not decrease much

# Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax \leq b + u\}$$

**properties** (we assume $p^\star(0)$ is finite)

- $p^\star(u) > -\infty$ everywhere (this follows from the global lower bound)
- the domain $\{u \mid p^\star(u) < +\infty\}$ is a polyhedron
- $p^\star(u)$ is piecewise-linear on its domain

(proof on next page)

*proof.* let $P$ be the dual feasible set, $K$ the recession cone of $P$:

$$P = \{z \mid A^T z + c = 0,\ z \geq 0\}, \qquad K = \{w \mid A^T w = 0, w \geq 0\}$$

- $p^\star(u) = +\infty$ (modified primal is infeasible) iff there exists a $w$ such that

$$A^T w = 0, \qquad w \geq 0, \qquad b^T w + u^T w < 0$$

  therefore $p^\star(u) < \infty$ if and only if

$$b^T w_k + u^T w_k \geq 0 \qquad \text{for all extreme rays } w_k \text{ of } K$$

  this is a finite set of linear inequalities in $u$

- if $p^\star(u)$ is finite,

$$p^\star(u) = \max_{z \in P} (-b^T z - u^T z) = \max_{k=1,\ldots,r} (-b^T z_k - u^T z_k)$$

  where $z_1, \ldots, z_r$ are the extreme points of $P$

# Local sensitivity analysis

let $x^\star$ be optimal for the unmodified problem, with active constraint set

$$J = \{i \mid a_i^T x^\star = b_i\}$$

assume $x^\star$ is a **nondegenerate extreme point**, *i.e.*,

- an extreme point: $A_J$ has full column rank ($\mathrm{rank}(A_J) = n$)
- nondegenerate: $|J| = n$ ($n$ active constraints)

then, for $u$ in a neighborhood of the origin, $x^\star(u)$ and $z^\star$ defined by

$$x^\star(u) = A_J^{-1}(b_J + u_J), \qquad z_J^\star = -A_J^{-T} c, \qquad z_i^\star = 0 \ \text{(for } i \notin J\text{)},$$

are primal, dual optimal for the modified problem

**note:** $x^\star(u)$ is affine in $u$ and $z^\star$ is independent of $u$

*proof*

**solution of original LP** $(u = 0)$

- since $A_J$ is square and nonsingular, we can express $x^\star$ as $x^\star = A_J^{-1} b_J$
- complementary slackness determines optimal $z^\star$ uniquely:

$$z_i^\star = 0 \quad i \notin J, \qquad A_J^T z_J^\star + c = 0$$

**solution of modified LP** (for sufficiently small $u$)

- $x^\star(u)$ satisfies inequalities indexed by $J$: $A_J x^\star(u) = b_J + u_J$ (for all $u$)
- $x^\star(u)$ satisfies the other inequalities ($i \notin J$) for sufficiently small $u$:

$$a_i^T x^\star(u) \leq b_i + u_i \quad \Longleftrightarrow \quad a_i^T A_J^{-1} u_J - u_i \leq b_i - a_i^T x^\star$$

  and $b_i - a_i^T x^\star > 0$
- $z^\star$ is dual feasible (for all $u$)
- $x^\star(u)$ and $z^\star$ satisfy complementary slackness conditions

# Derivative of optimal value function

under the assumptions of the local analysis (page 7–7),

$$
\begin{aligned}
p^\star(u) &= c^T x^\star(u) \\
&= c^T x^\star + c^T A_J^{-1} u_J \\
&= p^\star(0) - z_J^{\star T} u_J
\end{aligned}
$$

for $u$ in a neighborhood of the origin

- optimal value function is affine in $u$ for small $u$
- $-z_i^\star$ is derivative of $p^\star(u)$ with respect to $u_i$ at $u = 0$

# Outline

- sensitivity analysis
- **two-person zero-sum games**
- circuit interpretation

# Two-person zero-sum game (matrix game)

- player 1 chooses a number in $\{1, \ldots, m\}$ (one of $m$ possible actions)
- player 2 chooses a number in $\{1, \ldots, n\}$ ($n$ possible actions)
- players make their choices independently
- if P1 chooses $i$ and P2 chooses $j$, then P1 pays $A_{ij}$ to P2

  (negative $A_{ij}$ means P2 pays $-A_{ij}$ to P1)
- the $m \times n$-matrix $A$ is called the **payoff matrix**

# Mixed (randomized) strategies

players choose actions randomly according to some probability distribution

- P1 chooses randomly according to distribution $x$:

$$x_i = \text{probability that P1 selects action } i$$

- P2 chooses randomly according to distribution $y$:

$$y_j = \text{probability that P2 selects action } j$$

**expected payoff** (from P1 to P2), if they use mixed stragies $x$ and $y$,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

# Optimal mixed strategies

denote by $P_k = \{p \in \mathbf{R}^k \mid p \geq 0, \mathbf{1}^T p = 1\}$ the probability simplex in $\mathbf{R}^k$

- player 1: optimal strategy $x^\star$ is solution of the equivalent problems

$$\begin{array}{ll} \text{minimize} & \max_{y \in P_n} x^T A y \\ \text{subject to} & x \in P_m \end{array} \qquad \begin{array}{ll} \text{minimize} & \max_{j=1,\ldots,n} (A^T x)_j \\ \text{subject to} & x \in P_m \end{array}$$

- player 2: optimal strategy $y^\star$ is solution of

$$\begin{array}{ll} \text{maximize} & \min_{x \in P_m} x^T A y \\ \text{subject to} & y \in P_n \end{array} \qquad \begin{array}{ll} \text{maximize} & \min_{i=1,\ldots,m} (A y)_i \\ \text{subject to} & y \in P_n \end{array}$$

optimal strategies $x^\star$, $y^\star$ can be computed by linear optimization

# Exercise: minimax theorem

prove that

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

**some consequences**

- if $x^\star$ and $y^\star$ are the optimal mixed strategies, then

$$\min_{x \in P_m} x^T A y^\star = \max_{y \in P_n} x^{\star T} A y$$

- if $x^\star$ and $y^\star$ are the optimal mixed strategies, then

$$x^T A y^\star \geq x^{\star T} A y^\star \geq x^{\star T} A y \qquad \forall x \in P_m, \ \forall y \in P_n$$

*solution*

- optimal strategy $x^\star$ is the solution of the LP (with variables $x$, $t$)

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & x \geq 0 \\ & \mathbf{1}^T x = 1 \end{array}$$

- optimal strategy $y^\star$ is the solution of the LP (with variables $y$, $w$)

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \geq w\mathbf{1} \\ & y \geq 0 \\ & \mathbf{1}^T y = 1 \end{array}$$

- the two LPs can be shown to be duals

# Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

- note that

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

- optimal mixed strategies

$$x^\star = (0.37,\, 0.33,\, 0.3), \qquad y^\star = (0.4,\, 0,\, 0.13,\, 0.47)$$

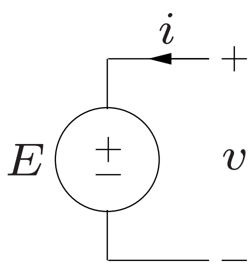
- expected payoff is $x^{\star T} A y^\star = 0.2$

# Outline

- sensitivity analysis
- two-person zero-sum games
- **circuit interpretation**

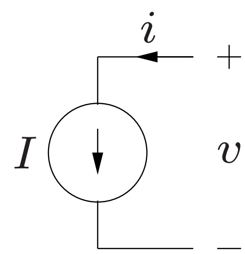
# Components

## voltage source



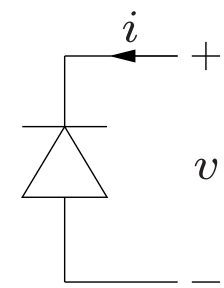

$$v = E$$

## current source




$$i = I$$

## ideal diode

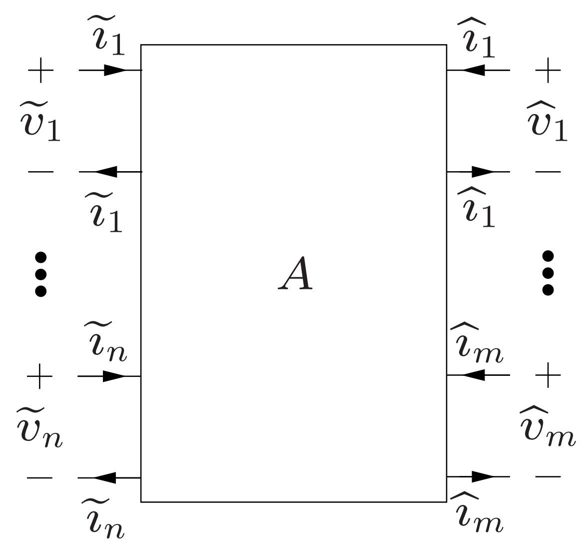



$$v \geq 0, \quad i \leq 0, \quad vi = 0$$

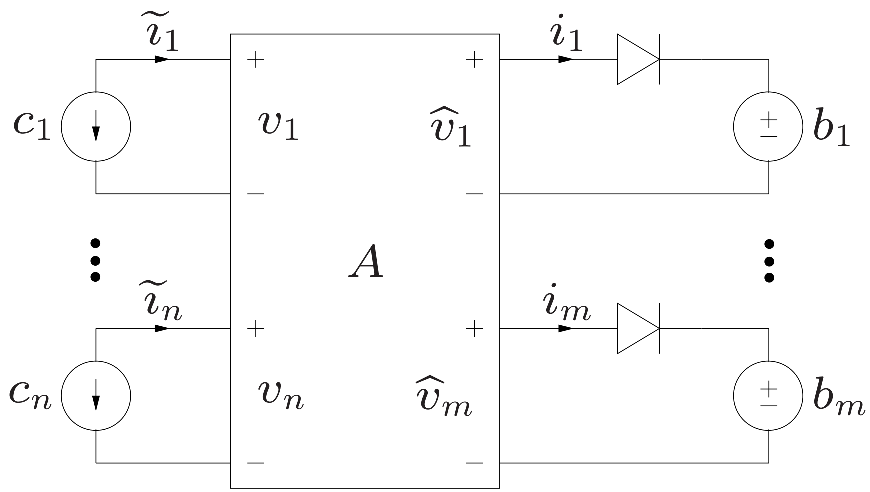
## multiterminal transformer

$$\widehat{v} = A\widetilde{v}, \qquad \widetilde{\imath} = -A^T\widehat{\imath}$$

with $A \in \mathbf{R}^{m \times n}$



# Example




circuit equations

- transformer:

$$\widehat{v} = Av, \quad \widetilde{i} = A^T i$$

- diodes and voltage souces:

$$\widehat{v} \leq b, \quad i \geq 0, \quad i^T(b - \widehat{v}) = 0$$

- current sources: $\widetilde{i} + c = 0$

these are the optimality conditions for the pair of primal and dual LPs

$$\begin{array}{ll} \text{minimize} & c^T v \\ \text{subject to} & Av \leq b \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T i \\ \text{subject to} & A^T i + c = 0, \;\; i \geq 0 \end{array}$$

# Variational description

two 'potential functions', **content** and **co-content** (in notation of p.7–16)

| | content (function of voltages) | co-content (function of currents) |
|---|---|---|
| current source | $Iv$ | $0$ if $i = I$<br>$-\infty$ otherwise |
| voltage source | $0$ if $v = E$<br>$\infty$ otherwise | $-Ei$ |
| diode | $0$ if $v \geq 0$<br>$\infty$ otherwise | $0$ if $i \leq 0$<br>$-\infty$ otherwise |
| transformer | $0$ if $\widehat{v} = A\widetilde{v}$<br>$\infty$ otherwise | $0$ if $\widetilde{i} = -A^T\widehat{\imath}$<br>$-\infty$ otherwise |

**optimization problems**

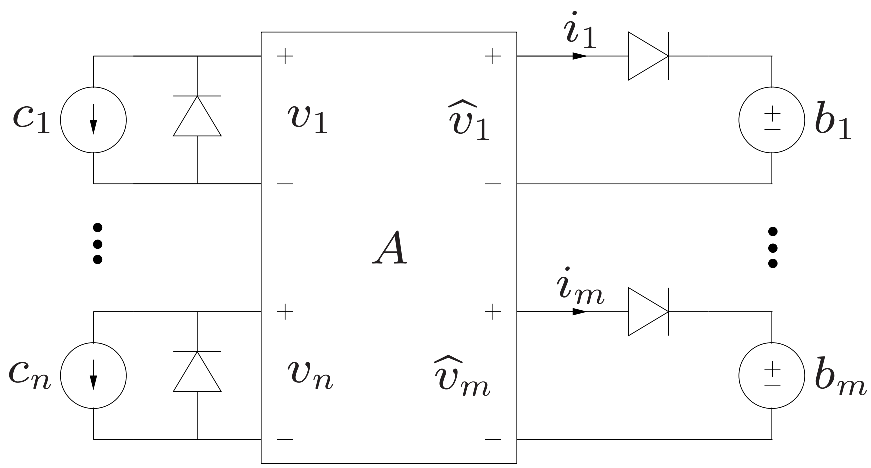
- primal: voltages minimize total content
- dual: currents maximize total co-content

# Example

**primal problem**

$$\begin{array}{ll} \text{minimize} & c^T v \\ \text{subject to} & Av \leq b \\ & v \geq 0 \end{array}$$

**equivalent circuit**




**dual problem**

$$\begin{array}{ll} \text{maximize} & -b^T i \\ \text{subject to} & A^T i + c \geq 0 \\ & i \geq 0 \end{array}$$