Lecture 16
Self-dual formulations

- self-dual linear programs
- self-dual embedding
- interior-point method for self-dual embedding

Optimality and infeasibility

minimize	$c^T x$	maximize	$-b^T z$
subject to	$Ax + s = b$	subject to	$A^T z + c = 0$
	$s \geq 0$		$z \geq 0$

- **optimality:** x, s, z are optimal if

$$Ax + s = b, \qquad A^T z + c = 0, \qquad c^T x + b^T z = 0, \qquad s \geq 0, \quad z \geq 0$$

- **primal infeasibility:** z certifies primal infeasibility if

$$A^T z = 0, \qquad z \geq 0, \qquad b^T z = -1$$

- **dual infeasibility:** x certifies dual infeasibility if

$$Ax \leq 0, \qquad c^T x = -1$$

Initialization and infeasibility detection

barrier method (lecture 14)

- requires a phase I to find strictly feasible x
- fails if problem is not strictly dual feasible (central path does not exist)

infeasible primal-dual method (lecture 15)

- does not require feasible starting points
- fails if problem is not primal and dual feasible

self-dual formulations (this lecture): embed LP in larger LP such that

- larger LP is primal and dual feasible, with known feasible points
- from solution can extract optimal solutions or certificates of infeasibility

Outline

- **self-dual linear programs**
- self-dual embedding
- interior-point method for self-dual embedding

Self-dual linear program

primal problem (variables u, v, w)

$$\begin{array}{ll} \text{minimize} & f^T u + g^T v \\ \text{subject to} & \begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} C & D \\ -D^T & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix} \\ & v \geq 0, \quad w \geq 0 \end{array}$$

C and E are skew-symmetric: $C = -C^T$, $E = -E^T$

dual problem (variables $\tilde{u}$, $\tilde{v}$, $\tilde{w}$)

$$\begin{array}{ll} \text{maximize} & -f^T \tilde{u} - g^T \tilde{v} \\ \text{subject to} & \begin{bmatrix} 0 \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} C & D \\ -D^T & E \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix} \\ & \tilde{v} \geq 0, \quad \tilde{w} \geq 0 \end{array}$$

derivation of dual:

- eliminate w and write primal problem as

$$\begin{array}{ll} \text{minimize} & f^Tu + g^Tv \\ \text{subject to} & \begin{bmatrix} -C & -D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = f \\ & \begin{bmatrix} D^T & -E \\ 0 & -I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \leq \begin{bmatrix} g \\ 0 \end{bmatrix} \end{array}$$

- apply dual from page 6–12 and use skew-symmetry

$$\begin{array}{ll} \text{maximize} & -f^T\tilde{u} - g^T\tilde{v} \\ \text{subject to} & \begin{bmatrix} C \\ -D^T \end{bmatrix} \tilde{u} + \begin{bmatrix} D & 0 \\ E & -I \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix} = 0 \\ & \tilde{v} \geq 0, \quad \tilde{w} \geq 0 \end{array}$$

Optimality condition

complementarity: feasible u, v, w are optimal if and only if

$$v^T w = 0$$

proof

- if (u, v, w) is primal optimal, then $(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w)$ is dual optimal
- from optimality conditions for LPs on page 16–4:

$$\tilde{w}^T v + \tilde{v}^T w = 0$$

for any primal optimal (u, v, w) and any dual optimal $(\tilde{u}, \tilde{v}, \tilde{w})$

Strict complementarity

if the self-dual LP is feasible, it has an optimal solution that satisfies

$$v^T w = 0, \qquad v + w > 0$$

- the LPs on p.16–4 have strictly complementary solutions (ex.72), with

$$v_i + \tilde{w}_i > 0, \qquad w_i + \tilde{v}_i > 0 \qquad \text{for all } i$$

- at the optimum, we also have $v^T w = 0$ and $\tilde{v}^T \tilde{w} = 0$ (page 16–6):

$$v_i w_i = 0, \qquad \tilde{v}_i \tilde{w}_i = 0 \qquad \text{for all } i$$

- this leaves only two possible sign patterns for every i

v_i	w_i	$\tilde{v}_i$	$\tilde{w}_i$
0	$+$	0	$+$
$+$	0	$+$	0

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Basic self-dual embedding

$$
\begin{array}{ll}
\text{minimize} & 0 \\
\text{subject to} & \begin{bmatrix} 0 \\ s \\ \kappa \end{bmatrix} = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ \tau \end{bmatrix} \\
& s \geq 0, \quad \kappa \geq 0, \quad z \geq 0, \quad \tau \geq 0
\end{array}
$$

variables s, κ, x, z, τ

- a self-dual LP with a trivial solution (all variables zero)
- all feasible points are optimal and satisfy $z^T s + \tau\kappa = 0$

  (to see this directly, take the inner product of each side with (x, z, τ))
- hence, problem cannot be strictly feasible

Classification of nonzero solution

let s, κ, x, z, τ be a **strictly complementary** solution:

$$s^T z + \kappa\tau = 0, \qquad s + z > 0, \qquad \kappa + \tau > 0$$

we distinguish two cases, depending on the sign of κ and τ

- **case 1** ($\tau > 0$ and $\kappa = 0$): define

  $$\hat{s} = s/\tau, \qquad \hat{x} = x/\tau, \qquad \hat{z} = z/\tau$$

  $\hat{x}$, $\hat{s}$, $\hat{z}$ are primal, dual optimal for the original LPs and satisfy

  $$\begin{bmatrix} 0 \\ \hat{s} \end{bmatrix} = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} + \begin{bmatrix} c \\ b \end{bmatrix}$$

  $$\hat{s} \geq 0, \qquad \hat{z} \geq 0, \qquad \hat{s}^T \hat{z} = 0$$

- **case 2** ($\tau = 0$, $\kappa > 0$): this implies

$$c^T x + b^T z < 0$$

so $c^T x < 0$ or $b^T z < 0$ or both

  - if $b^T z < 0$, then $\hat{z} = z/(-b^T z)$ is a certificate of primal infeasibility:

$$A^T \hat{z} = 0, \qquad b^T \hat{z} = -1, \qquad \hat{z} \geq 0$$

  - if $c^T x < 0$, then $\hat{x} = x/(-c^T x)$ is a certificate of dual infeasibility:

$$A\hat{x} \leq 0, \qquad c^T \hat{x} = -1$$

note: strict complementarity is only used to ensure $\kappa + \tau > 0$

Extended self-dual embedding

$$
\begin{array}{ll}
\text{minimize} & (m+1)\theta \\
\text{subject to} & \begin{bmatrix} 0 \\ s \\ \kappa \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & A^T & c & q_x \\ -A & 0 & b & q_z \\ -c^T & -b^T & 0 & q_\tau \\ -q_x^T & -q_z^T & -q_\tau & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ \tau \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ m+1 \end{bmatrix} \\
& s \geq 0, \quad \kappa \geq 0, \quad z \geq 0, \quad \tau \geq 0
\end{array}
$$

- variables s, κ, x, z, τ, θ
- q_x, q_z, q_τ are chosen so that the point

$$
(s, \kappa, x, z, \tau, \theta) = (s_0, 1, x_0, z_0, 1, \frac{z_0^T s_0 + 1}{m+1})
$$

is feasible, for some given $s_0 > 0$, x_0, $z_0 > 0$

Properties of extended self-dual embedding

- problem is strictly feasible by construction
- if s, κ, x, z, τ, θ satisfy the equality constraint, then

$$\theta = \frac{s^T z + \kappa\tau}{m+1}$$

  (take inner product with (x, z, τ, θ) of each side of the equality)

- at optimum, $s^T z + \kappa\tau = 0$ (from optimality conditions on page 16–6)
- at optimum, $\theta = 0$ and problem reduces to basic embedding (p.16–8)
- classification of p.16–9 also applies to solutions of extended embedding

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Central path for extended embedding

$$\begin{bmatrix} 0 \\ s \\ \kappa \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & A^T & c & q_x \\ -A & 0 & b & q_z \\ -c^T & -b^T & 0 & q_\tau \\ -q_x^T & -q_z^T & -q_\tau & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ \tau \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ m+1 \end{bmatrix}$$

$$(s, \kappa, z, \tau) \geq 0, \qquad s \circ z = \mu \mathbf{1}, \qquad \kappa\tau = \mu$$

- inner product with (x, z, τ, θ) shows that on the central path

$$\theta = \frac{z^T s + \kappa\tau}{m+1} = \mu$$

- by construction (q_x, q_z, q_τ on page 16–11), if $s_0 \circ z_0 = \mathbf{1}$, the point

$$(s,\, \kappa,\, x,\, z,\, \tau,\, \theta) = (s_0,\, 1,\, x_0,\, z_0,\, 1,\, (z_0^T s_0 + 1)/(m+1))$$

is on the central path with $\mu = (s_0^T z_0 + 1)/(m+1) = 1$

Simplified central path equations

$$\begin{bmatrix} 0 \\ s \\ \kappa \end{bmatrix} = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ \tau \end{bmatrix} + \mu \begin{bmatrix} q_x \\ q_z \\ q_\tau \end{bmatrix}$$

$$(s, \kappa, z, \tau) \geq 0, \qquad s \circ z = \mu \mathbf{1}, \qquad \kappa\tau = \mu$$

- we eliminated variable θ because $\theta = \mu$ on the central path
- we removed the 4th equality, because it is implied by the first three (this follows by taking inner product with (x, z, τ))
- can be viewed as a 'shifted central path' for basic embedding (p.16–8)

Basic update

let $\hat{s}$, $\hat{\kappa}$, $\hat{x}$, $\hat{z}$, $\hat{\tau}$ be the current iterates (with $\hat{s} > 0$, $\hat{\kappa} > 0$, $\hat{z} > 0$, $\hat{\tau} > 0$)

- determine Δs, $\Delta\kappa$, Δx, Δs, $\Delta\tau$ by linearizing central path equations

$$\begin{bmatrix} 0 \\ s \\ \kappa \end{bmatrix} = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ \tau \end{bmatrix} + \sigma\hat{\mu} \begin{bmatrix} q_x \\ q_z \\ q_\tau \end{bmatrix}$$

$$s \circ z = \sigma\hat{\mu}\,\mathbf{1}, \qquad \kappa\tau = \sigma\hat{\mu}$$

  where $\hat{\mu} = (\hat{s}^T\hat{z} + \hat{\kappa}\hat{\tau})/(m+1)$ and $\sigma \in [0,1]$

- make an update

$$(\hat{s}, \hat{\kappa}, \hat{x}, \hat{z}, \hat{\tau}) := (\hat{s}, \hat{\kappa}, \hat{x}, \hat{z}, \hat{\tau}) + \alpha\,(\Delta s, \Delta\kappa, \Delta x, \Delta z, \Delta\tau)$$

  that preserves positivity of $\hat{s}$, $\hat{\kappa}$, $\hat{z}$, $\hat{\tau}$

Linearized central path equations

a set of $2m+n+2$ equations in variables Δs, $\Delta\kappa$, Δx, Δz, $\Delta\tau$:

$$\begin{bmatrix} 0 \\ \Delta s \\ \Delta\kappa \end{bmatrix} - \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta\tau \end{bmatrix} = \sigma\hat{\mu} \begin{bmatrix} q_x \\ q_z \\ q_\tau \end{bmatrix} - \begin{bmatrix} r_x \\ r_z \\ r_\tau \end{bmatrix} \qquad (1)$$

$$\begin{aligned} \hat{s}\circ\Delta z + \hat{z}\circ\Delta s &= \sigma\hat{\mu}\,\mathbf{1} - \hat{s}\circ\hat{z} & (2) \\ \hat{\kappa}\Delta\tau + \hat{\tau}\Delta\kappa &= \sigma\hat{\mu} - \hat{\kappa}\hat{\tau} & (3) \end{aligned}$$

where

$$r = \begin{bmatrix} 0 \\ \hat{s} \\ \hat{\kappa} \end{bmatrix} - \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \\ \hat{\tau} \end{bmatrix}$$

note: $r = \hat{\mu}q$ for $(\hat{s},\hat{\kappa},\hat{x},\hat{z},\hat{\tau}) = (s_0, 1, x_0, z_0, 1)$ (by definition of q)

Properties of search direction

- from equations (2) and (3) and definition of $\hat{\mu}$:

$$\frac{\hat{s}^T \Delta z + \hat{z}^T \Delta s + \hat{\kappa} \Delta \tau + \hat{\tau} \Delta \kappa}{m+1} = -(1-\sigma)\hat{\mu}$$

- if $r = \hat{\mu} q$, primal and dual steps are orthogonal

$$\Delta s^T \Delta z + \Delta \kappa \Delta \tau = 0$$

(proof on next page)

- hence, gap depends linearly on stepsize

$$\frac{(\hat{s} + \alpha \Delta s)^T (\hat{z} + \alpha \Delta z) + (\hat{\kappa} + \alpha \Delta \kappa)(\hat{\tau} + \alpha \Delta \tau)}{m+1} = (1 - \alpha(1-\sigma))\hat{\mu}$$

proof of orthogonality

- if $r = \hat{\mu} q$, we can combine (1) and the definition of r to write

$$\begin{bmatrix} 0 \\ \Delta s + (1-\sigma)\hat{s} \\ \Delta\kappa + (1-\sigma)\hat{\kappa} \end{bmatrix} = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x + (1-\sigma)\hat{x} \\ \Delta z + (1-\sigma)\hat{z} \\ \Delta\tau + (1-\sigma)\hat{\tau} \end{bmatrix}$$

- the matrix on the right-hand side is skew-symmetric:

$$\begin{aligned} 0 &= \begin{bmatrix} \Delta x + (1-\sigma)\hat{x} \\ \Delta z + (1-\sigma)\hat{z} \\ \Delta\tau + (1-\sigma)\hat{\tau} \end{bmatrix}^T \begin{bmatrix} 0 \\ \Delta s + (1-\sigma)\hat{s} \\ \Delta\kappa + (1-\sigma)\hat{\kappa} \end{bmatrix} \\ &= \Delta s^T \Delta z + \Delta\kappa\Delta\tau + (1-\sigma)\big(\hat{s}^T \Delta z + \hat{z}^T \Delta s + \hat{\kappa}\Delta\tau + \hat{\tau}\Delta\kappa\big) \\ &\quad + (1-\sigma)^2 \big(\hat{s}^T \hat{z} + \hat{\kappa}\hat{\tau}\big) \\ &= \Delta s^T \Delta z + \Delta\kappa\Delta\tau \end{aligned}$$

last step follows from first property on page 16–17 and definition of $\hat{\mu}$

Gap and residual after update

notation: gap and residual as a function of steplength α

$$\hat{\mu}(\alpha) = \frac{(\hat{s} + \alpha\Delta s)^T(\hat{z} + \alpha\Delta z) + (\hat{\kappa} + \alpha\Delta\kappa)(\hat{\tau} + \alpha\Delta\tau)}{m+1}$$

$$r(\alpha) = \left[\begin{array}{c} 0 \\ \hat{s} + \alpha\Delta s \\ \hat{\kappa} + \alpha\Delta\kappa \end{array}\right] - \left[\begin{array}{ccc} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{array}\right]\left[\begin{array}{c} \hat{x} + \alpha\Delta x \\ \hat{z} + \alpha\Delta z \\ \hat{\tau} + \alpha\Delta\tau \end{array}\right]$$

properties: if $r = \hat{\mu}q$, then residual and gap decrease at the same rate

$$\hat{\mu}(\alpha) = (1 - \alpha(1-\sigma))\hat{\mu}, \qquad r(\alpha) = \hat{\mu}(\alpha)q$$

- first identity was already noted on page 16–17
- 2nd identity follows from definition of r and search directions (p.16–16)
- hence, update preserves relation $r = \hat{\mu}q$

Path-following algorithm

choose starting points $\hat{s}$, $\hat{x}$, $\hat{z}$, with $\hat{s} > 0$, $\hat{z} > 0$; set $\hat{\kappa} := 1$, $\hat{\tau} := 1$

1. **compute residuals and gap**

$$
\begin{aligned}
r &= \begin{bmatrix} 0 \\ \hat{s} \\ \hat{\kappa} \end{bmatrix} - \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \\ \hat{\tau} \end{bmatrix} \\
\hat{\mu} &= \frac{\hat{s}^T \hat{z} + \hat{\kappa}\hat{\tau}}{m+1}
\end{aligned}
$$

2. **evaluate stopping criteria:** terminate if
   - $\hat{x}/\hat{\tau}$ and $\hat{z}/\hat{\tau}$ are approximately optimal
   - or $\hat{z}$ is an approximate certificate of primal infeasibility
   - or $\hat{x}$ is an approximate certificate of dual infeasibility

3. **compute affine scaling direction:** solve the linear equation

$$\begin{aligned}
\begin{bmatrix} 0 \\ \Delta s_{\rm a} \\ \Delta\kappa_{\rm a} \end{bmatrix} - \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm a} \\ \Delta z_{\rm a} \\ \Delta\tau_{\rm a} \end{bmatrix} &= -r \\
\hat{s} \circ \Delta z_{\rm a} + \hat{z} \circ \Delta s_{\rm a} &= -\hat{s} \circ \hat{z} \\
\hat{\kappa}\Delta\tau_{\rm a} + \hat{\tau}\Delta\kappa_{\rm a} &= -\hat{\kappa}\hat{\tau}
\end{aligned}$$

4. **select barrier parameter:** find

$$\bar{\alpha} = \max\{\alpha \in [0,1] \mid (\hat{s}, \hat{\kappa}, \hat{z}, \hat{\tau}) + \alpha(\Delta s_{\rm a}, \Delta\kappa_{\rm a}, \Delta z_{\rm a}, \Delta\tau_{\rm a}) \geq 0\}$$

and take

$$\sigma := (1 - \bar{\alpha})^\delta$$

δ is an algorithm parameter (a typical value is $\delta = 3$)

5. **compute search direction:** solve the linear equation

$$\begin{aligned}
\begin{bmatrix} 0 \\ \Delta s \\ \Delta\kappa \end{bmatrix} - \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta\tau \end{bmatrix} &= -(1-\sigma)r \\
\hat{s} \circ \Delta z + \hat{z} \circ \Delta s &= \sigma\hat{\mu}\,\mathbf{1} - \hat{s} \circ \hat{z} \\
\hat{\kappa}\Delta\tau + \hat{\tau}\Delta\kappa &= \sigma\hat{\mu} - \hat{\kappa}\hat{\tau}
\end{aligned}$$

6. **update iterates:** find maximum step to the boundary

$$\bar{\alpha} = \max\{\alpha \in [0,1] \mid (\hat{s}, \hat{\kappa}, \hat{z}, \hat{\tau}) + \alpha(\Delta s, \Delta\kappa, \Delta z, \Delta\tau) \geq 0\}$$

and make an update with stepsize $\alpha = \min\{1, 0.99\bar{\alpha}\}$:

$$(\hat{s}, \hat{\kappa}, \hat{x}, \hat{z}, \hat{\tau}) := (\hat{s}, \hat{\kappa}, \hat{x}, \hat{z}, \hat{\tau}) + \alpha\,(\Delta s, \Delta\kappa, \Delta x, \Delta z, \Delta\tau)$$

return to step 1

Discussion

- the vector q is not used, but defined implicitly via $r = \hat{\mu} q$
- step 3: the linearized central path equation (page 16–16) with $\sigma = 0$
- step 4: same heuristic as on p.15–9, but simplified using p.16–17

$$\begin{aligned}\sigma &= \left(\frac{(\hat{s} + \bar{\alpha}\Delta s_{\mathrm{a}})^T(\hat{z} + \bar{\alpha}\Delta z_{\mathrm{a}}) + (\hat{\kappa} + \bar{\alpha}\Delta\kappa_{\mathrm{a}})(\hat{\tau} + \bar{\alpha}\Delta\tau_{\mathrm{a}})}{\hat{s}^T\hat{z} + \hat{\kappa}\hat{\tau}}\right)^\delta \\ &= \left(\frac{(1-\bar{\alpha})\hat{\mu}}{\hat{\mu}}\right)^\delta\end{aligned}$$

- step 5: the linearized central path equation (page 16–16), with $r = \hat{\mu} q$

Mehrotra correction

replace equation in step 5 with

$$\begin{bmatrix} 0 \\ \Delta s \\ \Delta\kappa \end{bmatrix} - \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta\tau \end{bmatrix} = -(1-\sigma)r$$

$$\begin{aligned} \hat{s} \circ \Delta z + \hat{z} \circ \Delta s &= \sigma\hat{\mu}\,\mathbf{1} - \hat{s} \circ \hat{z} - \Delta s_{\mathrm{a}} \circ \Delta z_{\mathrm{a}} \\ \hat{\kappa}\Delta\tau + \hat{\tau}\Delta\kappa &= \sigma\hat{\mu} - \hat{\kappa}\hat{\tau} - \Delta\kappa_{\mathrm{a}}\Delta\tau_{\mathrm{a}} \end{aligned}$$

- motivation for extra terms is the same as in lecture 15 (page 15–13)
- the important identity

$$\frac{\hat{s}^T\Delta z + \hat{z}^T\Delta s + \hat{\kappa}\Delta\tau + \hat{\tau}\Delta\kappa}{m+1} = -(1-\sigma)\hat{\mu}$$

  (see page 16–17) still holds because $\Delta s_{\mathrm{a}}^T\Delta z_{\mathrm{a}} + \Delta\kappa_{\mathrm{a}}\Delta\tau_{\mathrm{a}} = 0$

Linear algebra complexity

- essentially the same as for the method on page 15–8
- eliminating $\Delta\tau$, $\Delta\kappa$ in steps 3 and 5 requires solution of an extra system

$$\left[\begin{array}{cc} 0 & A^T \\ A & -SZ^{-1} \end{array}\right]\left[\begin{array}{c} \Delta\tilde{x} \\ \Delta\tilde{z} \end{array}\right] = \left[\begin{array}{c} c \\ b \end{array}\right]$$

  with $S = \mathbf{diag}(\hat{s})$, $Z = \mathbf{diag}(\hat{z})$

- this increases the number of linear equations solved per iteration to 3 (from 2 equations in the method on page 15–8)