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Distributed Structural Stabilization and Tracking for Formations of Dynamic Multi-Agents

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Abstract

In this paper, we provide a theoretical framework that consists of graph theoretical and Lyapunov-based approaches to stability analysis and distributed control of multi-agent formations. This framework relays on the notion of graph rigidity as a means of identifying the shape variables of a formation. Using this approach, we can formally define formations of multiple vehicles and three types of stabilization/tracking problems for dynamic multi-agent systems. We show how these three problems can be addressed mutually independent of each other for a formation of two agents. Then, we introduce a procedure called dynamic node augmentation that allows construction of a larger formation with more agents that can be rendered structurally stable in a distributed manner from some initial formation that is structurally stable. We provide two examples of formations that can be controlled using this approach, namely, the V-formation and the diamond formation.

1 Introduction

Formation stabilization and tracking for systems of multiple vehicles/agents are of primary interest in both military and industrial applications. Multi-agent systems arise in broad areas including formation flight of unmanned aerial vehicles (UAVs), coordination of satellite clusters, automated highways, understanding the coordination and movement of flocks of birds or schools of fish [1], coordination of underactuated underwater vehicles and surface vessels for search and rescue operations, and molecular conformation problems [2].

The applications that are of primary interest in our work include performing maneuvers by UAVs which (possibly) require doing split/rejoin maneuvers in case a group of vehicles come across an obstacle, or changing the communication configuration (or information flow [3]) in the network of vehicles due to the loss of line of sight and/or failure of a communication link (Fig. 1 (a)). In addition, we are interested in reconfiguration of the formation of a group of vehicles (Fig. 1 (b))

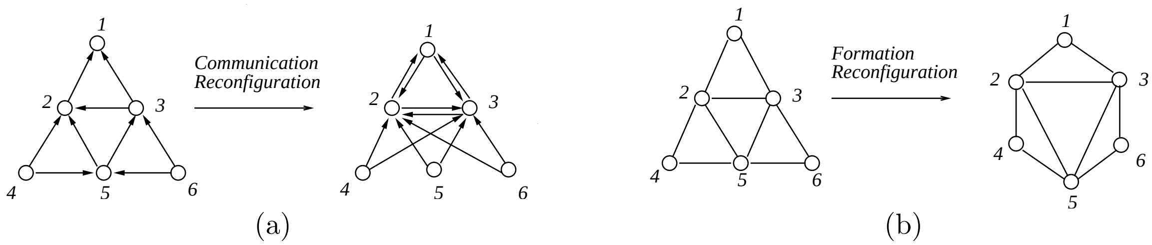

Figure 1: (a) communication reconfiguration for a group of 6 vehicles that leaves the formation unchanged and (b) Formation reconfiguration maneuver for a group of 6 vehicles moving in $\mathbb{R}^2$.

due to a change of the team-strategy in team-on-team competitive games like playing capture the flag using mobile robots known as the *RoboFlag* [4].

A sequence of operational maneuvers that changes the formation and/or information flow in a multi-agent system is shown in Fig. 2. A *crucial element* in performing the majority of these maneuvers is *the capability to solve stabilization/tracking problems for formations of multiple dynamic agents.*

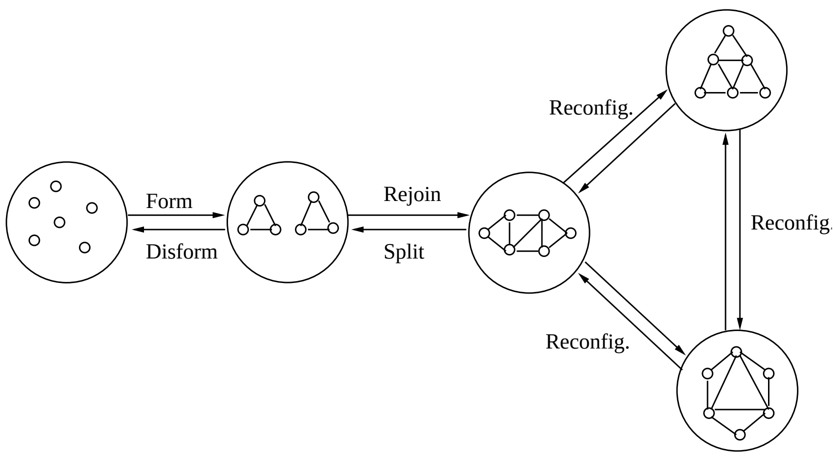

Figure 2: A Hybrid System representing the switching between multiple formations via performing a set of maneuvers.

The problem of *distributed structural stabilization of formations* of multiple vehicles using bounded control inputs is addressed in [5]. This is done by construction of a structural potential function from a minimally rigid graph that has a unique global minimum (up to rotation, translation, and folding [5]). In [1], a different type of potential function is used. In the context of that work, a unique global minimum of the overall potential function is not desirable. Furthermore, according to [1] construction of a global potential function with a unique minimum requires adding several virtual vehicles.

In both potential function based methods suggested in [5], [1], the information flow in the network of multiple agents is undirected and every two agents that are connected through a link either communicate with each other, or sense relative coordinates w.r.t. each other. This is in sharp contrast to the work in [3] which readily allows directed information flow. However, the mathematical framework in [3] does not apply to agents with nonlinear dynamics and/or performing operations like rotation of the attitude of a formation.

In summary, the main contribution of this work is to provide a means for performing stabilization/tracking in multi-agent systems in distributed and directed fashion that is capable of dealing with agents that have nonlinear dynamics and/or performing arbitrary rotations and translations. The key analytical tool is a separation principle that decouples structural stabilization from navigational tracking and the dynamic node augmentation procedure. This procedure allows construction of a larger formation with more agents that can be rendered structurally stable in a distributed manner from some initial formation that is structurally stable.

Here is an outline of the paper. In section 2, we define formations of multiple dynamic agents. In section 3, we give some background on graph rigidity and define minimally rigid graphs. In section 4, structural stabilization and tracking for a formation of two agents is presented. In section 5, we present our main result on dynamic node augmentation. Finally, we make concluding remarks.

2 Formations of Dynamic Multi-Agents

In this section, we define a formation of n-agents, the position, and the attitude of a formation. Consider a group of n agents ($n \geq 2$) each with the following dynamics

$$\left\{ \begin{array}{rcl} \dot{q}_i & = & p_i \\ \dot{p}_i & = & u_i \end{array} \right. \tag{1}$$

where $q_i, p_i, u_i \in \mathbb{R}^m$ for all $i \in \mathcal{I} = \{1, \ldots, n\}$. Therefore, each agent has a *linear dynamics.*

Remark 1. This assumption is made for the sake of presenting the main geometric and graph-theoretic ideas rather than getting involved in the technical details of dealing with nonlinear control of underactuated/nonholonomic mechanical systems. Control of formations of mobile robots that are underactuated or posses nonholonomic velocity constraints is the topic of a sequel of this paper.

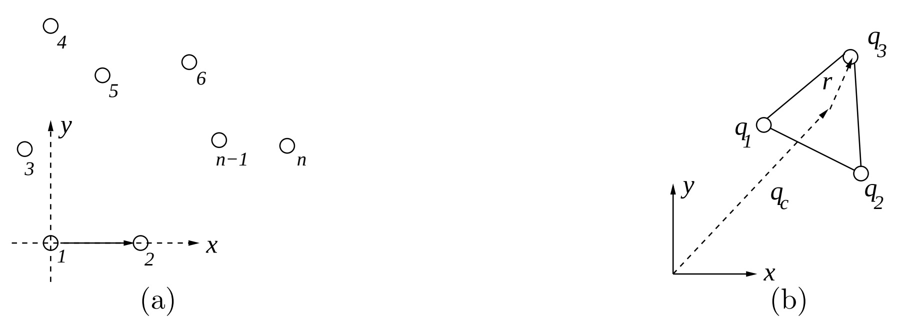

Figure 3: (a) A formation of $n \geq 2$ agents with a base $(1, 2)$ in $\mathbb{R}^2$, and (b) Position and attitude (q_c, r) of the formation of three vehicles.

We refer to a set of n points in $\mathbb{R}^m$ as an *n-grid.* The column vector $q = (q_1, \ldots, q_n)^c \in \mathbb{R}^{mn}$ is called the *configuration* of the n-grid. Identifying an agent $i \in \mathcal{I}$ by its position q_i, an agent can be viewed as a point in $\mathbb{R}^m$. Assume $\|q_2 - q_1\| > 0$ and connect the agents 1 and 2 by a directed partial-line e_{12} that is called the *base-edge* of the n-grid. An n-grid in which the distance between each two agents is greater than zero is called *collision-free*, i.e. $\|q_j - q_i\| > \gamma, \forall i, j \in \mathcal{I}, i \neq j$. Define $\gamma = \min_{1 \leq i < j \leq n} \|q_j - q_i\|$. We call γ the *safety margin* of the n-grid. Notice that in any collision-free n-grid, the safety margin is positive ($\gamma > 0$). In Fig. 3 (a), an n-grid of agents and its base-edge e_{12} is shown. For any n-grid in $\mathbb{R}^2$, a *body-axes* can be defined by taking e_{12} as the x-axis and $e_{12}^{\perp} = Te_{12}$ (read "orthogonal e_{12}") as the y-axis where T is a rotation by $\pi/2$ as the following

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Notice that for a vector $x = (x_1, x_2)^T$, $x^{\perp} = Tx = (-x_2, x_1)$. Let $\mathbf{n}(x) = x/\|x\|$ and for $q_i \neq q_j$ define

$$\mathbf{n}_{ij} = \mathbf{n}(q_j - q_i) = \frac{q_j - q_i}{\|q_j - q_i\|}$$

Then, $(\phi_1, \phi_2) = (\mathbf{n}_{12}, \mathbf{n}_{12}^{\perp})$ defines the bases of the *body-axes*. In general, in a collision-free n-grid, any two arbitrary agents can be used to define an orthonormal bases $(\mathbf{n}_{ij}, \mathbf{n}_{ij}^{\perp})$. For the special choice of e_{12} as the base-edge, the coordinates of points 1 and 2 in the body-axes are given by $(0,0)^T$ and $(0,l)^T$ (with $l = \|q_2 - q_1\|$), respectively. l is a single degree of freedom (DOF) that determines the distance between agents $1, 2$. The position of the remaining $(n-2)$ is each specified in the body-axes by their (x, y)-coordinates. Therefore, each remaining agent introduces 2 more degrees of freedom. Thus, the total number of degrees of freedom of an n-grid is $f = 1 + 2(n-2) = 2n - 3$ given that $n \geq 2$. In the body-axes, an n-grid is uniquely specified by a $(2n-3)$-dimensional vector

$$\varphi = (l, x_3, y_3, x_4, y_4, \ldots, x_n, y_n) \in Q := \mathbb{R}_{\geq 0} \times \mathbb{R}^{2n-4} \tag{2}$$

We refer to μ as the vector of *internal degrees of freedom* of an n-grid. Apparently, μ remains invariant under rotation and translation of all the points in an n-grid. Let $b \in \mathbb{R}^2$ denote the coordinates of agent 1 in the reference-frame and R be the rotation matrix by θ such that $\mathbf{n}_{12} = Re_1$ where $e_1 = (1,0)^T$ denotes the 1st base of the reference-frame. Then $(b, R) \in SE(2)$ (which is a 3-dimensional manifold) represents the 3 *external degrees of freedom* of an n-grid. We call (b, θ) the *navigational variables* of the n-grid. In general, any point $q_c = \sum_{i=1}^n \lambda_i q_i$ with a fixed set of λ_i's satisfying $\sum_{i=1}^n \lambda_i = 1$ and an arbitrary unit vector $\mathbf{r}$ satisfying

$$< \mathbf{r}, \mathbf{n}_{12} >= c_0 = \text{const.}$$

can be chosen to represent the *position and attitude* of the formation of an n-grid as $(q_c, \mathbf{r})$. The $(q_c, \mathbf{r})$ can be interpreted as the position and attitude of a *virtual agent* called *the navigation (virtual) agent* associated with the formation of n agents.

Remark 2. A special choice of λ_i's and c_0 is $\lambda_i = 1/n$ that gives the position of the center of mass of all agents and $c_0 = 0$ corresponding to $\mathbf{r} = \mathbf{n}_{12}^{\perp}$. Another interesting choice of the attitude $\mathbf{r}$ is $\mathbf{r} = (q_j - q_c)/\|q_j - q_c\|$ where q_c is the center of mass and agent j is an *attitude leader*.

Note. For the special case where all agents in an n-grid coincide, define $\mathbf{n}_{12} = e_1$. This case is excluded throughout the paper, unless otherwise is stated.

Definition 1. (formation) A *formation* φ of n-agents is a point on the manifold Q (defined in (2)) associated with the set of n-grids in $\mathbb{R}^2$. The *position and attitude* of a formation is defined as $\psi = (q_c, \mathbf{r})$.

Example 1. In Fig. 3 (b), an equilateral formation of three vehicles (i.e. a 3-grid) with its associated position and attitude $(q_c, \mathbf{r})$ is shown. Here, vehicle 3 is the attitude leader.

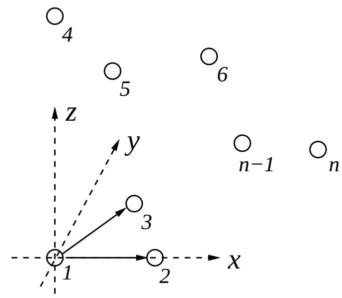

Figure 4: A formation of $n \geq 2$ agents with bases $(1, 2), (1, 3)$ in $\mathbb{R}^3$.

The method that is introduced here for representation of formations of planar n-grids can be directly generalized to any other dimension $m \geq 3$. Fig. 4 shows an n-grid in dimension $m = 3$ with two base-edges e_{12}, e_{13} that are constructed on a triangle with vertices $1, 2, 3$ and a nonzero area. In this case, the degrees of freedom of a formation is $f = 3 + 3(n - 3) = 3n - 6$. In general, for dimension $m \geq 2$, the degrees of freedom of the formation of an n-grid is

$$f = f(m, n) = m(m - 1)/2 + m(n - m) = mn - m(m + 1)/2$$

and the dimension of the navigation variables of the formation is $m(m + 1)/2$.The first term in f comes from the fact that in a group of m nodes that construct the set of $m - 1$ base-edges the distance between each two nodes introduces one degree of freedom, the remaining number of nodes are $n - m$ and the dimension of their coordinate vector in the body-axes is m. Thus the total number of degrees of freedom of the formation is $f = m(m - 1)/2 + m(n - m)$. This naturally raises the following question:

Question 1. *Is it possible to specify $f = f(m, n)$ algebraic constraints in the form of the distance between the points in an n-grid that uniquely determines the formation associated with the n-grid?*

The correct answer depends on the choice of m, n and what we mean by uniqueness.

3 Graph Rigidity and Shape Dynamics of a Graph

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a weighted graph with the set of vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$ (i.e. $|\mathcal{V}| = n$ where $|\cdot|$ denotes the number of elements of the set), the set of edges $\mathcal{E}$, and the set of weights $\mathcal{W}$. In addition, define $\mathcal{I} = \{1, 2, \ldots, n\}$ as the set of indices of the element of $\mathcal{V}$. Each agent in a multi-agent system can be viewed as a node of the graph $\mathcal{G}$ which represents the overall system.

Remark 3. Throughout this paper, we assume that controller of the multi-agent system is *distributed*. This means that each agent performs *sensing and communication* with all of its *neighbors* $J_i := \{j \in \mathcal{I} : e_{ij} \in \mathcal{E}\}$ in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. As a special case, this definition of a neighbor includes the case of *spatial neighbors* of an agent that are located within a distance $d > 0$ of each agent (see [1]).

Let $q_i \in \mathbb{R}^m$ denote the coordinates vector assigned to node v_i of the graph. Then $q = (q_1, \cdots, q_n)^c \in \mathbb{R}^{mn}$ is called a *realization* of $\mathcal{G}$ iff

$$\|q_j - q_i\| = w_{ij},\ \forall e_{ij} \in \mathcal{E}, q_i, q_j \in \mathbb{R}^m$$

where $\mathcal{W} = \{w_{ij}\}, \mathcal{E} = \{e_{ij}\}$. The pair $(\mathcal{G}, q)$ is called a *framework*. An *infinitesimal motion* is an assignment of a velocity vector p_i to the vertex v_i of the graph $\mathcal{G}$ such that

$$\langle p_j - p_i, q_j - q_i \rangle = 0, \quad \forall e_{ij} \in \mathcal{E} \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. A framework $(\mathcal{G}, q)$ is called *rigid* iff the only acceptable infinitesimal motions of the framework are due to rigid motions in $\mathbb{R}^m$. For the sake of this paper, we use the Laman's theorem as the definition of rigidity in $\mathbb{R}^2$. For a detailed treatment of graph rigidity in the context of formation stabilization of multi-agents we refer the reader to [6].

Definition 2. (Laman subgraph) A *Laman subgraph* of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ such that $\mathcal{V}_\mathcal{H} \subset \mathcal{V}$, $|\mathcal{V}_\mathcal{H}| \geq 2$, and $\mathcal{E}_\mathcal{H} = \mathcal{E}|_{\mathcal{V}_\mathcal{H}} := \{e_{ij} \in \mathcal{E} : v_i, v_j \in \mathcal{V}_\mathcal{H}\}$ (We read $\mathcal{E}_\mathcal{H}$ is the restriction of $\mathcal{E}$ to $\mathcal{V}_\mathcal{H}$)

Theorem. *(Laman, 1970 [7]) A planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $n \geq 2$ nodes is rigid iff there exists a subset $\mathcal{E}_\mathcal{H} \subset \mathcal{E}$ of $2n-3$ edges of $\mathcal{G}$ such that for the graph $\mathcal{H} = (\mathcal{V}, \mathcal{E}_\mathcal{H})$ with n nodes, each Laman subgraph $Y = (\mathcal{V}_Y, \mathcal{E}_Y)$ of $\mathcal{H}$ satisfies the property $|\mathcal{E}_Y| \leq 2|\mathcal{V}_Y| - 3$.*

Any rigid graph $\mathcal{G}$ with $n \geq 2$ nodes and $2n-3$ edges is called a *minimally rigid graph* (MRG). Apparently, any MRG is *the essential subgraph of itself*. In addition, *every edge of an MRG is independent.*

Due to computational and communications costs in a network of n-vehicles, we are interested in the least possible number of edges between the agents that creates a rigid graph and thus a locally stabilizing distributed control law for each vehicle [5]. This makes minimally rigid graphs the ideal choice for us. Moreover, it will become clear later that MRGs benefit from nice analytic properties that allow one to construct bigger graphs through connecting minimally rigid subgraphs. This is explained in complete details in [6].

The edges of a minimally rigid graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ define the following set of *shape variables* for the graph:

$$\eta_{ij} := \|q_j - q_i\| - w_{ij}, \quad \forall e_{ij} \in \mathcal{E} \tag{4}$$

We call the column vector η and manifold $Q(\mathcal{G})$ defined by

$$\eta = \{\eta_{ij}\} \in Q(\mathcal{G}) := \Pi_{e_{ij} \in \mathcal{E}}[-w_{ij}, \infty) \subset \mathbb{R}^{2n-3} \tag{5}$$

as the *shape configuration* and *shape manifold* of $\mathcal{G}$. *Any point at the boundary of $Q(G)$ corresponds to a collision between two agents.* The *shape velocity* of $\mathcal{G}$ is defined as $\nu = \{\nu_{ij}\}$ with elements $\nu_{ij} = \dot{\eta}_{ij}$ given by

$$\nu_{ij} := \frac{\langle q_j - q_i, p_j - p_i \rangle}{\|q_j - q_i\|} = \langle \mathbf{n}_{ij}, p_j - p_i \rangle, \quad \forall e_{ij} \in \mathcal{E} \tag{6}$$

By definition of ν, any infinitesimal motion of the graph maintains the shape velocity at zero. The *shape dynamics* of $\mathcal{G}$ is a set of equations in the form

$$\textit{edge dynamics of } e_{ij} : \left\{ \begin{array}{lcl} \dot{\eta}_{ij} & = & \nu_{ij} \\ \dot{\nu}_{ij} & = & (\|p_j - p_i\|^2 - \nu_{ij}^2)/\|q_j - q_i\| + \langle \mathbf{n}_{ij}, u_j - u_i \rangle \end{array} \right. \tag{7}$$

where $e_{ij} = (v_i, v_j) \in \mathcal{E}$ is an edge of the graph $\mathcal{G}$. The overall shape dynamics of $\mathcal{G}$ can be expressed as the following

$$\textit{shape dynamics of } \mathcal{G} : \left\{ \begin{array}{rcl} \dot{\eta} & = & \nu \\ \dot{\nu} & = & \Phi(\tilde{q}, \tilde{p}) + B(\tilde{q})\tilde{u} \end{array} \right. \tag{8}$$

where $\Phi(\tilde{q}, \tilde{p})$ vanishes at $\tilde{p} = 0$, $B(\tilde{q})$ is an $f \times 2f$ matrix ($f = 2n - 3$) and

$$\tilde{q} = \{(q_j - q_i)\}_{e_{ij} \in \mathcal{E}},\ \tilde{p} = \{(p_j - p_i)\}_{e_{ij} \in \mathcal{E}},\ \tilde{u} = \{(u_j - u_i)\}_{e_{ij} \in \mathcal{E}}$$

are column vectors of *relative positions*, *relative velocities*, and *relative controls* in $\mathbb{R}^{2f}$, respectively. The *structural potential function* of the graph $\mathcal{G}$ is defined as a smooth, proper, and positive definite function $V(\eta)$ that satisfies $V(0) = 0$. Two examples of $V(\eta)$ (or $V(q)$) are given in [5] as the following:

$$\begin{array}{rcl} V_1(\eta) & = & \sum_{e_{ij} \in \mathcal{E}} \eta_{ij}^2 \\ V_2(\eta) & = & \sum_{e_{ij} \in \mathcal{E}} [(1 + \eta_{ij}^2)^{\frac{1}{2}} - 1] \end{array} \tag{9}$$

Definition 3. (structural stabilization) By *(asymptotic) structural formation stabilization*, we mean (asymptotic) stabilization of the shape dynamics of $\mathcal{G}$ around the equilibrium point $(\eta, \nu) = 0$ such that for the closed loop system in (8),$(\eta, \nu) = 0$ is (asymptotically) stable in the sense of Lyapunov.

Remark 4. Clearly, $V_2(q) := V_2(\eta)$ has a bounded gradient w.r.t. q and this is the key in designing a bounded control input for structural formation stabilization in [5].

The following lemma shows that for a formation of $n = 2$ dynamic agents, exponential structural formation stabilization can be readily achieved.

Lemma 1. *The shape dynamics of $\mathcal{G}$ in* (8) *for $n = 2$ agents satisfying $\tilde{u} = \alpha_{12} \cdot \mathbf{n}_{12}$ with a scalar control input $\alpha_{12} \in \mathbb{R}$ is fully-actuated.*

Proof. For $n = 2$, $\tilde{u} = u_2 - u_1 \in \mathbb{R}^2$. Applying the invertible change of control

$$\tilde{u} = [\gamma_{12} - \Phi(\tilde{q}, \tilde{p})]\mathbf{n}_{12}$$

where $\gamma_{12} \in \mathbb{R}$ is the new control, we get

$$\left\{ \begin{array}{rcl} \dot{\eta}_{12} & = & \nu_{12} \\ \dot{\nu}_{12} & = & \gamma_{12} \end{array} \right.$$

which is a fully-actuated system with a single degree of freedom.

In this paper, our approach is to define and achieve structural stabilization and navigational stabilization/tracking for a formation of $n = 2$ dynamic agents. Then, we augment this formation with further agents and demonstrate a three-way *separation principle* in control design for both current and successively added agents. This process in a graph theoretical setting is called *node augmentation* and in [6] it is proved that node augmentation preserves minimal rigidity property of the obtained graph.

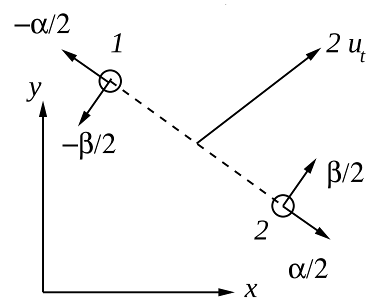

Figure 5: The control inputs applied to each agent in a two-agent formation.

4 Stabilization and Tracking for $n = 2$ Agents

In this section, we demonstrate that structural stabilization, position tracking, and attitude tracking for a formation of $n = 2$ agents can be reduced to three separate stabilization problems.

Theorem 1. *(structural stabilization and navigation control separation) For a formation of $n = 2$ agents, shape, rotational, and translational dynamics can be decoupled as the following:*

$$\begin{array}{rl} \textit{shape:} & \left\{ \begin{array}{rcl} \dot{\eta} & = & \nu \\ \dot{\nu} & = & u_s \end{array} \right. \\ \textit{rotation:} & \left\{ \begin{array}{rcl} \dot{R} & = & R\hat{\omega} \\ \dot{\omega} & = & u_r \end{array} \right. \\ \textit{translation:} & \left\{ \begin{array}{rcl} \dot{x} & = & v \\ \dot{v} & = & u_t \end{array} \right. \end{array} \tag{10}$$

where $\eta = \|q_2 - q_1\| - w_{12}$, $R = [\mathbf{n}_{12}|\mathbf{n}_{12}^{\perp}]$, $x = (q_1 + q_2)/2$, $\hat{\omega} \in \mathfrak{so}(2)$, and u_1, u_2 are given by

$$\begin{array}{rcl} u_1 & = & -\frac{\alpha}{2}\mathbf{n}_{12} - \frac{\beta}{2}\mathbf{n}_{12}^{\perp} + u_t \\ u_2 & = & +\frac{\alpha}{2}\mathbf{n}_{12} + \frac{\beta}{2}\mathbf{n}_{12}^{\perp} + u_t \end{array} \tag{11}$$

and $u_s, u_t \in \mathbb{R}$ are, respectively, obtained from $\alpha, \beta \in \mathbb{R}$ by applying an invertible change of control and $u_t \in \mathbb{R}^2$.

Proof. The forces applied to each agent are shown in Fig. 5. We have $u_1 + u_2 = 2u_t$, thus $\ddot{x} = u_t$. Moreover, since $u_2 - u_1 = \alpha\mathbf{n}_{12} + \beta\mathbf{n}_{12}^{\perp}$, the term $\langle u_2 - u_1, \mathbf{n}_{12}\rangle = \alpha$ does not depend on the choice of β. For the shape dynamics of the edge e_{12}, we obtain

$$\begin{array}{rcl} \dot{\eta} & = & \nu \\ \dot{\nu} & = & \phi_{12} + \alpha \end{array} \tag{12}$$

where $\phi_{12} = \Phi(\tilde{q}, \tilde{p})$. After applying the change of control

$$\alpha = u_s - \frac{\|p_2 - p_1\|^2 - \nu^2}{\|q_2 - q_1\|}$$

one gets $\ddot{\eta} = u_s$ which determines the structural dynamics of the formation. It remains to establish the connection between β and the control of the attitude dynamics u_r. For doing so, observe that

$$\hat{\omega} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

and from $\dot{R} = R\hat{\omega}$, we have $\dot{\mathbf{n}}_{12} = -\omega \mathbf{n}_{12}^{\perp}$ and $\dot{\mathbf{n}}_{12}^{\perp} = \omega \mathbf{n}_{12}$. Thus, after differentiating both sides of the following equation

$$\|q_2 - q_1\| \mathbf{n}_{12} = q_2 - q_1$$

with respect to t, we get

$$\nu \mathbf{n}_{12} - \omega \|q_2 - q_1\| \mathbf{n}_{12}^{\perp} = p_2 - p_1$$

which means

$$\omega = -\frac{\langle p_2 - p_1, \mathbf{n}_{12}^{\perp} \rangle}{\|q_2 - q_1\|} \tag{13}$$

after differentiating

$$\|q_2 - q_1\| \omega = -\langle p_2 - p_1, \mathbf{n}_{12}^{\perp} \rangle \tag{14}$$

w.r.t. t, we obtain

$$\nu\omega + \|q_2 - q_1\| \dot{\omega} = -\langle u_2 - u_1, \mathbf{n}_{12}^{\perp} \rangle - \langle p_2 - p_1, \omega \mathbf{n}_{12} \rangle \tag{15}$$

Noticing that $\langle p_2 - p_1, \omega \mathbf{n}_{12} \rangle = \nu\omega$ and $\langle u_2 - u_1, \mathbf{n}_{12}^{\perp} \rangle = \beta$ does not depend on the choice of α (or u_t), one gets

$$u_r = \dot{\omega} = -\frac{\beta + 2\nu\omega}{\|q_2 - q_1\|} \tag{16}$$

or

$$\beta = -(\|q_2 - q_1\| u_r + 2\nu\omega) \tag{17}$$

This change of control is invertible as long as $q_1 \neq q_2$. The overall control input for agent-i ($i = 1, 2$) takes the following explicit form:

$$u_i = \frac{(-1)^i}{2} \left(u_s - \frac{\|p_2 - p_1\|^2 - \nu^2}{\|q_2 - q_1\|} \right) \mathbf{n}_{12} + \frac{(-1)^{i+1}}{2} (\|q_2 - q_1\| u_r + 2\nu\omega) \mathbf{n}_{12}^{\perp} + u_t \tag{18}$$

where the controls u_s, u_r, u_t can be determined mutually independent of each other. □

Theorem 2. *Collision-free exponential structural stabilization and navigational tracking can be achieved for a formation of $n = 2$ dynamic agents.*

Proof. Exponential structural formation stabilization can be achieved using the following controller

$$u_s = k_s(\eta, \nu) := -c_1 \eta - c_2 \nu, \quad c_1, c_2 > 0 \tag{19}$$

that is linear in (η, ν) and nonlinear as a function of relative positions and velocities $\tilde{q}, \tilde{p}$. Apparently, u_s can be chosen to be bounded $u_s = -\sigma(c_1\eta + c_2\nu)$ where $\sigma(\cdot)$ is a bounded sigmoidal function (e.g. saturation or $\tanh(\cdot)$). Set $y = (\eta, \nu)^T \in \mathbb{R}^2$ and let $V_s(\eta, \nu) = y^T P y$ where $P > 0$ is a positive definite matrix be the Lyapunov function for the closed-loop shape dynamics. Then, all the level sets of $V_s(\eta, \nu)$ in $\mathbb{R}^2$ the lie on the right hand side of the vertical line $\eta = -w_{12}$ correspond to collision-free trajectories of the system. The overall region of attraction of the closed-loop shape dynamics that is collision-free is a bigger set than the initial conditions starting at the internal part of the larges level set of $V_s(y)$ that is tangent to $\eta = -w_{12}$. This is schematically demonstrated in Fig. 6. Clearly, exponential collision-free structural stabilization with a margin of $\epsilon > 0$ is achieve locally for initial conditions in a set that contains $\Omega = \{(\eta, \nu) \in \mathbb{R}^2 : V_s(\eta, \nu) \leq c_0(\epsilon)\}$ with $c_0(\epsilon) = -w_{12} + \epsilon$ and $0 < \epsilon < w_{12}$. According to the separation principle in Theorem 1, whether the trajectories of a formation of $n = 2$ vehicles is collision-free or not has to do nothing with the choice of its attitude and position controllers and is solely determined by the controller of the shape dynamics. The size of Ω depends on the values of the parameters $w_{12}, \epsilon > 0$ and the choice of $c_1, c_2 > 0$. Let θ denote the angle between $\mathbf{n}_{12}$ and the x-axis of the reference frame. Then,

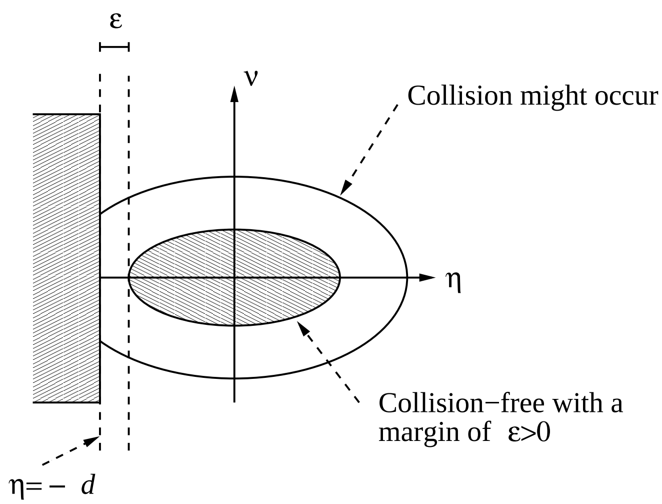

Figure 6: The collision-free region of attraction of the closed-loop shape dynamics of the formation.

$\ddot{\theta} = u_r$. Let $x_d(\cdot), \theta_d(\cdot)$ denote the desired position and attitude for tracking. Then, the following control laws achieve exponential tracking of these trajectories

$$\begin{aligned} u_r &= -c_3(\theta - \theta_d) - c_4(\omega - \dot{\theta}_d) + \ddot{\theta}_d, \quad c_3, c_4 > 0 \\ u_t &= -c_5(x - x_d) - c_6(v - \dot{x}_d) + \ddot{x}_d, \quad c_5, c_6 > 0 \end{aligned} \tag{20}$$

For exponential stabilization to an equilibrium point (x_d, θ_d), simply set $(\dot{x}_d, \ddot{x}_d, \dot{\theta}_d, \ddot{\theta}_d) = 0$ in (20). Substituting the *structural control* u_s from equation (19) and the *navigation controls* u_r, u_t from equation (20) in (18) gives the overall control input for each dynamic agent. □

Remark 5. According to the separation principle in Theorem 1, whether the trajectories of a formation of $n = 2$ vehicles is collision-free or not has to do nothing with the choice of its attitude and position controllers and is solely determined by the controller of the shape dynamics.

Remark 6. Despite the fact that the structural and navigation controllers in equations (19), (20) are *linear*, the controller of each agent in (18) is inherently a *nonlinear controller.*

Remark 7. For coordinate-independent exponential attitude tracking for $SO(2)$ and $SO(3)$ matrices, we refer the reader to the important work of Bullo [8] for the kinematic equation $\dot{R} = R\hat{\omega}$ and its generalization in [9, pp. 179–184] to the dynamic case.

5 Main Result: Dynamic Node Augmentation

One possible way to view a multi-agent formation or group of vehicles with $n \geq 3$ agents is to start with a formation of $n = 2$ agents and then successively add more agents to the formation. This process is formally described in [6] and is called *node augmentation*. Roughly speaking, each new agent, say agent-k, establishes two edges with two exiting agents, say agent-i and agent-j, in the graph representing the formation. This is shown schematically in Fig. 7. In [6], it is proved that a minimally rigid graph remains minimally rigid under node augmentation.

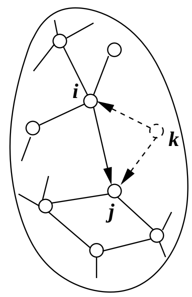

Figure 7: Augmentation of the node k at the edge (i, j) of a multi-node graph representing a multi-agent formation.

Two examples of graphs created using node-augmentation are shown in Fig. 8. Here is a sequence of nodes and edges that create a (minimally) rigid graph shown in Fig. 8 (a), (b).

$$\begin{array}{ll} \mathcal{G}_a: & 1; 2, (2,1); 3, (3,1), (3,2); 4, (4,2), (4,3). \\ \mathcal{G}_b: & 1; 2, (2,1); 3, (3,1), (3,2); 4, (4,2), (4,3); 5, (5,3), (5,4); \\ & 6, (6,4), (6,5); 7, (7,5), (7,6). \end{array} \tag{21}$$

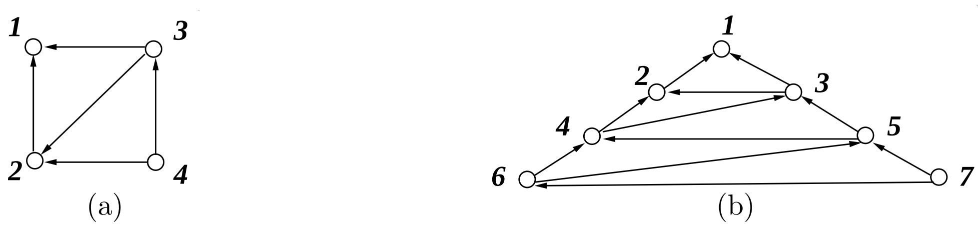

Figure 8: (a) A rigid graph $\mathcal{G}_a$ with $n = 4$ nodes and $n_e = 5$ edges representing a diamond formation and (b) A rigid graph $\mathcal{G}_b$ representing a V-formation of $n = 7$ vehicles with $n_e = 11$ edges.

In this section, we describe the process of control design for the augmented agent, called agent-k, to achieve structural stabilization of the obtained formation with the use of control inputs of agent-i and agent-j. The edge dynamics associated with the augmented edges e_{ki} and e_{kj} can be expressed as follows:

$$\begin{array}{l} e_{ki}: \left\{ \begin{array}{rcl} \dot{\eta}_{ki} & = & \nu_{ki} \\ \dot{\nu}_{ki} & = & \phi_{ki} + \langle \mathbf{n}_{ki}, u_i - u_k \rangle \end{array} \right. \\ e_{kj}: \left\{ \begin{array}{rcl} \dot{\eta}_{kj} & = & \nu_{kj} \\ \dot{\nu}_{kj} & = & \phi_{kj} + \langle \mathbf{n}_{kj}, u_j - u_k \rangle \end{array} \right. \end{array} \tag{22}$$

where

$$\phi_{ab} = \frac{\|p_b - p_a\|^2 - \nu_{ab}^2}{\|q_b - q_a\|} \tag{23}$$

for all indices $a, b \in \{i, j, k\}, a \neq b$.

Assumption 1. *Suppose initially dynamic agent-k is not collinear with agent-i and agent-j and in addition agent-k applies a control input in the following form:*

$$u_k = \gamma_i \mathbf{n}_{ki} + \gamma_j \mathbf{n}_{kj} \tag{24}$$

where $\gamma_i, \gamma_j \in \mathbb{R}$ are new controls for agent-k.

Define the following quantities:

$$\begin{array}{rcl} \lambda_i & = & \phi_{ki} + \langle \mathbf{n}_{ki}, u_i \rangle \\ \lambda_j & = & \phi_{kj} + \langle \mathbf{n}_{kj}, u_j \rangle \end{array} \tag{25}$$

Under Assumption 1, we obtain the following dynamics for the augmented edges:

$$\begin{array}{l} e_{ki}: \left\{ \begin{array}{rcl} \dot{\eta}_{ki} & = & \nu_{ki} \\ \dot{\nu}_{ki} & = & \lambda_i - \gamma_i - \langle \mathbf{n}_{ki}, \mathbf{n}_{kj} \rangle \gamma_j \end{array} \right. \\ e_{kj}: \left\{ \begin{array}{rcl} \dot{\eta}_{kj} & = & \nu_{kj} \\ \dot{\nu}_{kj} & = & \lambda_j - \langle \mathbf{n}_{kj}, \mathbf{n}_{ki} \rangle \gamma_i - \gamma_j \end{array} \right. \end{array} \tag{26}$$

Defining $\eta = (\eta_{ki}, \eta_{kj})^T$, $\nu = (\nu_{ki}, \nu_{kj})^T$, $\lambda = (\lambda_i, \lambda_j)^T$, $\gamma = (\gamma_i, \gamma_j)^T$, and

$$S = \begin{bmatrix} 1 & \langle \mathbf{n}_{ki}, \mathbf{n}_{kj} \rangle \\ \langle \mathbf{n}_{ki}, \mathbf{n}_{kj} \rangle & 1 \end{bmatrix} \tag{27}$$

The shape dynamics associated with the augmented edges in (26) can be rewritten as

$$\begin{array}{rcl} \dot{\eta} & = & \nu \\ \dot{\nu} & = & \lambda - S \cdot \gamma \end{array} \tag{28}$$

where $\eta, \nu, \lambda, \gamma \in \mathbb{R}^2$ and $S = S^T$ is an invertible matrix. The invertibility of S follows from Assumption 1 and the fact that $\det(S) = 1 - \langle \mathbf{n}_{ki}, \mathbf{n}_{kj} \rangle^2 > 0$ for three *non-collinear agents* i, j, and k.

Theorem 3. *(dynamic node augmentation) Suppose each agent in a group of n dynamic agents applies a control input that guarantees structural stabilization of a desired formation φ_d with an associated minimally rigid graph $\mathcal{G}$. Let agent-k be a new agent that is augmented to the exiting group of agents (represented by $\mathcal{G}$) by two new edges and call the augmented graph $\mathcal{G}_a$. Suppose agent-k satisfies Assumption 1, then applying the distributed control law $u_k = \gamma_i \mathbf{n}_{ki} + \gamma_j \mathbf{n}_{kj}$ by agent-k with*

$$\begin{bmatrix} \gamma_i \\ \gamma_j \end{bmatrix} = S^{-1}(c_p \eta + c_d \nu + \lambda), \quad c_p, c_d > 0 \tag{29}$$

achieves structural stabilization of the shape dynamics of the augmented graph $\mathcal{G}_a$.

Proof. The closed-loop shape dynamics of the augmented edges takes the form:

$$\begin{array}{rcl} \dot{\eta} & = & \nu \\ \dot{\nu} & = & -c_p\eta - c_d\nu \end{array} \tag{30}$$

and therefore $(\eta, \nu) = 0$ is (locally) exponentially stable. A collision-free region of attraction Ω for the shape dynamics of the augmented edges can be obtained in a similar way that is discussed in the proof of Theorem 2. □

For a general formation of n-agents with $n \geq 3$ that can be constructed using successive node augmentations satisfying the non-collinearity condition in Assumption 1, the *dynamic node augmentation procedure* can be summarized as follows. The first two agents are used to solve structural stabilization and navigational tracking problems for a formation of $n = 2$ agents. Then, each new agent solves the structural stabilization for the shape dynamics of the augmented edges. This procedure leads to a distributed control law with a sensing and communication pattern shown in Fig 9. Here, the source might or might not be an agent and it plays the role of a *task command center* for the formation of the vehicles. Apparently, the sensing pattern (flow) required to implement the controllers obtained using dynamic node augmentation procedure is uni-direction.

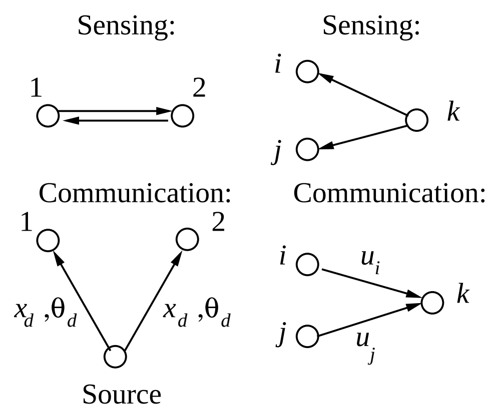

Figure 9: The information flow in dynamic node augmentation in terms of inter-agent directions of sensing and communication.

6 Conclusion

In this paper, we provided a theoretical framework that is a mix of graph theoretical and Lyapunov-based approaches to stability analysis and distributed formation control for dynamic multi-agent systems. The notion of graph rigidity and minimally rigid graph turned out to be crucial in identifying the shape variables of a formation. Graph rigidity allowed us to formally define formations of multiple dynamic agents and three types of stabilization/tracking problems for multi-agent systems. We stated a separation principle that allows addressing structural stability and navigational tracking problems independently for a formation of two agents. Then, we introduced a procedure called dynamic node augmentation that allowed construction of a larger formation with more agents that can be rendered structurally stable in a distributed manner given that the initial formation is structurally stable. We provided two examples of formations that can be controlled in a distributed fashion using this approach. Namely, the diamond formation and the V-formation. One of the main advantages of this framework is that it can be directly generalized to formation control in $\mathbb{R}^3$. Moreover, the sensing performed by each agent in dynamic node augmentation is uni-directional as supposed to bi-directional sensing as a result of using potential functions.

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