

10. Equality constrained minimization

- equality constrained minimization
- eliminating equality constraints
- Newton's method with equality constraints
- infeasible start Newton method
- implementation

Equality constrained minimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- f convex, twice continuously differentiable; hence, $\text{dom } f$ is an open set
- $A \in \mathbf{R}^{p \times n}$ with **rank** $A = p$
- we assume the optimal value p^* is finite and attained

Optimality conditions: x is optimal if and only if

$$x \in \text{dom } f, \quad Ax = b, \quad \nabla f(x) + A^T \nu = 0 \quad (1)$$

for some $\nu \in \mathbf{R}^p$

Equality constrained quadratic minimization

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Px + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

where $P \in \mathbf{S}_+^n$ and $\mathbf{rank}(A) = p$

- optimality conditions from previous page:

$$Px + q + A^T v = 0, \quad Ax = b$$

- this is a set of $n + p$ linear equations in $n + p$ variables

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

coefficient matrix is called Karush–Kuhn–Tucker (KKT) matrix

Nonsingular KKT matrix

each of the following three conditions is equivalent to nonsingularity of

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix}$$

(assuming $P \succeq 0$ and $\mathbf{rank}(A) = p$)

1. the first block column has full column rank

$$\mathbf{rank}\left(\begin{bmatrix} P \\ A \end{bmatrix}\right) = n \quad (2)$$

2. P is positive definite on the nullspace of A

$$Ax = 0, \quad x \neq 0 \quad \implies \quad x^T Px > 0 \quad (3)$$

3. the matrix $P + A^T A$ is positive definite

Exercise

show that the three conditions are equivalent to nonsingularity of the KKT matrix

Solution (condition 1)

- clearly, (2) is necessary (first n columns of KKT matrix must be independent)
- to show it is sufficient, we assume that (2) holds and show that

$$Px + A^T y = 0, \quad Ax = 0 \quad (4)$$

holds only if $x = 0, y = 0$

- inner product of x and first equation of (4) gives $x^T Px = x^T (Px + A^T y) = 0$
- $x^T Px = 0$ if and only if $Px = 0$ (for positive semidefinite P)
- hence (4) is equivalent to $Px = 0, Ax = 0, A^T y = 0$
- by the rank property (2) and $\mathbf{rank}(A) = p$, this holds only if $x = 0, y = 0$

Exercise

Condition 2

- (3) means the same as

$$x^T P x = 0, \quad A x = 0 \quad \implies \quad x = 0$$

- for $P \succeq 0$, can replace $x^T P x = 0$ with $P x = 0$, so (3) is the same as (2)

Condition 3

- the matrix $P + A^T A$ is positive definite if

$$x^T (P + A^T A) x = x^T P x + \|A x\|_2^2 > 0 \quad \text{for all } x \neq 0$$

- for $P \succeq 0$, this is the same as condition 2 ($x^T P x > 0$ for nonzero x with $A x = 0$)

Eliminating equality constraints

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array} \quad \longrightarrow \quad \begin{array}{ll} \text{minimize} & f(Fz + \hat{x}) \end{array}$$

- the affine set defined by $Ax = b$ is represented as translate of range of F

$$\{x \mid Ax = b\} = \{Fz + \hat{x} \mid z \in \mathbf{R}^{n-p}\}$$

- \hat{x} is (any) particular solution of the linear equation $Ax = b$
- $F \in \mathbf{R}^{n \times (n-p)}$ is a full-rank matrix with range equal to nullspace of A
- the reformulated problem is unconstrained with variable $z \in \mathbf{R}^{n-p}$
- from solution z^\star , solution of optimality conditions (1) is

$$x^\star = Fz^\star + \hat{x}, \quad v^\star = -(AA^T)^{-1}A\nabla f(x^\star)$$

- elimination step can be expensive, obscure structure in A (e.g., sparsity)

Newton step

we extend the definition of Newton step (p. 9.18) to equality-constrained problems

- assume x is feasible ($x \in \text{dom } f$ and $Ax = b$)
- define Newton step Δx_{nt} at x as the solution v of

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix}$$

- $x + \Delta x_{\text{nt}}$ solves problem if f is replaced with 2nd order approximation \hat{f} at x

$$\begin{array}{ll} \text{minimize (over } v) & \hat{f}(x + v) = f(x) + \nabla f(x)^T v + (1/2)v^T \nabla^2 f(x)v \\ \text{subject to} & A(x + v) = b \end{array}$$

- $x + \Delta x_{\text{nt}}$ solves optimality conditions (1), linearized at x :

$$A(x + v) = b, \quad \nabla f(x + v) + A^T w \approx \nabla f(x) + \nabla^2 f(x)v + A^T w = 0$$

Newton decrement

for the equality-constrained problem, we define the Newton decrement as

$$\begin{aligned}\lambda(x) &= (-\nabla f(x)^T \Delta x_{\text{nt}})^{1/2} \\ &= (\Delta x_{\text{nt}}^T \nabla^2 f(x) \Delta x_{\text{nt}})^{1/2}\end{aligned}$$

- $\lambda(x)^2$ is directional derivative of f at x in Newton direction Δx_{nt} :

$$\left. \frac{d}{dt} f(x + t \Delta x_{\text{nt}}) \right|_{t=0} = -\lambda(x)^2$$

- $\lambda(x)$ is norm of Newton step in quadratic Hessian norm
- $\lambda(x)$ gives estimate of $f(x) - p^*$, estimated using quadratic approximation \hat{f} :

$$f(x) - \inf_{Ay=b} \hat{f}(y) = \frac{1}{2} \lambda(x)^2$$

- in general,

$$\lambda(x) \neq (\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x))^{1/2}$$

Newton's method with equality constraints

given: a starting point $x \in \text{dom } f$ with $Ax = b$, tolerance $\epsilon > 0$

repeat

1. *Newton step*: compute Newton step Δx_{nt} and Newton decrement $\lambda(x)$
2. *stopping criterion*: quit if $\lambda^2/2 \leq \epsilon$
3. *line search*: choose step size t by backtracking line search
4. *update*: $x := x + t\Delta x_{\text{nt}}$

- a feasible descent method: iterates $x^{(k)}$ are feasible and $f(x^{(k+1)}) < f(x^{(k)})$
- affine-invariant

Newton's method and elimination

Newton's method for unconstrained optimization (after eliminating $Ax = b$)

$$\text{minimize } g(z) = f(Fz + \hat{x})$$

suppose Newton method for g , started at $z^{(0)}$, generates iterates $z^{(k)}$

Newton's method with equality constraints (method on page 10.10)

when started at $x^{(0)} = Fz^{(0)} + \hat{x}$, iterates are

$$x^{(k+1)} = Fz^{(k)} + \hat{x}$$

hence, we don't need separate convergence analysis for the method on p. 10.10

Newton step at infeasible points

2nd interpretation of page 10.8 extends to infeasible x (i.e., with $Ax \neq b$)

linearizing optimality conditions at infeasible x (with $x \in \text{dom } f$) gives

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax - b \end{bmatrix} \quad (5)$$

Interpretation

- optimality condition (1) is nonlinear equation $r(x, \nu) = 0$, where

$$r(x, \nu) = \begin{bmatrix} \nabla f(x) + A^T \nu \\ Ax - b \end{bmatrix}$$

- linearizing $r(y) = 0$ gives

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\text{nt}} \\ \Delta \nu_{\text{nt}} \end{bmatrix} = - \begin{bmatrix} \nabla f(x) + A^T \nu \\ Ax - b \end{bmatrix}$$

this is the same as (5) with $w = \nu + \Delta \nu_{\text{nt}}$

Infeasible start Newton method

given: a starting point $x \in \text{dom } f$, v , tolerance $\epsilon > 0$, $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$

repeat

1. *Newton step*: compute primal and dual Newton steps Δx_{nt} , Δv_{nt}

2. *backtracking line search*:

$t := 1$

while $\|r(x + t\Delta x_{\text{nt}}, v + t\Delta v_{\text{nt}})\|_2 > (1 - \alpha t)\|r(x, v)\|_2$

$t := \beta t$

3. *update*: $x := x + t\Delta x_{\text{nt}}$, $v := v + t\Delta v_{\text{nt}}$

until $Ax = b$ and $\|r(x, v)\|_2 \leq \epsilon$

- not a descent method: $f(x^{(k+1)}) > f(x^{(k)})$ is possible
- therefore we use norm of r as merit function in line search
- directional derivative of norm of r , at $y = (x, v)$, in direction $\Delta y = (\Delta x_{\text{nt}}, \Delta v_{\text{nt}})$ is

$$\left. \frac{d}{dt} \|r(y + t\Delta y)\|_2 \right|_{t=0} = -\|r(y)\|_2$$

Solving KKT systems

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} g \\ h \end{bmatrix}$$

Solution methods

- use matrix factorization for symmetric indefinite matrices (LDL^T factorization)
- if H is positive definite, solve by block elimination: two equations

$$AH^{-1}A^T w = h - AH^{-1}g, \quad Hv = -(g + A^T w)$$

- if H is not positive definite, first write KKT system as

$$\begin{bmatrix} H + A^T Q A & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} g + A^T Q h \\ h \end{bmatrix}$$

with $Q \succeq 0$ chosen so that $H + A^T Q A \succ 0$; then apply block elimination

Equality constrained analytic centering

Primal and dual problems

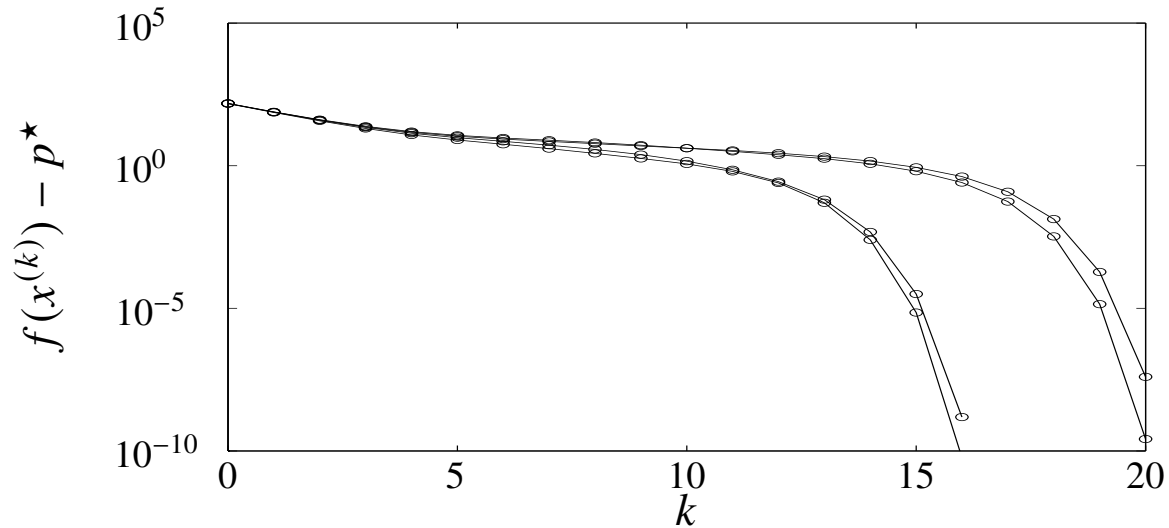
$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^n \log x_i \\ \text{subject to} & Ax = b \end{array} \qquad \begin{array}{ll} \text{maximize} & -b^T \nu + \sum_{i=1}^n \log(A^T \nu)_i + n \end{array}$$

Algorithms

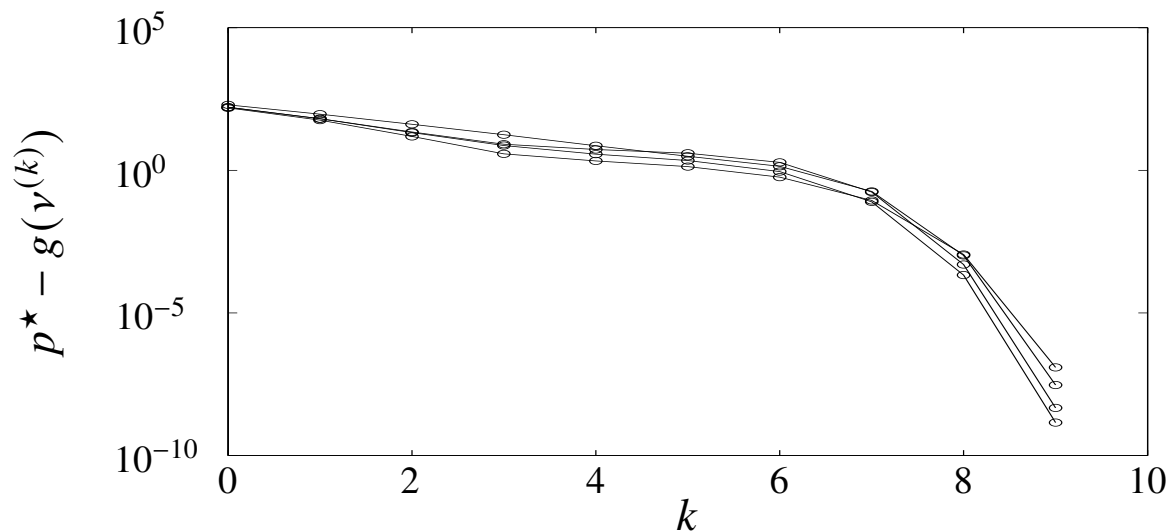
- we compare three versions of Newton's method
- $A \in \mathbf{R}^{100 \times 500}$
- starting points are different for the three methods

Equality constrained analytic centering

1. Newton method with equality constraints (requires $x^{(0)} \succ 0$, $Ax^{(0)} = b$)

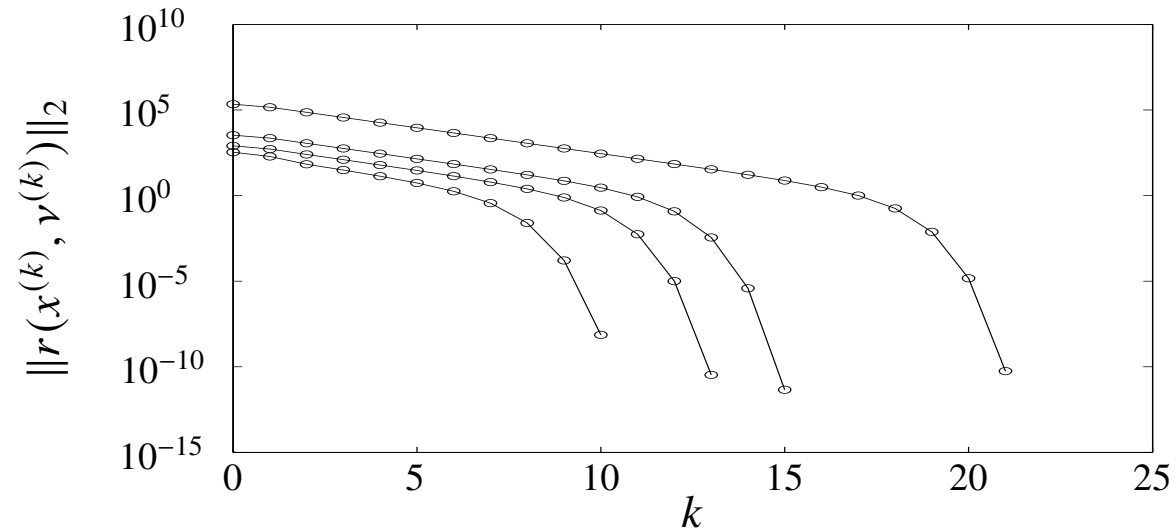


2. (unconstrained) Newton method applied to dual problem (requires $A^T v^{(0)} \succ 0$)



Equality constrained analytic centering

3. infeasible start Newton method (requires $x^{(0)} \succ 0$)



Equality constrained analytic centering

complexity per iteration of the three methods is identical

1. use block elimination to solve KKT system

$$\begin{bmatrix} \mathbf{diag}(x)^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{diag}(x)^{-1} \mathbf{1} \\ 0 \end{bmatrix}$$

reduces to solving $A \mathbf{diag}(x)^2 A^T w = b$

2. solve Newton system $A \mathbf{diag}(A^T v)^{-2} A^T \Delta v = -b + A \mathbf{diag}(A^T v)^{-1} \mathbf{1}$

3. use block elimination to solve KKT system

$$\begin{bmatrix} \mathbf{diag}(x)^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \end{bmatrix} = \begin{bmatrix} \mathbf{diag}(x)^{-1} \mathbf{1} \\ Ax - b \end{bmatrix}$$

reduces to solving $A \mathbf{diag}(x)^2 A^T w = 2Ax - b$

conclusion: in each case, solve $ADA^T w = h$ with D positive diagonal

Network flow optimization

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \phi_i(x_i) \\ \text{subject to} & Ax = b \end{array}$$

- directed graph with n arcs, $p + 1$ nodes
- x_i is flow through arc i ; ϕ_i is cost flow function for arc i (with $\phi_i''(x) > 0$)
- node-incidence matrix $\tilde{A} \in \mathbf{R}^{(p+1) \times n}$ defined as

$$\tilde{A}_{ij} = \begin{cases} 1 & \text{arc } j \text{ leaves node } i \\ -1 & \text{arc } j \text{ enters node } i \\ 0 & \text{otherwise} \end{cases}$$

reduced node-incidence matrix $A \in \mathbf{R}^{p \times n}$ is \tilde{A} with last row removed

- $b \in \mathbf{R}^p$ is (reduced) source vector
- **rank** $A = p$ if graph is connected

KKT system for network flow problem

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} g \\ h \end{bmatrix}$$

- $H = \mathbf{diag}(\phi''_1(x_1), \dots, \phi''_n(x_n))$, positive diagonal
- can be solved by block elimination:

$$AH^{-1}A^T w = h - AH^{-1}g, \quad Hv = -(g + A^T w)$$

sparsity pattern of coefficient matrix $AH^{-1}A^T$ is given by graph connectivity

$$\begin{aligned} (AH^{-1}A^T)_{ij} \neq 0 &\iff (AA^T)_{ij} \neq 0 \\ &\iff \text{nodes } i \text{ and } j \text{ are adjacent} \end{aligned}$$

Analytic center of linear matrix inequality

$$\begin{array}{ll} \text{minimize} & -\log \det X \\ \text{subject to} & \text{tr}(A_i X) = b_i, \quad i = 1, \dots, p \end{array}$$

- variable is $X \in \mathbf{S}^n$
- includes an implicit constraint $X \succ 0$

Optimality conditions

$$X \succ 0, \quad -X^{-1} + \sum_{j=1}^p \nu_j A_j = 0, \quad \text{tr}(A_i X) = b_i, \quad i = 1, \dots, p$$

Newton equation at feasible X

$$X^{-1} \Delta X X^{-1} + \sum_{j=1}^p w_j A_j = X^{-1}, \quad \text{tr}(A_i \Delta X) = 0, \quad i = 1, \dots, p$$

- follows from linear approximation $(X + \Delta X)^{-1} \approx X^{-1} - X^{-1} \Delta X X^{-1}$
- a set of $\frac{1}{2}n(n+1) + p$ linear equations in $\frac{1}{2}n(n+1) + p$ variables $\Delta X, w$

Solution by block elimination

- eliminate ΔX from first equation:

$$\Delta X = X - \sum_{j=1}^p w_j X A_j X \quad (6)$$

- substitute expression for ΔX in second equation

$$\sum_{j=1}^p \text{tr}(A_i X A_j X) w_j = b_i, \quad i = 1, \dots, p \quad (7)$$

this is a dense positive definite set of linear equations $Gw = b$ with

$$G_{ij} = \text{tr}(A_i X A_j X), \quad i, j = 1, \dots, p$$

- first solve (7) by Cholesky factorization of G ; then substitute w in (6) to get ΔX

Complexity of block elimination method

the dominant terms in a flop count:

- Cholesky factorization $X = LL^T$ $((1/3)n^3$ flops)
- form p products $L^T A_j L$ $((3/2)pn^3$ flops)
- compute $p(p + 1)/2$ elements of G

$$G_{ij} = \text{tr}((L^T A_i L)(L^T A_j L))$$

$((1/2)p^2 n^2$ flops)

- solve (7) via Cholesky factorization $((1/3)p^3$ flops)

complexity is *cubic* in n (although KKT system has $\frac{1}{2}n(n + 1) + p$ variables)