1. (a) i. First, let's find $V_{n, out}^2$. The noise contributions from $R_S$, $R_D$, and $M_1$ all appear as uncorrelated current sources flowing through $R_S$ || $R_D$ || $R_{in}$ (where $R_{in}$ is the input resistance excluding $R_S$ and $R_D$), which then experience a gain of 1 to the output (since $M_2$ is biased by an ideal current source). The noise current of $M_2$ flows to the output directly and gets converted to a voltage via $R_{out}$.

\[
V_{n, out}^2 = \left( \frac{T_{R_S}^2}{R_S} + \frac{T_{R_D}^2}{R_D} + \frac{T_{D_1}^2}{R_{D_1}} \right) (R_S || R_D || R_{in})^2 + \frac{T_{D_2}^2 R_{out}^2}{R_{out}}
\]

\[= (4kT/R_S + 4kT/R_D + 4kT\gamma g_{m1}) (R_S || R_D || R_{in})^2 + 4kT\gamma g_{m2} R_{out}^2 \]

We can see that $R_{in} = 1/g_{m1}$, since $v_{gs2} = 0$ in the small-signal model means $M_1$ is effectively diode-connected for small-signal purposes.

Fig. 1 shows the test setup we'll use to find $R_{out}$.

![Test Setup Diagram]

**Figure 1:** Test setup for finding $R_{out}$

\[i_s = -g_{m2} v_{gs2} \]
\[v_{gs2} = -g_{m1} v_{gs1} (R_S || R_D) - v_i \]
\[v_{gs1} = v_i \]
\[v_{gs2} = -v_i \left[ g_{m1} (R_S || R_D) + 1 \right] \]
\[i_s = g_{m2} v_i \left[ g_{m1} (R_S || R_D) + 1 \right] \]
\[R_{out} = \frac{v_i}{i_s} = \frac{1}{g_{m2} \left[ g_{m1} (R_S || R_D) + 1 \right]} \]

Thus, we have:

\[V_{n, out}^2 = 4kT \left[ \left( \frac{1}{R_S} + \frac{1}{R_D} + \gamma g_{m1} \right) \left( R_S || R_D || \frac{1}{g_{m1}} \right)^2 + \gamma g_{m2} \left( \frac{1}{g_{m2} \left[ g_{m1} (R_S || R_D) + 1 \right]} \right)^2 \right] \]

\[N F = \frac{V_{n, out}^2}{A^2} \frac{1}{4kTR_S} \]

\[A = \frac{R_D || \frac{1}{g_{m1}}}{R_D || \frac{1}{g_{m1}}} + R_S \]

\[N F = \frac{1}{R_S} \left( 1 + \frac{R_S}{R_D || \frac{1}{g_{m1}}} \right)^2 \left[ \left( \frac{1}{R_S || R_D} + \gamma g_{m1} \right) \left( R_S || R_D || \frac{1}{g_{m1}} \right)^2 + \gamma \frac{1}{g_{m2} \left[ g_{m1} (R_S || R_D) + 1 \right]^2} \right] \]

\[= R_S \left( \frac{1}{R_S || R_D || \frac{1}{g_{m1}}} \right)^2 \left[ \left( \frac{1}{R_S || R_D} + \gamma g_{m1} \right) \left( R_S || R_D || \frac{1}{g_{m1}} \right)^2 + \gamma \frac{1}{g_{m2} \left[ g_{m1} (R_S || R_D) + 1 \right]^2} \right] \]
\[
= 1 + \frac{R_S}{R_D} + \gamma g_m_1 R_S + \frac{\gamma R_S (1/R_S + 1/R_D + g_m_1)^2}{g_m_2 (R_S || R_D) + 1} \\
= \frac{1 + \frac{R_S}{R_D} + \gamma g_m_1 R_S + \frac{\gamma R_S}{g_m_2 (R_S || R_D)^2}}{\sqrt{\gamma g_m_1 R_S}}
\]

As we let \( R_D \to \infty \), the contribution to the noise figure from \( R_D \) goes to zero, and the contribution to the noise figure from \( M_2 \) decreases. Intuitively, we know that the noise current squared from \( R_D \) goes as \( 1/R_D \) while the input resistance (which the noise flows through) is limited by \( (R_S || 1/g_m_1) \). Thus, as \( R_D \) increases, the noise current drops out while the input resistance stays finite, causing the noise contribution to drop out.

Similar intuition applies to the decrease in noise from \( M_2 \). We see that \( R_{out} \) decreases as \( R_D \) increases. Since the noise current of \( M_2 \) flows through \( R_{out} \), we can see that as \( R_D \) increases, the noise contribution from \( M_2 \) will decrease.

ii. Let \( R_{in} \) be the input resistance excluding \( R_S \), \( R_{out} \) the output resistance excluding \( R_D \), and \( A_v \) the gain excluding \( R_S \) (i.e., from the source to the drain of \( M_1 \)).

\[
V_{r_{in, out}}^2 = \left( \frac{\gamma^2}{R_S} + \frac{\gamma^2}{R_D} \right) (R_S || R_{in})^2 A_v^2 + \left( \frac{\gamma}{R_D} (R_D || R_{out}) \right)^2 + \left\{(\frac{\gamma}{R_D} (R_S || R_{in}) A_v - (R_D || R_{out})) \right\}^2
\]

\[
= \left( \frac{\gamma^2}{R_S} + \frac{\gamma^2}{R_D} \right) (R_S || R_{in})^2 A_v^2 + \frac{\gamma^2}{R_D} (R_D || R_{out})^2 + \left\{(\frac{\gamma}{R_D} (R_S || R_{in}) A_v - (R_D || R_{out})) \right\}^2
\]

We need to find \( R_{in} \), \( R_{out} \), and \( A_v \). Let’s use Fig. 2 to find \( R_{in} \).

![Figure 2: Test setup for finding \( R_{in} \)](image)

\[
\begin{align*}
\dot{i}_t &= -g_{m_1} v_{gs_1} - g_{m_2} v_{gs_2} \\
v_{gs_1} &= v_t \\
v_{gs_2} &= -g_{m_1} v_{gs_1} R_D - v_t \\
&= g_{m_1} R_D v_t - v_t \\
&= v_t (g_{m_1} R_D - 1) \\
i_t &= g_{m_1} v_t - g_{m_2} v_t (g_{m_1} R_D - 1) \\
&= v_t [g_{m_1} + g_{m_2} (1 - g_{m_1} R_D)] \\
\frac{v_t}{i_t} &= \frac{1}{g_{m_1} + g_{m_2} (1 - g_{m_1} R_D)}
\end{align*}
\]

Let’s use Fig. 3 to find \( R_{out} \).
Figure 3: Test setup for finding $R_{\text{out}}$

\[ \begin{align*}
    i_t &= g_{m1} v_{gs1} \\
    -v_{gs1} &= (i_t + g_{m2} v_{gs2}) R_S \\
    v_{gs2} &= v_t + v_{gs1} \\
    -v_{gs1} &= (i_t + g_{m2} v_t + g_{m2} v_{gs1}) R_S \\
    v_{gs1} &= \frac{1}{1 + g_{m2} R_S} (1 + g_{m1} R_S) v_t \\
    i_t &= -g_{m1} \left( \frac{(i_t + g_{m2} v_t) R_S}{1 + g_{m2} R_S} \right) \\
    \frac{v_t}{i_t} &= \frac{1 + g_{m1} R_S}{g_{m1} g_{m2} R_S} \\
    R_{\text{out}} &= \frac{v_t}{i_t} = \frac{1 + g_{m1} R_S}{g_{m1} g_{m2} R_S}
\end{align*} \]

Let's use Fig. 4 to find $A_v$.

Figure 4: Test setup for finding $A_v$

\[ \begin{align*}
    v_{\text{out}} &= -g_{m1} v_{gs1} R_D \\
    v_{gs1} &= -v_{\text{in}} \\
    v_{\text{out}} &= g_{m1} R_D v_{\text{in}} \\
    A_v &= \frac{v_{\text{out}}}{v_{\text{in}}} = g_{m1} R_D
\end{align*} \]
Now we can write $V_{n, out}^2$:

$$R_S \parallel R_{in} = \frac{R_S}{R_S + \frac{g_{m1} + g_{m2}}{(1 - g_{m1} R_D) R_S}}$$

$$R_D \parallel R_{out} = \frac{R_D}{R_D - \frac{1}{1 + g_{m1} g_{m2} R_S R_D}}$$

$$V_{n, out}^2 = 4kT \left\{ \left( \frac{1}{R_S} + \gamma g_{m2} \right) \left[ \frac{g_{m1} R_S R_D}{R_D - (g_{m1} + g_{m2}) R_S} \right] \right\}^2$$

$$+ \frac{1}{R_D} \left[ \frac{g_{m1} R_S R_D - R_D - (g_{m1} + g_{m2}) R_S}{1 + (g_{m1} + g_{m2}) R_S R_D} \right]^2$$

$$+ \gamma g_{m1} \left[ \frac{g_{m1} R_S R_D - R_D - (g_{m1} + g_{m2}) R_S}{1 + (g_{m1} + g_{m2}) R_S R_D} \right]^2$$

$$NF = \frac{V_{n, out}^2}{A^2 4kT R_S}$$

$$A = \frac{R_{in}}{R_S + R_{in}} \sqrt{\frac{g_{m1} R_D}{1 + (g_{m1} + g_{m2}) R_S - g_{m1} g_{m2} R_S R_D}}$$

$$NF = \frac{1}{(g_{m1} R_D)^2 R_S} \left\{ \left( \frac{1}{R_S} + \gamma g_{m2} \right) \left( g_{m1} R_S R_D \right)^2 + R_D \left[ 1 + (g_{m1} + g_{m2}) R_S \right]^2 + \gamma g_{m1} R_D^2 \left[ 1 + g_{m2} R_S \right]^2 \right\}$$

$$= 1 + \gamma g_{m2} R_S + \frac{\gamma}{g_{m1} R_S} (1 + g_{m2} R_S)^2 + \frac{1}{g_{m1} R_S R_D} \left[ 1 + (g_{m1} + g_{m2}) R_S \right]^2$$

$$\lim_{R_D \to \infty} NF = 1 + \gamma g_{m2} R_S + \frac{\gamma}{g_{m1} R_S} (1 + g_{m2} R_S)^2$$

As we let $R_D \to \infty$, the contribution to the noise figure from $R_D$ goes to zero, while all other terms remain the same. Intuitively, the noise current squared of $R_D$ goes as $1/R_D$, while the output resistance is limited by $R_{out}$, which is independent of $R_D$ (note again that $R_{out}$ refers to the output resistance neglecting $R_D$, so the total output resistance would be $R_D \parallel R_{out}$). Thus, as $R_D$ goes to infinity, its noise current goes to zero while the resistance it flows through approaches a finite value, meaning its noise contribution goes to zero.
(b) i. 

$$\lim_{g_{m2} \to 0} NF = \infty$$

As we let \( g_{m2} \to 0 \), the noise figure goes to infinity. Intuitively, the noise current squared of \( M_2 \) goes as \( g_{m2} \), but the output resistance goes at \( 1/g_{m2} \). Since the noise contribution of \( M_2 \) is obtained by multiplying the square of the noise current by the square of the output resistance, the noise contribution of \( M_2 \) goes as \( 1/g_{m2} \). Thus, as \( g_{m2} \) goes to zero, the noise contribution of \( M_2 \) goes to infinity, causing the noise figure to go to infinity.

ii. 

$$\lim_{g_{m2} \to 0} NF = \frac{1}{1 + \frac{\gamma}{g_{m1}R_S} + \frac{(1 + g_{m1}R_S)^2}{g_{m1}^2R_S^2R_D}}$$

As we let \( g_{m2} \to 0 \), the contribution to the noise figure from \( M_2 \) goes to zero and the contributions from \( M_1 \) and \( R_D \) decrease. Intuitively, the noise current from \( M_2 \) flows through a finite input resistance, and since the noise current from \( M_2 \) goes to zero as \( g_{m2} \to 0 \), the noise contribution from \( M_2 \) goes to zero.

We also see that the output resistance will decrease to \( R_D \) as \( g_{m2} \to \infty \) (note that it actually is a decrease, not an increase, since \( R_{\text{out}} < 0 \), where again \( R_{\text{out}} \) is the output resistance neglecting \( R_D \)). Since the noise currents of \( M_1 \) and \( R_D \) flow through this output resistance, their noise contributions will decrease.

2. (a) Using \( W_2 = 13.7 \mu \text{m} \) causes the drain of \( M_1 \) to resonate at 5.2 GHz. The AC response of the circuit is shown in Fig. 5.

(b) Let \( R_{\text{out}1} \) be the output resistance of the first stage at 5.2 GHz, \( R_{\text{in}1} \) be the resistance seen at the source of \( M_1 \) at 5.2 GHz, and \( A_{\text{out}} \) be the gain from the source of \( M_1 \) to the drain of \( M_1 \).

$$V_{n,\text{out}}^2 = \frac{1}{R_{\text{in}1} R_{\text{out}1} A_{\text{out}}^2} + \frac{1}{R_{\text{P}1} R_{\text{out}1}} + \frac{1}{R_D} (R_{\text{in}1} A_{\text{out}} - R_{\text{out}1})^2$$

We can use SPICE to measure \( R_{\text{in}1} \), \( R_{\text{out}1} \), \( A_{\text{out}} \), and \( g_{m1} \). We can estimate \( \gamma \) using SPICE as well by taking the noise measured from \( M_1 \) and dividing it by \( 4kT g_{m1} \).

$$R_{\text{in}1} = 48.870 \ \Omega$$

$$R_{\text{out}1} = 2.5872 \ \text{k} \Omega$$

$$A_{\text{out}} = 27.5236$$

$$g_{m1} = 18.1324 \ \text{mS}$$

$$\gamma = 0.81819$$

$$R_{\text{P}1} = 1.3069 \ \text{k} \Omega$$

$$T = 298.15 \ \text{K}$$

$$k = 1.38065 \times 10^{-16} \ \text{J/K}$$

$$V_{n,\text{out}}^2 = 1.0570 \times 10^{-15} \ \text{V}^2/\text{Hz}$$

$$NF = \frac{V_{n,\text{out}}^2}{4kT R_S}$$

$$A = 27.708$$

$$NF = \frac{V_{n,\text{out}}^2}{4kT R_S}$$

$$NF = 1.67235$$

Using SPICE to find the noise figure (by dividing the total output noise by the output noise contributed by the source resistance), we get \( NF = 1.78516 \). The calculated and simulated values differ by about 6.75 %, which is relatively good agreement.
(c) The noise figure of the overall amplifier is \( NF = 1.8087 \). The second stage increases the noise figure by only about 1.32%. This makes sense intuitively because the gain of the first stage is relatively large \( (A = 27.708) \). This reduces the contribution of the second stage to the noise figure, as can be seen from the Friis equation.

This is the SPICE netlist used for this problem:

* EE215C HW2 Problem 2

```
.inc '215a.sp'

.param W2=13.7u
.param PI=3.14159265358979323846
.param Q=5
.param FREQ=5.2e9
.param L1=8n
.param RP1='2*PI*L1*FREQ/Q'
.param L2=6n
.param RP2='2*PI*L2*FREQ/Q'

vdd vdd gnd 1.8V
vin vin gnd AC 1
rs vin i 50
cl i vs1 1u
il vs1 gnd 2.5m
m1 vdi vdd vs1 gnd CMOSN W=50u L=0.18u
+PS='2*5e-5+1.2e-6' PD='2*5e-5+1.2e-6' AS='5e-5+0.6e-6' AD='5e-5+0.6e-6'
l1 vdd vd1 L1
rp1 vdd vd1 RP1

m2 vout vd1 vs2 gnd CMOSN W=W2 L=0.18u
+PS='2*W2+1.2e-6' PD='2*W2+1.2e-6' AS='W2+0.6e-6' AD='W2+0.6e-6'
i2 vs2 gnd 2m
c2 vs2 gnd 750f
c3 vout gnd 60f
l2 vdd vout L2
rp2 vdd vout RP2

* Used for finding the resonance frequency.
.ac lin 1000 5G 5.4G

* Used for finding the noise figure.
**.noise v(vout) vin 1
.option post nomod accurate
.end
```