Feedback Stabilization of Uncertain Systems Using a Stochastic Digital Link (TAC 04-001)

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Abstract

We study the stabilizability of uncertain stochastic systems in the presence of finite capacity feedback. Motivated by the structure of communication networks, we consider a stochastic digital link that sends words whose size is governed by a random process. Such link is used to transmit state measurements between the plant and the controller. We derive necessary and sufficient conditions for internal and external stabilizability of the feedback loop. In accordance with previous publications, stabilizability of unstable plants is possible if and only if the link's average transmission rate is above a positive critical value. In our formulation the plant and the link can be stochastic. In addition, stability in the presence of uncertainty in the plant is analyzed using a small-gain argument. We show that the critical average transmission rate, for stabilizability, depends on the description of uncertainty and the statistical properties of the plant as well as the link.

I. Introduction

With a wide range of formulations, control in the presence of communication constraints has been the focus of intense research. The need to remotely control one or more systems from a central location, has stimulated the study of stabilizability of unstable plants when the information flow in the feedback loop is finite. Such limitation results from the use of an analog communication channel or a digital link as a way to transmit information about the state of the plant. It can also be viewed as an abstraction of computational constraints created by several systems sharing a common decision center.

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Various publications in this field have introduced necessary and sufficient conditions for the stabilizability of unstable plants in the presence of data-rate constraints. The construction of a stabilizing controller requires that the data-rate of the feedback loop is above a non-zero critical value [20], [21], [17], [18], [13]. Different notions of stability have been investigated, such as containability [23], [24], moment stability [18] and stability in the almost sure sense [20]. The last two are different when the state is a random variable. That happens when disturbances are random or if the communication link is stochastic. In [20] it is shown that the necessary and sufficient condition for almost sure stabilizability of finite dimensional linear and time-invariant systems is given by an inequality of the type $\mathcal{C} > \mathcal{R}$. The parameter $\mathcal{C}$ represents the average data-rate of the feedback loop and $\mathcal{R}$ is a quantity that depends on the eigenvalues of A, the dynamic matrix of the system. If a well defined channel is present in the feedback loop then $\mathcal{C}$ may be taken as the Shannon Capacity. If it is a digital link then $\mathcal{C}$ is the average transmission rate. Different notions of stability may lead to distinct requirements for stabilization. For tighter notions of stability, such as in the m-th moment sense, the knowledge of $\mathcal{C}$ may not suffice. More informative notions, such as higher moments or any-time capacity [18], are necessary. Results for the problem of state estimation in the presence of information constraints can be found in [23], [19] and [12].

A. *Main contributions of the paper*

In this paper we study the moment stabilizability of uncertain time-varying stochastic systems in the presence of a stochastic digital link. In contrast with [16], we consider systems whose time-variation is governed by an identically and independently distributed (i.i.d.) process which may be defined over a continuous and unbounded alphabet. We also provide complementary results to [16], [6], [10] because we consider a more general problem formulation where we consider external disturbances and uncertainty on the plant and a stochastic digital link.

Our work provides a unified framework for the necessary and sufficient conditions for robust stabilizability by establishing that the average transmission rate must satisfy $C > R + \alpha + \beta$, where $\alpha, \beta \geq 0$ are constants that quantify the influence of randomness in the link and the plant, respectively. As a consequence, C must be higher than R to compensate for randomness both in the plant and the digital link. The conclusion that $C > R$ does not suffice in the presence of a stochastic link was originally derived by [19]. We quantify such difference for stochastic digital links. The work of [19] was an important motivation for our work and the treatment of the nominal stabilization of first order linear systems, using a parameterized notion of capacity, denoted as anytime capacity, can be found there. If the plant and the link are deterministic, we get $\beta, \alpha = 0$ which is consistent with the condition $C > R$ derived by [20]. We also show that model uncertainty in the plant can be tolerated. By using an appropriate measure, we prove that an increase in C leads to higher tolerance to uncertainty. All of our conditions for stability are expressed as simple inequalities where the terms depend on the description of uncertainty in the plant as well as the statistics of the system and the digital link. A different approach to dealing with robustness, with respect to transmission rates, can be found in [11].

In order to focus on the fundamental issues and keep clarity, we start by deriving our results for first order linear systems. Subsequently, we provide an extension to a class of multi-state linear systems. As pointed out in [16], non-commutativity creates difficulties in the study of arbitrary time-varying stochastic systems. Results for the fully-observed Markovian case over finite alphabets, in the presence of a deterministic link, can be found in [16].

Besides the introduction, the paper has five sections and one Appendix: section II comprises the problem formulation and preliminary definitions; in section III we prove sufficiency conditions by constructing a stabilizing feedback scheme for first order systems; section IV contains the proof of the necessary condition for stability; some of the quanties, introduced in the paper, are given a detailed interpretation in section V and section VI extends the sufficient conditions to a class of multi-state linear systems. The Appendix motivates and discusses implementation issues associated with the communication link used in the paper.

The following notation is adopted:

- Whenever that is clear from the context we refer to a sequence of real numbers $x(k)$ simply as x. In such cases we may add that $x \in \mathbb{R}^\infty$.
- Random variables are represented using boldface letters, such as $\mathbf{w}$
- if $\mathbf{w}(k)$ is a stochastic process, then we use $w(k)$ to indicate a specific realization. According to the convention used for sequences, we may denote $\mathbf{w}(k)$ just as $\mathbf{w}$ and $w(k)$ as w.
- the expectation operator over $\mathbf{w}$ is written as $\mathcal{E}[\mathbf{w}]$
- if E is a probabilistic event, then its probability is indicated as $\mathcal{P}(E)$
- we write $\log_2(.)$ simply as $\log(.)$
- if $x \in \mathbb{R}^\infty$, then $\|x\|_1 = \sum_{i=0}^{\infty} |x(i)|$ and $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x(i)|$

Definition 1.1: Let $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$ be an upper-bound for the memory horizon of an operator. If $G_f : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is a causal operator then we define $\|G_f\|_{\infty(\varrho)}$ as:

$$\|G_f\|_{\infty(\varrho)} = \sup_{k \geq 0, x \neq 0} \frac{|G_f(x)(k)|}{\max_{j \in \{k-\varrho+1,\dots,k\}} |x(j)|} \tag{1}$$

Note that, since G_f is causal, $\|G_f\|_{\infty(\infty)}$ is the infinity induced norm $\|G_f\|_\infty = sup_{x \neq 0} \frac{\|G_f(x)\|_\infty}{\|x\|_\infty}$.

II. Problem Formulation

We study the stabilizability of uncertain stochastic systems under communication constraints. Motivated by the type of constraints that arise in most computer networks, we consider the following class of stochastic links:

Definition 2.1: (**Stochastic Link**) Consider a link that, at every instant k, transmits $\mathbf{r}(k)$ bits. We define it to be a stochastic link, provided that $\mathbf{r}(k) \in \{0, \dots, \bar{r}\}$ is an independent and identically distributed (i.i.d.) random process satisfying:

$$\mathbf{r}(k) = \mathcal{C} - \mathbf{r}^\delta(k) \tag{2}$$

where $\mathcal{E}[\mathbf{r}^\delta(k)] = 0$ and $C \geq 0$. The term $\mathbf{r}^\delta(k)$ represents a fluctuation in the transfer rate of the link. More specifically, the link is a stochastic truncation operator $\mathcal{F}_k^l : \{0,1\}^{\bar{r}} \to \bigcup_{i=0}^{\bar{r}} \{0,1\}^i$ defined as:

$$\mathcal{F}_k^l(b_1, \dots, b_{\bar{r}}) = (b_1, \dots, b_{\mathbf{r}(k)}), \ \ b_i \in \{0,1\} \tag{3}$$

Given $x(0) \in [-\frac{1}{2}, \frac{1}{2}]$ and $\bar{d} \geq 0$, we consider nominal systems of the form:

$$\mathbf{x}(k+1) = \mathbf{a}(k)\mathbf{x}(k) + \mathbf{u}(k) + \mathbf{d}(k) \tag{4}$$

with $|\mathbf{d}(k)| \leq \bar{d}$ and $\mathbf{x}(i) = 0$ for $i < 0$.

A. Description of Uncertainty in the Plant

Let $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, $\bar{z}_f \in [0,1)$ and $\bar{z}_a \in [0,1)$ be given constants, along with the stochastic process $\mathbf{z}_a$ and the operator $G_f : \mathbb{R}^\infty \to \mathbb{R}^\infty$ satisfying:

$$|\mathbf{z}_a(k)| \leq \bar{z}_a \tag{5}$$

$$G_f \text{ causal and } \|G_f\|_{\infty(\varrho)} \leq \bar{z}_f \tag{6}$$

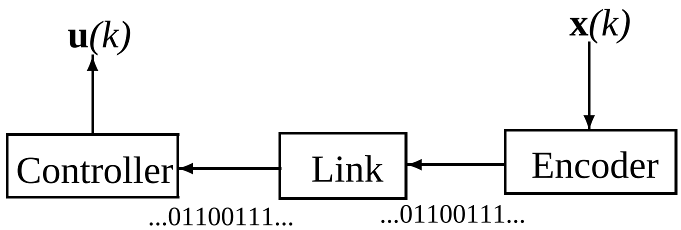

Fig. 1. Structure of the Feedback Interconnection

Given $x(0) \in [-\frac{1}{2}, \frac{1}{2}]$ and $\bar{d} \geq 0$, we study the existence of stabilizing feedback schemes for the following perturbed plant:

$$\mathbf{x}(k+1) = \mathbf{a}(k)\left(1 + \mathbf{z}_a(k)\right)\mathbf{x}(k) + \mathbf{u}(k) + G_f(\mathbf{x})(k) + \mathbf{d}(k) \tag{7}$$

where the perturbation processes $\mathbf{z}_a$ and $G_f(\mathbf{x})$ satisfy (5)-(6). Notice that $\mathbf{z}_a(k)$ may represent uncertainty in the knowledge of $\mathbf{a}(k)$, while $G_f(\mathbf{x})(k)$ may portray the output of a feedback uncertainty block G_f. We chose this structure because it allows the representation of a wide class of model uncertainty. It is also allows the construction of a suitable stabilizing scheme.

Example 2.1: If $G_f(\mathbf{x})(k) = \sum_{i=0}^{n-1} \mu_i \mathbf{x}(k-i)$ then $\|G_f\|_{\infty(\varrho)} = \begin{cases} \sum_{i=0}^{n-1} |\mu_i| & \text{if } \varrho \geq n \\ \infty & \text{otherwise} \end{cases}$.
In general, the operator G_f may be nonlinear and time-varying.

B. Statistical Description of $\mathbf{a}(k)$

The process $\mathbf{a}(k)$ is i.i.d. and independent of $\mathbf{r}(k)$, meaning that it carries no information about the link nor the initial state. In addition, for convenience, we use the same representation as in (2) and write:

$$\log(|\mathbf{a}(k)|) = \mathcal{R} + \mathbf{l}_a^\delta(k) \tag{8}$$

where $\mathcal{E}[\mathbf{l}_a^\delta(k)] = 0$. Notice that $\mathbf{l}_a^\delta(k)$ is responsible for the stochastic behavior, if any, of the plant. Since $\mathbf{a}(k)$ is ergodic, we also assume that $\mathcal{P}\left(a(k) = 0\right) = 0$, otherwise the system is trivially stable. Such assumption is also realistic if we assume that (7) comes from the discretization of a continuous-time system.

C. Functional Structure of the Feedback Interconnection

We assume that the feedback loop has the structure depicted in Fig 1, also referred to as the information pattern [25]. The blocks denoted as encoder and controller are stochastic operators whose domain and image are uniquely determined by the diagram. At any given time k, we assume that both the encoder and the controller have access to $a(0), \dots, a(k)$ and $r(k-1)$ as well as the constants ϱ, $\bar{z}_f$, $\bar{z}_a$ and $\bar{d}$. The encoder and the controller are described as:

- The encoder is a function $\mathcal{F}_k^e : \mathbb{R}^{k+1} \to \{0,1\}^{\bar{r}}$ that has the following dependence on observations:

$$\mathcal{F}_k^e(x(0), \mathbf{x}(1), \dots, \mathbf{x}(k)) = (\mathbf{b}_1, \dots, \mathbf{b}_{\bar{r}}) \tag{9}$$

- The control action results from a map, not necessarily memoryless, $\mathcal{F}_k^c : \bigcup_{i=0}^{\bar{r}} \{0,1\}^i \to \mathbb{R}$ exhibiting the following functional dependence:

$$\mathbf{u}(k) = \mathcal{F}_k^c(\vec{\mathbf{b}}(k)) \tag{10}$$

where $\vec{b}(k)$ are the bits successfully transmitted through the link, i.e.:

$$\vec{\mathbf{b}}(k) = \mathcal{F}_k^l\left(\mathbf{b}_1, \dots, \mathbf{b}_{\bar{r}}\right) = \left(\mathbf{b}_1, \dots, \mathbf{b}_{\mathbf{r}(k)}\right) \tag{11}$$

As such, $\mathbf{u}(k)$ can be equivalently expressed as $\mathbf{u}(k) = (\mathcal{F}_k^c \circ \mathcal{F}_k^l \circ \mathcal{F}_k^e)(x(0), \mathbf{x}(1), \dots, \mathbf{x}(k))$

Definition 2.2: (**Feedback Scheme**) We define a feedback scheme as the collection of a controller $\mathcal{F}_k^c$ and an encoder $\mathcal{F}_k^e$.

D. Problem Statement and M-th Moment Stability

Definition 2.3: (**Worst Case Envelope**) Let $\mathbf{x}(k)$ be the solution to (7) under a given feedback scheme. Given any realization of the random variables $\mathbf{r}(k)$, $\mathbf{a}(k)$, $G_f(\mathbf{x})(k)$, $\mathbf{z}_a(k)$ and $\mathbf{d}(k)$, the worst case envelope $\bar{\mathbf{x}}(k)$ is the random variable whose realization is defined by:

$$\bar{x}(k) = \sup_{x(0) \in [-\frac{1}{2}, \frac{1}{2}]} |x(k)| \tag{12}$$

Consequently, $\bar{\mathbf{x}}(k)$ is the smallest envelope that contains every trajectory generated by an initial condition in the interval $x(0) \in [-\frac{1}{2}, \frac{1}{2}]$. We adopted the interval $[-\frac{1}{2}, \frac{1}{2}]$ to make the paper more readable. All results are valid if it is replaced by any other symmetric bounded interval.

Our problem is to determine necessary and sufficient conditions that guarantee the existence of a stabilizing feedback scheme. The results are derived for the following notion of stability.

Definition 2.4: **(m-th Moment Robust Stability)** Let $m > 0$, $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, $\bar{z}_f \in [0, 1)$, $\bar{z}_a \in [0, 1)$ and $\bar{d} \geq 0$ be given. The system (7), under a given feedback scheme, is m-th moment (robustly) stable provided that the following holds:

$$\begin{cases} \lim_{k\to\infty} \mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right] = 0 & \text{if } \bar{z}_f = \bar{d} = 0 \\ \exists b \in \mathbb{R}_+ \text{ s.t. } \limsup_{k\to\infty} \mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right] < b & \text{otherwise} \end{cases} \tag{13}$$

The first limit in (13) is an internal stability condition while the second is an external stability condition. The constant b must be such that $\limsup_{k\to\infty} \mathcal{E}\left[\bar{\mathbf{x}}(k)^m\right] < b$ holds for all allowable perturbations $\mathbf{z}_a$ and $G_f(\mathbf{x})$ satisfying (5)-(6).

E. Digression over the main results and Conclusions

The main results of the paper are the sufficiency theorems 3.2 and 3.4 proven in section III as well as the extension to the multi-state case given in section VI. The sufficiency conditions are proven constructively by means of the stabilizing feedback scheme of definition 3.3. The necessary and sufficient conditions can be expressed as inequalities involving $\mathcal{C}$ and $\mathcal{R}$ plus a few auxiliary quantities that depend on the statistical behavior of the plant and the link as well as the descriptions of uncertainty. The intuition behind, the auxiliary quantities, is given in section V. In order to preserve stability, the presence of randomness must be offset by an increase of the average transmission rate $\mathcal{C}$. In addition, we find that the higher $\mathcal{C}$ the larger the tolerance to uncertainty in the plant. Our results extend to a class of multi-state systems.

III. Sufficiency Conditions for the Robust Stabilization of First Order Linear Systems

In this section, we derive constructive sufficient conditions for the existence of a stabilizing feedback scheme. We start with the deterministic case in subsection III-A, while III-B deals with random $\mathbf{r}$ and $\mathbf{a}$. We stress that our proofs hold under the framework of section II. The strength of our assumptions can be accessed from the discussion in section . The following definition introduces the main idea behind the construction of a stabilizing feedback scheme.

Definition 3.1: **(Upper-bound Sequence)** Let $\bar{z}_f \in [0, 1)$, $\bar{z}_a \in [0, 1)$, $\bar{d} \geq 0$ and $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$ be given. Define the upper-bound sequence as:

$$\mathbf{v}(k+1) = |\mathbf{a}(k)| 2^{-\mathbf{r}_e(k)} \mathbf{v}(k) + \bar{z}_f \max\{\mathbf{v}(k-\varrho+1), \dots, \mathbf{v}(k)\} + \bar{d}, \tag{14}$$

where $v(i) = 0$ for $i < 0$, $v(0) = \frac{1}{2}$ and $\mathbf{r}_e(k)$ is an effective rate given by:

$$\mathbf{r}_e(k) = -\log(2^{-\mathbf{r}(k)} + \bar{z}_a) \tag{15}$$

Definition 3.2: Following the definition of $\mathbf{r}(k)$ we also define C_e and $\mathbf{r}_e^\delta(k)$ such that:

$$\mathbf{r}_e(k) = C_e - \mathbf{r}_e^\delta(k) \tag{16}$$

where $\mathcal{E}[\mathbf{r}_e^\delta(k)] = 0$.

We adopt $v(0) = \frac{1}{2}$ to guarantee that $|x(0)| \leq v(0)$. If $x(0) = 0$ then we can select $v(0) = 0$. Notice that the multiplicative uncertainty $\bar{z}_a$ acts by reducing the effective rate $\mathbf{r}_e(k)$. After inspecting (15), we find that $\mathbf{r}_e(k) \leq \min\{\mathbf{r}(k), -\log(\bar{z}_a)\}$. Also, notice that:

$$\bar{z}_a = 0 \implies \mathbf{r}_e(k) = \mathbf{r}(k), \mathbf{r}_e^\delta(k) = \mathbf{r}^\delta(k) \text{ and } C = C_e \tag{17}$$

We make use of the sequence specified in definition 3.1. Notice that $\mathbf{v}(k)$ can be constructed at the controller and the encoder because both have access to ϱ, $\bar{z}_f$, $\bar{z}_a$, $\bar{d}$, $\mathbf{r}(k-1)$ and $\mathbf{a}(k-1)$.

Definition 3.3: (**Stabilizing feedback scheme**) The feedback scheme is defined as:

- **Encoder:** Measures $x(k)$ and computes $b_i \in \{0, 1\}$ such that:

$$(b_1, \dots, b_{\bar{r}}) = arg \max_{\sum_{i=1}^{\bar{r}} b_i \frac{1}{2^i} \leq \left(\frac{\mathbf{x}(k)}{2\mathbf{v}(k)} + \frac{1}{2}\right)} \sum_{i=1}^{\bar{r}} b_i \frac{1}{2^i} \tag{18}$$

  Place $(b_1, \dots, b_{\bar{r}})$ for transmission. For any $r(k) \in \{0, \dots, \bar{r}\}$, the above construction provides the following centroid approximation $\hat{x}(k)$ for $x(k) \in [-v(k), v(k)]$:

$$\hat{x}(k) = 2v(k)(\sum_{i=1}^{r(k)} b_i \frac{1}{2^i} + \frac{1}{2^{r(k)+1}} - \frac{1}{2}) \tag{19}$$

  which satisfies $|x(k) - \hat{x}(k)| \leq 2^{-r(k)} v(k)$.

- **Controller:** From the $\bar{r}$ bits placed for transmission in the stochastic link, only $\mathbf{r}(k)$ bits go through. Compute $\mathbf{u}(k)$ as:

$$\mathbf{u}(k) = -\mathbf{a}(k)\hat{\mathbf{x}}(k) \tag{20}$$

As expected, the transmission of state information through a finite capacity medium requires quantization. The encoding scheme of definition 3.3 is not an exception and is structurally identical to the ones used by [2], [20], where sequences were already used to upper-bound the state of the plant.

The following lemma suggests that, in the construction of stabilizing controllers, we may choose to focus on the dynamics of the sequence $v(k)$. That simplifies the analysis in the presence of uncertainty because the dynamics of $v(k)$ is described by a first-order difference equation.

Lemma 3.1: Let $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given. If $\mathbf{x}(k)$ is the solution of (7) under the feedback scheme of definition 3.3, then the following holds:

$$\bar{\mathbf{x}}(k) \leq \mathbf{v}(k)$$

for all $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, every choice $G_f \in \Delta_{f,\varrho}$ and $|\mathbf{z_a}(k)| \leq \bar{z}_a$, where

$$\Delta_{f,\varrho} = \{G_f : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty : \|G_f\|_{\infty(\varrho)} \leq \bar{z}_f\} \tag{21}$$

Proof: We proceed by induction, assuming that $\bar{x}(i) \leq v(i)$ for $i \in \{0,\dots,k\}$ and proving that $\bar{x}(k+1) \leq v(k+1)$. From (7), we get:

$$|x(k+1)| \leq |a(k)||x(k) + \frac{u(k)}{a(k)}| + |z_a(k)||a(k)||x(k)| + |G_f(x)(k)| + |d(k)| \tag{22}$$

The way the encoder constructs the binary expansion of the state, as well as (20), allow us to conclude that $|x(k) + \frac{u(k)}{a(k)}| \leq 2^{-r(k)} v(k)$. Now we recall that $|z_a(k)| \leq \bar{z}_a$, $|G_f(x)(k)| \leq \bar{z}_f \max\{v(k-\varrho+1),\dots,v(k)\}$ and that $|d(k)| \leq \bar{d}$, so that (22) implies:

$$|x(k+1)| \leq |a(k)|(2^{-r(k)} + \bar{z}_a)v(k) + \bar{z}_f \max\{v(k-\varrho+1),\dots,\mathbf{v}(k)\} + \bar{d} \tag{23}$$

The proof is concluded once we realize that $|x(0)| \leq v(0)$. $\square$

A. The Deterministic Case

We start by deriving a sufficient condition for the existence of a stabilizing feedback scheme in the deterministic case, i.e., $\mathbf{r}(k) = \mathcal{C}$ and $\log(|\mathbf{a}(k)|) = \mathcal{R}$. Subsequently, we move for the stochastic case where we derive a sufficient condition for stabilizability.

Theorem 3.2: (**Sufficiency conditions for Robust Stability**) Let $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given and $h(k)$ be defined as

$$h(k) = 2^{k(\mathcal{R} - \mathcal{C}_e)},\ k \geq 0$$

where $\mathcal{C}_e = r_e = -\log(2^{-\mathcal{C}} + \bar{z}_a)$.

Consider that $\mathbf{x}(k)$ is the solution of (7) under the feedback scheme of definition 3.3 as well as the following conditions:

- **(C 1)** $\mathcal{C}_e > \mathcal{R}$

- **(C 2)** $\bar{z}_f \|h\|_1 < 1$

If conditions **(C 1)** and **(C 2)** are satisfied then the following holds for all $|\mathbf{d}(t)| \leq \bar{d}$, $G_f \in \Delta_{f,\varrho}$ and $|\mathbf{z}_a(k)| \leq \bar{z}_a$:

$$\bar{x}(k) \leq \|h\|_1 \left(\bar{z}_f \frac{\|h\|_1 \bar{d} + \frac{1}{2}}{1 - \|h\|_1 \bar{z}_f} + \bar{d} \right) + h(k) \frac{1}{2} \tag{24}$$

where $\Delta_{f,\varrho}$ is given by $\Delta_{f,\varrho} = \{G_f : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty : \|G_f\|_{\infty(\varrho)} \leq \bar{z}_f\}$.

Proof: From definition 3.1, we know that, for arbitrary $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, the following is true:

$$v(k+1) = 2^{\mathcal{R} - \mathcal{C}_e} v(k) + \bar{z}_f \max\{v(k - \varrho + 1), \ldots, v(k)\} + \bar{d} \tag{25}$$

Solving the difference equation gives:

$$v(k) = 2^{k(\mathcal{R} - \mathcal{C}_e)} v(0) + \sum_{i=0}^{k-1} 2^{(k-i-1)(\mathcal{R} - \mathcal{C}_e)} \left(\bar{z}_f \max\{v(i - \varrho + 1), \ldots, v(i)\} + \bar{d} \right), \ k \geq 1 \tag{26}$$

which, using $\|\Pi_k v\|_\infty = \max\{v(0), \ldots, v(k)\}$, leads to:

$$v(k) \leq \|h\|_1 (\bar{z}_f \|\Pi_k v\|_\infty + \bar{d}) + 2^{k(\mathcal{R} - \mathcal{C}_e)} v(0) \tag{27}$$

But we also know that $2^{k(\mathcal{R} - \mathcal{C}_e)}$ is a decreasing function of k, so that:

$$\|\Pi_k v\|_\infty \leq \|h\|_1 (\bar{z}_f \|\Pi_k v\|_\infty + \bar{d}) + v(0) \tag{28}$$

which implies:

$$\|\Pi_k v\|_\infty \leq \frac{\|h\|_1 \bar{d} + v(0)}{1 - \|h\|_1 \bar{z}_f} \tag{29}$$

Direct substitution of (29) in (27) leads to $v(k) \leq \|h\|_1 \left(\bar{z}_f \frac{\|h\|_1 \bar{d} + v(0)}{1 - \|h\|_1 \bar{z}_f} + \bar{d} \right) + 2^{k(\mathcal{R} - \mathcal{C}_e)} v(0)$. The proof is complete once we make $v(0) = \frac{1}{2}$ and use lemma 3.1 to conclude that $\bar{x}(k) \leq v(k)$. $\square$

B. Sufficient Condition for the Stochastic Case

The following lemma provides a sequence, denoted by $v_m(k)$, which is an upper-bound for the m-th moment of $\bar{\mathbf{x}}(k)$. We show that v_m is propagated according to a first-order difference equation that is suitable for the analysis in the presence of uncertainty.

Lemma 3.3: (**M-th moment boundedness**) Let $\varrho \in \mathbb{N}_+$, $\bar{z}_f \in [0, 1)$, $\bar{z}_a \in [0, 1)$ and $\bar{d} \geq 0$ be given along with the following set:

$$\Delta_{f,\varrho} = \{G_f : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty : \|G_f\|_{\infty(\varrho)} \leq \bar{z}_f\} \tag{30}$$

Given m, consider the following sequence:

$$v_m(k) = h_m(k)v_m(0) + \sum_{i=0}^{k-1} h_m(k-i-1)\left(\varrho^{\frac{1}{m}}\bar{z}_f \max\{v_m(i-\varrho+1),\dots,v_m(i)\} + \bar{d}\right),\ k \geq 1 \tag{31}$$

where $v_m(i) = 0$ for $i < 0$, $v_m(0) = \frac{1}{2}$, $h_m(k)$ is the impulse response given by:

$$h_m(k) = \left(\mathcal{E}[2^{m(\log(|a(k)|)-r_e(k))}]\right)^{\frac{k}{m}},\ k \geq 0 \tag{32}$$

and $r_e(k) = -\log\left(2^{-r(k)} + \bar{z}_a\right)$. If $\mathbf{x}(k)$ is the solution of (7) under the feedback scheme of definition 3.3, then the following holds

$$\mathcal{E}[\bar{\mathbf{x}}(k)^m] \leq v_m(k)^m$$

for all $|\mathbf{d}(t)| \leq \bar{d}$, $G_f \in \Delta_{f,\varrho}$ and $|\mathbf{z}_a(k)| \leq \bar{z}_a$.

Proof: Since lemma 3.1 guarantees that $\bar{x}(k+1) \leq v(k+1)$, we only need to show that $\mathcal{E}[\mathbf{v}(k+1)^m]^{\frac{1}{m}} \leq v_m(k+1)$. Again, we proceed by induction by noticing that $v(0) = v_m(0)$ and by assuming that $\mathcal{E}[\mathbf{v}(i)^m]^{\frac{1}{m}} \leq v_m(i)$ for $i \in \{1,\dots,k\}$. The induction hypothesis is proven once we establish that $\mathcal{E}[\mathbf{v}(k+1)^m]^{\frac{1}{m}} \leq v_m(k+1)$. From definition 3.1, we know that:

$$\mathcal{E}[\mathbf{v}(k+1)^m]^{\frac{1}{m}} = \mathcal{E}[\left(2^{\log(|\mathbf{a}(k)|)-\mathbf{r}_e(k)}\mathbf{v}(k) + \bar{z}_f \max\{\mathbf{v}(k-\varrho+1),\dots,\mathbf{v}(k)\} + \bar{d}\right)^m]^{\frac{1}{m}} \tag{33}$$

Using Minkowsky's inequality [7] as well as the fact that $\mathbf{v}(i)$ is independent of $\mathbf{a}(j)$ and $\mathbf{r}_e(j)$ for $j \geq i$, we get:

$$\mathcal{E}[\mathbf{v}(k+1)^m]^{\frac{1}{m}} \leq \mathcal{E}[2^{m(\log(|\mathbf{a}(k)|)-\mathbf{r}_e(k))}]^{\frac{1}{m}}\mathcal{E}[\mathbf{v}(k)^m]^{\frac{1}{m}} + \bar{z}_f\mathcal{E}[\max\{\mathbf{v}(k-\varrho+1),\dots,\mathbf{v}(k)\}^m]^{\frac{1}{m}} + \bar{d} \tag{34}$$

which, using the inductive assumption, implies the following inequality:

$$\mathcal{E}[\mathbf{v}(k+1)^m]^{\frac{1}{m}} \leq \mathcal{E}[2^{m(\log(|\mathbf{a}(k)|)-\mathbf{r}_e(k))}]^{\frac{1}{m}}v_m(k) + \varrho^{\frac{1}{m}}\bar{z}_f \max\{v_m(k-\varrho+1),\dots,v_m(k)\} + \bar{d} \tag{35}$$

where we used the fact that, for arbitrary random variables $\mathbf{s}_1,\dots,\mathbf{s}_n$, the following holds:

$$\mathcal{E}[\max\{|\mathbf{s}_1|,\dots,|\mathbf{s}_n|\}^m] \leq \mathcal{E}[\sum_{i=1}^{n}|\mathbf{s}_i|^m] \leq n\max\{\mathcal{E}[|\mathbf{s}_1|^m],\dots,\mathcal{E}[|\mathbf{s}_n|^m]\}$$

The proof follows once we notice that the right hand side of (35) is just $v_m(k+1)$. □

Theorem 3.4: (**Sufficient Condition**) Let m , $\varrho \in \mathbb{N}_+$, $\bar{z}_f \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given along with the quantities bellow:

$$\beta(m) = \frac{1}{m}\log\mathcal{E}\left[2^{m\mathbf{l}_a^{\delta}(k)}\right]$$

$$\alpha_e(m) = \frac{1}{m}\log \mathcal{E}\left[2^{m\mathbf{r}_e^\delta(k)}\right]$$
$$h_m(k) = 2^{k(\mathcal{R}+\beta(m)+\alpha_e(m)-\bar{\mathcal{C}}_e)},\ k \geq 0$$

where $\mathbf{r}_e^\delta$ comes from (16). Consider that $\mathbf{x}(k)$ is the solution of (7) under the feedback scheme of definition 3.3 as well as the following conditions:

- **(C 3)** $C_e > \mathcal{R} + \beta(m) + \alpha_e(m)$
- **(C 4)** $\varrho^{\frac{1}{m}}\bar{z}_f\|h_m\|_1 < 1$

If conditions **(C 3)** and **(C 4)** are satisfied, then the following holds for all $|\mathbf{d}(t)| \leq \bar{d}$, $G_f \in \Delta_{f,\varrho}$ and $|\mathbf{z}_a(k)| \leq \bar{z}_a$:

$$\mathcal{E}[\bar{\mathbf{x}}(k)^m]^{\frac{1}{m}} \leq \|h_m\|_1\left(\varrho^{\frac{1}{m}}\bar{z}_f\frac{\|h_m\|_1\bar{d}+\frac{1}{2}}{1-\varrho^{\frac{1}{m}}\bar{z}_f\|h_m\|_1}+\bar{d}\right)+h_m(k)\frac{1}{2} \tag{36}$$

where $\Delta_{f,\varrho} = \{G_f : \mathbb{R}^\infty \to \mathbb{R}^\infty : \|G_f\|_{\infty(\varrho)} \leq \bar{z}_f\}$.

Proof: Using v_m from lemma 3.3, we arrive at:

$$v_m(k) \leq h_m(k)v_m(0) + \|h_m\|_1\left(\varrho^{\frac{1}{m}}\bar{z}_f\|\Pi_k v_m\|_\infty + \bar{d}\right) \tag{37}$$

where we use $\|\Pi_k v_m\|_\infty = \max\{v_m(0),\dots,v_m(k)\}$. But from (37), we conclude that:

$$\|\Pi_k v_m\|_\infty \leq v_m(0) + \|h_m\|_1\left(\varrho^{\frac{1}{m}}\bar{z}_f\|\Pi_k v_m\|_\infty + \bar{d}\right) \tag{38}$$

or equivalently:

$$\|\Pi_k v_m\|_\infty \leq \frac{v_m(0)+\|h_m\|_1\bar{d}}{1-\|h_m\|_1\varrho^{\frac{1}{m}}\bar{z}_f} \tag{39}$$

Substituting (39) in (37), gives:

$$v_m(k) \leq h_m(k)v_m(0) + \|h_m\|_1\left(\varrho^{\frac{1}{m}}\bar{z}_f\frac{v_m(0)+\|h_m\|_1\bar{d}}{1-\|h_m\|_1\varrho^{\frac{1}{m}}\bar{z}_f}+\bar{d}\right) \tag{40}$$

The proof follows from lemma 3.3 and by noticing that $h_m(k)$ can be rewritten as:

$$h_m(k) = \left(\mathcal{E}[2^{m(\log(|\mathbf{a}(k)|)-\mathbf{r}_e(k))}\right)^{\frac{k}{m}} = 2^{k(\mathcal{R}+\beta(m)+\alpha_e(m)-C_e)},\ k \geq 0$$

□

IV. NECESSARY CONDITIONS FOR THE EXISTENCE OF STABILIZING FEEDBACK SCHEMES

Consider that $\bar{z}_a = \bar{z}_f = \bar{d} = 0$. We derive necessary conditions for the existence of an internally stabilizing feedback scheme. We emphasize that the proofs in this section use the m-th moment stability as a stability criteria and that they are valid regardless of the encoding/decoding scheme. They follow from a counting argument[1] which is identical to the one used by [21]. Necessary conditions for stability were also studied for the Gaussian channel in [22] and for other stochastic channels in [18], [19]. A necessary condition for the almost sure stability of general stochastic channels is given by [20]. We include our treatment, because it provides necessary conditions for m-th moment stability, which are inequalities involving directly the defined quantities $\alpha(m)$ and $\beta(m)$. Such quantities are an important aid on the derivation of the conclusions presented in section V. In section VI-H, we show that the necessary condition of Theorem 4.1 is not conservative.

We derive the necessary condition for the following class of state-space representations:

$$\mathbf{x}(k) = \mathbf{U}(k)\mathbf{x}(k) + B\mathbf{u}(k) \tag{41}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{n_b}$, $B \in \mathbb{R}^{n \times n_b}$ and $\mathbf{U}(k)$ is a block upper-triangular matrix of the form:

$$\mathbf{U}(k) = \begin{bmatrix} \mathbf{a}(k)\mathbf{Rot}(k) & \cdots & \cdots & \\ 0 & \mathbf{a}(k)\mathbf{Rot}(k) & \ddots & \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & \mathbf{a}(k)\mathbf{Rot}(k) \end{bmatrix} \tag{42}$$

and $\mathbf{Rot}$ is a sequence of random rotation matrices satisfying $det(Rot(k)) = 1$. We also assume that $\mathbf{Rot}$ is independent of $\mathbf{r}$.

Theorem 4.1: Let $\mathbf{x}(k)$ be the solution of the state-space equation (41) along with $\alpha(m)$ and $\beta(m)$ given by:

$$\alpha(m) = \frac{1}{m}\log(E[2^{m\mathbf{r}^{\delta}(k)}]) \tag{43}$$

$$\beta(m) = \frac{1}{m}\log(E[2^{m\mathbf{l}_a^{\delta}(k)}]) \tag{44}$$

[1]We also emphasize that this proof is different from what we had originally. The present argument was suggested by a reviewer of one of our publications

and the norm on the vector $x(k)$ be represented as:

$$\|x(k)\|_\infty = \max_i |[x(k)]_i|, \tag{45}$$

where $[x(k)]_i$ are components of the vectors $x(k)$.

If the state satisfies the following:

$$\sup_k E[\sup_{x(0)\in[-1/2,1/2]^n} \|\mathbf{x}(k)\|_\infty^m] < \infty \tag{46}$$

then the following must hold:

$$C - \alpha(\frac{m}{n}) > n\beta(m) + nR \tag{47}$$

Proof: Consider a specific realization of $\mathbf{Rot}$, $\mathbf{r}$ and $\mathbf{a}$ along with the following sets:

$$\bar{\Omega}_k = \{\Pi_{i=0}^{k-1} U(i)x(0) : x(0) \in [-1/2, 1/2]^n\} \tag{48}$$

$$\Omega_k(\{u(i)\}_{i=0}^{k-1}) = \{x(k) : x(0) \in [-1/2, 1/2]^n, \{u(i)\}_{i=0}^{k-1} = \mathcal{F}(x(0), k)\} \tag{49}$$

where $u(k)$ is obtained through a fixed feedback law $\mathcal{F}(x(0), k)$. Since $x(k)$ is given by (41) and $\{u(i)\}_{i=0}^{k-1}$ can take, at most, $2^{\sum_{i=0}^{k-1} r(i)}$ values, we find that:

$$\frac{Vol(\bar{\Omega}_k)}{\max_{\{u(i)\}_{i=0}^{k-1}} Vol(\Omega_k(\{u(i)\}_{i=0}^{k-1}))} \leq 2^{\sum_{i=0}^{k-1} r(i)} \tag{50}$$

Computing bounds for the volumes, we get:

$$Vol(\bar{\Omega}_k) = 2^{\sum_{i=0}^{k-1} \log |det(U(i))|} \tag{51}$$

$$Vol(\Omega_k(\{u(i)\}_{i=0}^{k-1})) \leq v^n(k) \tag{52}$$

where $v(k) = 2\sup_{x(0)\in[-1/2,1/2]^n} \|x(k)\|_\infty$. Consequently, using (50) we infer that:

$$2^{\sum_{i=0}^{k-1} \log |det(U(i))|} 2^{-\sum_{i=0}^{k-1} r(i)} \leq 2^n v^n(k) \tag{53}$$

By taking expectations, the m-th moment stability assumption leads to:

$$\limsup_{k\to\infty} (\mathcal{E}[2^{\frac{m}{n}\log|det(\mathbf{U}(k))|}]\mathcal{E}[2^{-\frac{m}{n}\mathbf{r}(k)}])^k \leq 2^m \limsup_{k\to\infty} \mathcal{E}[\mathbf{v}^m(k)] < \infty \tag{54}$$

which implies that:

$$C > \alpha(\frac{m}{n}) + n\beta(m) + nR \tag{55}$$

where we used the fact that $\mathcal{E}[2^{\frac{m}{n}\log|det(\mathbf{U}(k))|}]\mathcal{E}[2^{-\frac{m}{n}\mathbf{r}(k)}] < 1$ must hold and that $\log|det(\mathbf{U}(k))| = n\log|\mathbf{a}(k)|$. $\square$

Corollary 4.2: Let $\mathbf{x}(k)$ be the solution of the following linear and time-invariant system:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) \tag{56}$$

If the state satisfies the following:

$$\sup_k E[\sup_{x(0)\in[-1/2,1/2]^n} \|\mathbf{x}(k)\|_\infty^m] < \infty \tag{57}$$

then the following must hold:

$$C - \alpha(\frac{m}{n_{unstable}}) > \sum_{i=1}^{n} \max\{\log|\lambda_i(A)|, 0\} \tag{58}$$

where $n_{unstable}$ is the number of unstable eigenvalues.

Proof: The proof is a direct adaptation of the proof of Theorem 4.1. □

V. PROPERTIES OF THE MEASURES $\alpha(m)$ AND $\beta(m)$

Consider that $\mathbf{a}$ and $\mathbf{r}$ are stochastic processes, that there is no uncertainty in the plant and no external disturbances, i.e., $\bar{z}_f = \bar{z}_a = \bar{d} = 0$. In such situation, (7) can be written as:

$$\mathbf{x}(k+1) = \mathbf{a}(k)\mathbf{x}(k) + \mathbf{u}(k) \tag{59}$$

For a given m, the stability condition of definition 2.4 becomes:

$$lim_{k\to\infty}\mathcal{E}[\bar{\mathbf{x}}(k)^m] = 0 \tag{60}$$

If $\mathbf{v}$ is a real random variable then Jensen's inequality [4] implies:

$$\mathcal{E}[2^{\mathbf{v}}] \geq 2^{\mathcal{E}[\mathbf{v}]}$$

where equality is attained if and only if $\mathbf{v}$ is a deterministic constant. As such, $\log(\mathcal{E}[2^{\mathbf{v}}]2^{-\mathcal{E}[\mathbf{v}]}) \geq 0$ can be used as a measure of "randomness" which can be taken as an alternative to variance. Notice that such quantity may be more informative than variance because it depends on higher moments of $\mathbf{v}$. We use this concept to interpret our results and express our conditions in a way that is amenable to a direct comparison with other publications. Along these lines, the following are randomness measures for $\log(|\mathbf{a}(k)|)$ and $\mathbf{r}(k)$:

$$\beta(m) = \frac{1}{m}\log(\mathcal{E}[2^{m\mathbf{l}_a^\delta(k)}]) \tag{61}$$

$$\alpha(m) = \frac{1}{m}\log(\mathcal{E}[2^{m\mathbf{r}^{\delta}(k)}]) \tag{62}$$

where $\mathbf{l}_a^{\delta}(k)$ and $\mathbf{r}^{\delta}(k)$ are given by (repeated here for convenience):

$$\log(|\mathbf{a}(k)|) = \mathcal{E}[\log(|\mathbf{a}(k)|)] + \mathbf{l}_a^{\delta}(k) = R + \mathbf{l}_a^{\delta}(k) \tag{63}$$

$$\mathbf{r}(k) = \mathcal{E}[\mathbf{r}(k)] - \mathbf{r}^{\delta}(k) = C - \mathbf{r}^{\delta}(k) \tag{64}$$

The following equivalence is a direct consequence of the necessary and sufficient conditions proved in theorems 4.1 and 3.4:

$$\exists \text{ feedback scheme s.t. } \lim_{k\to\infty}\mathcal{E}[\bar{\mathbf{x}}(k)^m] = 0 \Longleftrightarrow \mathcal{C} > \mathcal{R} + \alpha(m) + \beta(m) \tag{65}$$

After examining (65), we infer that $\alpha(m)$ and $\beta(m)$ encompass the influence of m on the stability condition, while C and R are independent of m. The condition (65) suggests that $\alpha(m)$ is the right intuitive measure of quality, of a stochastic link, for the class considered in this paper. The following are properties of $\alpha(m)$ and $\beta(m)$:

- note that Jensen's inequality implies that $\alpha(m) \geq 0$ and $\beta(m) \geq 0$, where equality is achieved only if the corresponding random variable is deterministic. Accordingly, (65) shows that randomness in $\mathbf{r}(k)$ implies that $\mathcal{C} > R + \alpha(m)$ is necessary for stabilization. The fact that randomness in the channel creates the need for capacity larger than R, was already established, but quantified differently, in [18]. In addition, we find that randomness in the system adds yet another factor $\beta(m)$.
- by means of a Taylor expansion and taking limits, we get

$$\lim_{m\searrow 0}\alpha(m) = \lim_{m\searrow 0}\beta(m) = 0 \tag{66}$$

  Under the above limit, the necessary and sufficient condition (65) becomes $C > R$. That is the weakest condition of stability and coincides with the one derived by [20] for almost sure stability. By means of (65) and (66) we can also conclude that if $C > R$, i.e. the feedback scheme is almost surely stabilizable [20], then it is m-th moment stabilizable for some $m > 0$.
- the opposite limiting case, gives

$$\lim_{m\to\infty}\alpha(m) = C - r_{min} \tag{67}$$

$$\lim_{m\to\infty} \beta(m) = \log(a^{sup}) - R \tag{68}$$

where

$$r_{min} = \min\{r \in \{0, \dots, \bar{r}\} : \mathcal{P}(\mathbf{r}(k) = r) \neq 0\}$$

$$a^{sup} = \sup\{\tilde{a} : \mathcal{P}(|\mathbf{a}(k)| \geq \tilde{a}) \neq 0\}$$

- $\alpha(m)$ and $\beta(m)$ are non-decreasing functions of m

From the previous properties of $\alpha(m)$ and $\beta(m)$ we find that

- a feedback scheme is stabilizing for all moments, i.e., $\forall m, \sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] < \infty$ if and only if $r_{min} > \log(a^{sup})$.
- if $r_{min} = 0$ then there exists m_0 such that $\forall m \geq m_0, \sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] = \infty$. This is the case of the erasure channel suggested by [18]. This conclusion was already reported in [18] (see Example 5.2).
- similarly, if $\log(a^{sup}) = \infty$ then there exists m_0 such that $\forall m \geq m_0, \sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] = \infty$ (see example 5.1). Notice that a^{sup} can be larger than one and still $\mathcal{E}[\mathbf{a}(k)^m] < 1$ for some m. Even more, in example 5.1, we have $a^{sup} = \infty$ and $\mathcal{E}[|\mathbf{a}(k)|^m] < \infty$ for all m.

Example 5.1: Consider that $|\mathbf{a}|$ is log-normally distributed, i.e., $\log|\mathbf{a}(k)|$ is normally distributed. An example where $\mathbf{a}(k)$ is log-normally distributed is given by [3]. If $Var(|\mathbf{a}(k)|)$ is the variance of $|\mathbf{a}(k)|$ then $\beta(m)$ is given by:

$$\beta(m) = \frac{m}{2} \log\left(1 + \frac{Var(|\mathbf{a}(k)|)}{\left(\mathcal{E}\left[|\mathbf{a}(k)|\right]\right)^2}\right) \tag{69}$$

where the expression is obtained by direct integration. Note that $\beta(m)$ grows linearly with m. It illustrates a situation where, given $Var(|\mathbf{a}(k)|) > 0$, $\mathcal{C}$ and $\alpha(m)$, there always exist m large enough such that the necessary and sufficient condition $\mathcal{C} > \mathcal{R} + \beta(m) + \alpha(m)$ is violated.

The above analysis stresses the fact that feedback, using a stochastic link, acts by increasing m^{max} for which $\forall m \leq m^{max}, \sup_k \mathcal{E}[\bar{\mathbf{x}}(k)^m] < \infty$ is satisfied. In some cases one may get $m^{max} = \infty$.

1) The Exponential Statistic: Directly from (61) and (62), we derive the equivalence below:

$$\mathcal{C} > \mathcal{R} + \alpha(m) + \beta(m) \iff \mathcal{E}[|\mathbf{a}(k)|^m]\mathcal{E}[2^{-m\mathbf{r}(k)}] < 1 \tag{70}$$

The equivalences expressed in (65) and (70) show that all the information we need to know about the link is $\alpha(m)$ and $\mathcal{C}$ or, equivalently, $\mathcal{E}[2^{-m\mathbf{r}(k)}]$.

Example 5.2: (From [18]) The binary erasure channel is a particular case of the class of stochastic links considered. It can be described by taking $r(k) = 1$ with probability $1 - p_e$ and $r(k) = 0$ with probability of erasure p_e. In that case, $\mathcal{E}[2^{-m\mathbf{r}(k)}] = 2^{-m}(1 - p_e) + p_e$. After working through the formulas, one may use (70) and (65) to get the same result as in [18]. In particular, the necessary and sufficient condition for the existence of a stabilizing feedback, for the time-invariant system with $\mathbf{a}(k) = a$, is given by

$$0 \leq p_e < 1 - \frac{|a|^m - 1}{|a|^m(1 - 2^{-m})}$$

A. *Determining the decay of the probability distribution function of* $\bar{\mathbf{x}}$

In this subsection, we explore (65) as a way to infer the decay of the probability distribution of $\bar{\mathbf{x}}(k)$. From Markov's inequality (pp. 80 of [1]), we have that:

$$\forall m > 0, \forall k, \mathcal{P}(\bar{\mathbf{x}}(k) > \vartheta) \leq \vartheta^{-m}\mathcal{E}[\bar{\mathbf{x}}(k)^m] \tag{71}$$

On the other hand, for any given m, if $\bar{\mathbf{x}}(k)$ has a probability density function then:

$$\exists \varepsilon, \delta > 0, \forall k \geq 0, \forall \vartheta > 0, \mathcal{P}(\bar{\mathbf{x}}(k) > \vartheta) < \varepsilon\vartheta^{-(m+\delta+1)} \implies \limsup_{k \to \infty} \mathcal{E}[\bar{\mathbf{x}}(k)^m] < \infty \tag{72}$$

As such, we infer that (65) and (71)-(72) lead to:

$$\mathcal{C} > \mathcal{R} + \alpha(m) + \beta(m) \implies \exists \varepsilon > 0, \forall k, \forall \vartheta, \mathcal{P}(\bar{\mathbf{x}}(k) > \vartheta) \leq \varepsilon\vartheta^{-m} \tag{73}$$

$$\mathcal{C} < \mathcal{R} + \alpha(m) + \beta(m) \implies \forall \varepsilon, \delta > 0, \exists k, \exists \vartheta, \mathcal{P}(\bar{\mathbf{x}}(k) > \vartheta) \geq \varepsilon\vartheta^{-(m+\delta+1)} \tag{74}$$

B. *Uncertainty Interpretation of the Statistical Description of the Stochastic Link*

We suggest that $\alpha(m)$ can be viewed not only as a measure of the quality of the link, in the sense of how $\mathbf{r}(k)$ is expected to fluctuate over time, but it can also be modified to encapsulate a **description of uncertainty**. To be more precise, consider that Δ_l is an uncertainty set of stochastic links and that the "nominal" link has a deterministic data-rate $\mathbf{r}^o(k) = \mathcal{C}$. The elements of Δ_l are the following probability mass functions:

$$\Delta_l \subset \{p_l : \{0, \ldots, \bar{r}\} \to [0, 1] : \sum_{i=0}^{\bar{r}} p_l(i) = 1, \sum_{i=0}^{\bar{r}} i \times p_l(i) = \mathcal{C}\}$$

where $p_l \in \Delta_l$ represents a stochastic link by specifying its statistics, i.e., $\mathcal{P}(r(k)=i) = p_l(i)$. The following is a measure of uncertainty in the link:

$$\bar{\alpha}(m) = \sup_{p_l \in \Delta_l} \frac{1}{m} \log(\mathcal{E}[2^{m\mathbf{r}^{\delta}(k)}]) \tag{75}$$

In this situation, (65) implies that the following is a necessary and sufficient condition for the existence of a feedback scheme that is stabilizing for all stochastic links in the uncertainty set Δ_l:

$$\mathcal{C} - \mathcal{R} - \beta(m) > \bar{\alpha}(m)$$

The authors suggest that $\mathcal{C} - \mathcal{R} - \beta(m) > \bar{\alpha}(m)$ should be viewed as a stability margin condition well adapted to this type of uncertainty. If the plant and the link are time-invariant then $\mathcal{C} - \mathcal{R} > 0$ is necessary and sufficient for stabilizability. Stability is preserved for any stochastic link in Δ_l characterized by $\bar{\alpha}(m) < \mathcal{C} - \mathcal{R}$. This shows that the results by [20], [21] are robust to stochastic links with average transmission rate $\mathcal{C}$ and $\alpha(m) > 0$ sufficiently small.

C. Issues on the Stabilization of Linearizable Non-Linear Systems

In this section, we prove that a minimum rate must be guaranteed at all times in order to achieve stabilization in the sense of Lyapunov[2]. The fact that the classical erasure channel cannot be used to achieve stability in the sense of Lyapunov could already be inferred from [19]. Consider that the following is a state-space representation which corresponds to the linearization of a non-linear system around an equilibrium point:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) \text{ , } \mathbf{y}(k) = C\mathbf{x}(k) \tag{76}$$

If the linearized system is stable in the sense of Lyapunov then (76) must also be stable in the sense of Lyapunov. Consequently, stability in the sense of Lyapunov implies that:

$$\sup_{k} \sup_{x(0)\in[-1/2,1/2]^n} \|x(k)\|_\infty < \infty \tag{77}$$

where $\|x(k)\|_\infty = max_i |[x(k)]_i|$ and $[x(k)]_i$ are the components of $x(k)$. But (77) implies that $\mathbf{x}(k)$ is stable for all moments, so Corollary 4.2 leads to:

$$\forall m, C - \alpha(\frac{m}{n_{unstable}}) > \sum_i \max\{\log|\lambda_i(A)|, 0\} \tag{78}$$

[2]Also denoted as $\epsilon - \delta$ stability

which also implies that:

$$r_{min} > \sum_i \max\{\log|\lambda_i(A)|, 0\} \tag{79}$$

where we have used (67). As a consequence, local stabilization imposes a minimum rate which has to be guaranteed at all times. The classical packet-erasure channel is characterized by $r_{min} = 0$ and, as such, it cannot be used for stabilization in the sense of Lyapunov. This is an important issue in the control of non-linear systems because, frequently, it is necessary to keep the state in a bounded set. That may arise from a physical limitation or as a way to stay in a region of model validity. The stabilization of non-linear systems in this framework was studied in [14], for deterministic channels. Consequently, we conclude that, in the study of stabilization in the sense of Lyapunov, it is sufficiently general to consider deterministic links with rate r_{min}.

VI. Sufficient Conditions for a Class of Multi-state Systems

The results, derived in section III, can be extended, in specific cases, to systems of order higher than one. In the subsequent analysis, we outline how and suggest one case when such extension can be attained. Our results do not generalize to arbitrary stochastic systems of order $n > 1$. We emphasize that the proofs in this section are brief as they follow the same structure of the proofs of section III [3].

A. Notation

We use n as the order of a linear system whose state is indicated by $x(k) \in \mathbb{R}^n$. The following is a list of the adaptations, of the notation and definitions of section I, to the multi-state case: If $x \in \mathbb{R}^n$ then we indicate its components by $[x]_i$, with $i \in \{1, \ldots, n\}$. In a similar way, if M is a matrix then we represent the element located in the i-th row and j-th column as $[M]_{ij}$. We also use $|M|$ to indicate the matrix whose elements are obtained as $[|M|]_{ij} = |[M]_{ij}|$. $\mathbb{R}^{n\times\infty}$ is used to represent the set of sequences of n-dimensional vectors, i.e., $x \in \mathbb{R}^{n\times\infty} \implies x(k) \in \mathbb{R}^n, k \in \mathbb{N}$. The infinity norm in $\mathbb{R}^{n\times\infty}$ is defined as $\|x\|_\infty = \sup_i \max_j |[x(i)]_j|$. It follows that if $x \in \mathbb{R}^n$ then $\|x\|_\infty = \max_{j\in\{1,\ldots,n\}} |[x]_j|$. Accordingly, if $x \in \mathbb{R}^{n\times\infty}$ we use $\|x(k)\|_\infty = \max_{j\in\{1,\ldots,n\}} |[x(k)]_j|$ to indicate the norm of a single vector, at time k, in contrast with $\|x\|_\infty = \sup_i \max_j |[x(i)]_j|$. The convention for random variables remains unchanged, e.g., $[\mathbf{x}(k)]_j$ is the jth component of a

[3] The authors suggest the reading of section III first

n-dimensional random sequence whose realizations lie on $\mathbb{R}^{n\times\infty}$. If H is a sequence of matrices, with $H(k)\in\mathbb{R}^{n\times n}$, then $\|H\|_1=\max_i\sum_{k=0}^{\infty}\sum_{j=1}^{n}|[H(k)]_{ij}|$. For a particular matrix $H(k)$, we also use $\|H(k)\|_1=\max_i\sum_{j=1}^{n}|[H(k)]_{ij}|$. We use $\vec{1}\in\mathbb{R}^n$ to indicate a vector of ones, i.e., $[\vec{1}]_j=1$ for $j\in\{1,\dots,n\}$.

B. Description of uncertainty and robust stability

Let $\varrho\in\mathbb{N}_+\bigcup\{\infty\}$, $\bar{d}\geq 0$, $\bar{z}_f^x,\bar{z}_f^u\in[0,1)$ and $\bar{z}_a>0$ be given constants then we study the stabilizability of the following uncertain system:

$$\mathbf{x}(k+1)=A(I+\mathbf{Z}_a(k))\mathbf{x}(k)+\mathbf{u}(k)+\mathbf{d}(k)+G_f(\mathbf{x},\mathbf{u})(k) \tag{80}$$

where A is upper triangular, with identical real diagonal elements denoted as a:

$$A=\begin{bmatrix} a & \cdots & \\ 0 & \ddots & \vdots \\ 0 & 0 & a\end{bmatrix} \tag{81}$$

In addition, we assume that $\|\mathbf{d}\|_\infty\leq\bar{d}$, $|[\mathbf{Z}_a(k)]_{ij}|\leq\bar{z}_a$ and that $G_f(\mathbf{x},\mathbf{u})(k)$ satisfies:

$$\|G_f(\mathbf{x},\mathbf{u})(k)\|_\infty\leq\bar{z}_f^x\max\{\|\mathbf{x}(k-\varrho+1)\|_\infty,\dots,\|\mathbf{x}(k)\|_\infty\}+ \\ \bar{z}_f^u\max\{\|\mathbf{u}(k-\varrho+1)\|_\infty,\dots,\|\mathbf{u}(k)\|_\infty\} \tag{82}$$

We refer to Chapter 2 of [15], where we use a modification of the rotation operator, defined by [20], to characterize a class of stochastic state-space representations, which can be re-written in a form similar to (80)-(81). In particular, the results in this section hold for stochastic, linear second order systems for which $\mathbf{A}(k)$ is a i.i.d. sequence of random matrices with complex eigenvalues [15]. If the dynamic matrix is a time-invariant real Jordan form or block upper triangular, as in (42), then the state-space representation can be written in the form (80)-(81) [15]. We emphasize that, in (80), we incorporate $\mathbf{u}$ in the description of the feedback uncertainty. As it will be evident from the subsequent discussion, such generalization can be treated with the same techniques used in section III. We decide for including $\mathbf{u}$ in G_f because that allows for a richer description of uncertainty.

A given feedback scheme is robustly stabilizing if it satisfies the following definition.

Definition 6.1: (**m-th Moment Robust Stability**) Let $m > 0$, $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, $\bar{z}_f^x, \bar{z}_f^u \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given. The system (80), under a given feedback scheme, is m-th moment (robustly) stable provided that the following holds:

$$\begin{cases} \lim_{k\to\infty} \mathcal{E}\left[\|\bar{\mathbf{x}}(k)\|_\infty^m\right] = 0 & \text{if } \bar{z}_f^x = \bar{z}_f^u = \bar{d} = 0 \\ \exists b \in \mathbb{R}_+ \text{ s.t. } \limsup_{k\to\infty} \mathcal{E}\left[\|\bar{\mathbf{x}}(k)\|_\infty^m\right] < b & \text{otherwise} \end{cases} \tag{83}$$

where $\bar{\mathbf{x}}(k)$ is given by $[\bar{\mathbf{x}}(k)]_i = \sup_{x(0)\in[-1/2,1/2]^n} |[\mathbf{x}(k)]_i|$.

C. Feedback structure and channel usage assumptions

In order to study the stabilization of systems of order n, higher than one, we assume the existence of a channel allocation $\vec{\mathbf{r}}(k) \in \{0,\dots,\bar{r}\}^n$, satisfying:

$$\sum_{j=1}^{n} [\vec{\mathbf{r}}(k)]_j = \mathbf{r}(k) \tag{84}$$

where $\mathbf{r}(k)$ is the instantaneous rate sequence. We also assume that $\vec{\mathbf{r}}(k)$ is i.i.d..

Using the same notation of section I, we define C_j and $[\vec{\mathbf{r}}^\delta(k)]_j$ as:

$$[\vec{\mathbf{r}}(k)]_j = C_j - [\vec{\mathbf{r}}^\delta(k)]_j \tag{85}$$

Similarly, we also define $\alpha_i(m) = \frac{1}{m} \log \mathcal{E}[2^{m[\vec{\mathbf{r}}^\delta(k)]_i}]$.

In the general case, the allocation problem is difficult because it also entails a change of the encoding process described in the Appendix. The encoding must be such that each $[\vec{\mathbf{r}}^\delta(k)]_i$ corresponds to the instantaneous rate of a truncation operator. In section VI-H we solve the allocation problem for the class of systems described by (80)-(81).

As in the one dimensional case, we assume that both the encoder and the controller have access to $\vec{r}(k-1)$ as well as the constants ϱ, $\bar{z}_f^x$, $\bar{z}_f^u$, $\bar{z}_a$ and $\bar{d}$. The encoder is a function $\mathcal{F}_k^e : \mathbb{R}^{n\times(k+1)} \to \{0,1\}^{n\times\bar{r}}$ that has the following dependence on observations: $\mathcal{F}_k^e(x(0),\dots,\mathbf{x}(k)) = (\mathbf{b}_1,\dots,\mathbf{b}_{\bar{r}})$, where $\mathbf{b}_i \in \{0,1\}^n$. The control action results from a map, not necessarily memoryless, $\mathcal{F}_k^c : \bigcup_{i=0}^{\bar{r}} \{0,1\}^{n\times i} \to \mathbb{R}$ exhibiting the following functional dependence: $\mathbf{u}(k) = \mathcal{F}_k^c(\vec{\mathbf{b}}(k))$, where $\vec{\mathbf{b}}(k)$ is the vector for which, each component $[\vec{\mathbf{b}}(k)]_j$, comprises a string of $[\vec{\mathbf{r}}(k)]_j$ bits successfully transmitted through the link, i.e., $[\vec{\mathbf{b}}(k)]_j = [\mathcal{F}_k^l(\mathbf{b}_1,\dots,\mathbf{b}_{\bar{r}})]_j = \big([\mathbf{b}_1]_j,\dots,[\mathbf{b}_{[\vec{\mathbf{r}}(k)]_j}]_j\big)$. As such, $\mathbf{u}(k)$ can be equivalently expressed as $\mathbf{u}(k) = (\mathcal{F}_k^c \circ \mathcal{F}_k^l \circ \mathcal{F}_k^e)(x(0),\mathbf{x}(1),\dots,\mathbf{x}(k))$.

D. Construction of a stabilizing feedback scheme

The construction of a stabilizing scheme follows the same steps used in section III. The following is the definition of a multidimensional upper-bound sequence.

Definition 6.2: **(Upper-bound Sequence)** Let $\bar{z}_f^x, \bar{z}_f^u \in [0,1)$, $\bar{z}_a \in [0,1)$, $\bar{d} \geq 0$ and $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$ be given. Define the upper-bound sequence $\mathbf{v}(k)$, with $v(k) \in \mathbb{R}^n$, as:

$$\mathbf{v}(k+1) = \mathbf{A}_{cl}(k)\mathbf{v}(k) + (\bar{z}_f^x + \bar{z}_f^u \|A\|_1) \max\{\|\mathbf{v}(k-\varrho+1)\|_\infty, \dots, \|\mathbf{v}(k)\|_\infty\}\vec{1} + \bar{d}\vec{1}, \quad (86)$$

where $[|A|]_{ij} = |[A]_{ij}|$, $v(i) = 0$ for $i < 0$, $[v(0)]_j = \frac{1}{2}$ and $A_{cl}(k)$ is given by:

$$A_{cl}(k) = |A| \left(diag(2^{-[\vec{r}(k)]_1}, \dots, 2^{-[\vec{r}(k)]_n}) + \bar{z}_a \vec{1}\vec{1}^T \right) \quad (87)$$

Adopt the feedback scheme of definition 3.3, mutatis mutandis, for the multi-dimensional case. By measuring the state $x(k)$ and using $[\vec{r}(k)]_j$ bits (at time k) to encode each component $[x(k)]_j$, we construct $[\hat{x}(k)]_j$ such that

$$|[x(k)]_j - [\hat{x}(k)]_j| \leq 2^{-[\vec{r}(k)]_j}[v(k)]_j \quad (88)$$

Accordingly, $\mathbf{u}(k)$ is defined as:

$$\mathbf{u}(k) = -A\hat{\mathbf{x}}(k) \quad (89)$$

The following lemma establishes that the stabilization of $\mathbf{v}(k)$ is sufficient for the stabilization of $\mathbf{x}(k)$.

Lemma 6.1: Let $\bar{z}_f^x, \bar{z}_f^u \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given. If $\mathbf{x}(k)$ is the solution of (80) under the feedback scheme given by (88)-(89), then the following holds:

$$[\bar{\mathbf{x}}(k)]_j \leq [\mathbf{v}(k)]_j$$

for all $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, $\|\mathbf{d}(k)\|_\infty \leq \bar{d}$, every choice $|[Z_a]_{ij}| \leq \bar{z}_a$ and G_f satisfying:

$$\|G_f(\mathbf{x},\mathbf{u})(k)\|_\infty \leq \bar{z}_f^x \max\{\|\mathbf{x}(k-\varrho+1)\|_\infty, \dots, \|\mathbf{x}(k)\|_\infty\} + \bar{z}_f^u \max\{\|\mathbf{u}(k-\varrho+1)\|_\infty, \dots, \|\mathbf{u}(k)\|_\infty\} \quad (90)$$

Proof: The proof follows the same steps as in lemma 3.1. We start by assuming that $[\bar{x}(i)]_j \leq [v(i)]_j$ for $i \in \{0, \dots, k\}$ and proceed to prove that $[\bar{x}(k+1)]_j \leq [v(k+1)]_j$. From (80) and the feedback scheme (88)-(89), we find that:

$$\begin{bmatrix} |[x(k+1)]_1| \\ \vdots \\ |[x(k+1)]_n| \end{bmatrix} \underset{element-wise}{\leq} |A| \begin{bmatrix} |[v(k)]_1| 2^{-\vec{r}_1(k)} \\ \vdots \\ |[v(k)]_n| 2^{-\vec{r}_n(k)} \end{bmatrix} + |A| \; |Z_a(k)| \begin{bmatrix} |[v(k)]_1| \\ \vdots \\ |[v(k)]_n| \end{bmatrix} + \bar{d}\vec{1} +$$

$$\bar{z}_f^x \vec{1} \max\{\|v(k-\varrho+1)\|_\infty, \dots, \|v(k)\|_\infty\} + \bar{z}_f^u \vec{1} \max\{\|u(k-\varrho+1)\|_\infty, \dots, \|u(k)\|_\infty\} \quad (91)$$

In order to address the dependence on $\mathbf{u}$, we notice that (89) implies that $|[\mathbf{u}(k)]_j| \leq [|A| \; |\mathbf{v}(k)|]_j$, which by substituting in (91) leads to the conclusion of the proof. $\square$

E. *Sufficiency for the stochastic case*

We derive the multi-dimensional version of the sufficiency results in section III-B. The results presented below are the direct generalizations of lemma 3.3 and theorem 3.4.

Definition 6.3: (Upper-bound sequence for the stochastic case) Let $\varrho \in \mathbb{N}_+$, $\bar{z}_f^x, \bar{z}_f^u \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given. Given m, consider the following sequence:

$$v_m(k+1) = A_{cl,m} v_m(k) + \left((n\varrho)^{\frac{1}{m}} (\bar{z}_f^x + \bar{z}_f^u \bar{a}_1) \max\{\|v_m(k-\varrho+1)\|_\infty, \dots, \|v_m(k)\|_\infty\} + \bar{d} \right) \vec{1} \quad (92)$$

where $v_m(i) = 0$ for $i < 0$, $v_m(0) = \frac{1}{2}\vec{1}$, $\bar{a}_1 = \|A\|_1$ and $A_{cl,m}$ is defined as

$$A_{cl,m} = |A| \left(diag(2^{-C_1+\alpha_1(m)}, \dots, 2^{-C_n+\alpha_n(m)}) + \bar{z}_a \vec{1}\vec{1}^T \right)$$

Lemma 6.2: (**M-th moment boundedness**)

If $\mathbf{x}(k)$ is the solution of (80) under the feedback scheme of (88)-(89), then the following holds

$$\mathcal{E}[[\bar{\mathbf{x}}(k)]_i^m]^{\frac{1}{m}} \leq [v_m(k)]_i$$

Proof: We start by showing that $\mathcal{E}[[\mathbf{v}(k)]_i^m]^{\frac{1}{m}} \leq [v_m(k)]_i$. We proceed by induction, by assuming that $\mathcal{E}[[\mathbf{v}(j)]_i^m]^{\frac{1}{m}} \leq [v_m(j)]_i$ holds for $j \in \{0, \dots, k\}$ and proving that $\mathcal{E}[[\mathbf{v}(k+1)]_i^m]^{\frac{1}{m}} \leq [v_m(k+1)]_i$. Let $\mathbf{z}$, $\mathbf{s}$ and $\mathbf{g}$ be random variables with $\mathbf{z}$ independent of $\mathbf{s}$. By means of the Minkovsky inequality, we know that $\mathcal{E}[|\mathbf{zs}+\mathbf{g}|^m]^{\frac{1}{m}} \leq \mathcal{E}[|\mathbf{z}|^m]^{\frac{1}{m}} \mathcal{E}[|\mathbf{s}|^m]^{\frac{1}{m}} + \mathcal{E}[|\mathbf{g}|^m]^{\frac{1}{m}}$. Using such property, the following inequality is a consequence of (86):

$$\begin{bmatrix} \mathcal{E}[[\mathbf{v}(k+1)]_1^m]^{\frac{1}{m}} \\ \vdots \\ \mathcal{E}[[\mathbf{v}(k+1)]_n^m]^{\frac{1}{m}} \end{bmatrix} \underset{element-wise}{\leq} A_{cl,m} \begin{bmatrix} \mathcal{E}[[\mathbf{v}(k)]_1^m]^{\frac{1}{m}} \\ \vdots \\ \mathcal{E}[[\mathbf{v}(k)]_n^m]^{\frac{1}{m}} \end{bmatrix} + (\bar{z}_f^x + \bar{z}_f^u \|A\|_1)\mathcal{E}[\max\{\|\mathbf{v}(k-\varrho+1)\|_\infty, \dots, \|\mathbf{v}(k)\|_\infty\}^m]^{\frac{1}{m}}\vec{1} + \bar{d}\vec{1} \quad (93)$$

But using the inductive assumption that $\mathcal{E}[[\mathbf{v}(j)]_i^m]^{\frac{1}{m}} \leq [v_m(j)]_i$ holds for $j \in \{0, \dots, k\}$ and that:

$$\mathcal{E}[\max\{\|\mathbf{v}(k-\varrho+1)\|_\infty, \dots, \|\mathbf{v}(k)\|_\infty\}^m]^{\frac{1}{m}} \leq (n\varrho)^{\frac{1}{m}} \max_{j\in[k-\varrho+1,k]} \max_{i\in[1,n]} \mathcal{E}[[\mathbf{v}(j)]_i^m]^{\frac{1}{m}} \quad (94)$$

we can rewrite (93) as:

$$\begin{bmatrix} \mathcal{E}[[\mathbf{v}(k+1)]_1^m]^{\frac{1}{m}} \\ \vdots \\ \mathcal{E}[[\mathbf{v}(k+1)]_n^m]^{\frac{1}{m}} \end{bmatrix} \underset{element-wise}{\leq} A_{cl,m} v_m(k) + \left((n\varrho)^{\frac{1}{m}}(\bar{z}_f^x + \bar{z}_f^u \|A\|_1) \max\{\|v_m(k-\varrho+1)\|_\infty, \dots, \|v_m(k)\|_\infty\} + \bar{d}\right)\vec{1} \quad (95)$$

Since the induction hypothesis is verified, we can use lemma 6.1 to finalize the proof. □

Theorem 6.3: (**Sufficiency conditions for Robust m-th moment Stability**) Let A be the dynamic matrix of (80), $\varrho \in \mathbb{N}_+$, $\bar{z}_f^x, \bar{z}_f^u \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given and $H_m(k)$ be defined as

$$H_m(k) = A_{cl,m}^k, \ k \geq 0$$

where $A_{cl,m} = |A| \left(diag(2^{-C_1+\alpha_1(m)}, \dots, 2^{-C_n+\alpha_n(m)}) + \bar{z}_a \vec{1}\vec{1}^T\right)$ and $[|A|]_{ij} = |[A]_{ij}|$. Consider that $\mathbf{x}(k)$ is the solution of (80) under the feedback scheme of (88)-(89) as well as the following conditions:

- **(C 1)** $\max_i \lambda_i(A_{cl,m}) < 1$
- **(C 2)** $(n\varrho)^{\frac{1}{m}}(\bar{z}_f^x + \|A\|_1 \bar{z}_f^u)\|H_m\|_1 < 1$

If conditions **(C 1)** and **(C 2)** are satisfied then the following holds:

$$\mathcal{E}[[\bar{\mathbf{x}}(k)]_i^m]^{\frac{1}{m}} \leq \|H_m\|_1 \left((n\varrho)^{\frac{1}{m}}(\bar{z}_f^x + \|A\|_1 \bar{z}_f^u) \frac{\|H_m\|_1 \bar{d} + \|\tilde{g}_m\|_\infty \frac{1}{2}}{1 - (n\varrho)^{\frac{1}{m}}(\bar{z}_f^x + \|A\|_1 \bar{z}_f^u)\|H_m\|_1} + \bar{d}\right) + \|\tilde{g}_m(k)\|_\infty \frac{1}{2} \quad (96)$$

where $\tilde{g}_m(k) = A_{cl,m}^k \vec{1}$.

Proof: We start by noticing that the condition (C1) is necessary and sufficient to guarantee that $\|H_m\|_1$ is finite. From definition 6.3, we have that:

$$v_m(k) = A_{cl,m}^k v(0) + \sum_{i=0}^{k-1} A_{cl,m}^{k-i-1} \left((n\varrho)^{\frac{1}{m}} (\bar{z}_f^x + \bar{z}_f^u \|A\|_1) \max\{\|v_m(i-\varrho+1)\|_\infty, \ldots, \|v_m(i)\|_\infty\} + \bar{d} \right) \vec{1} \quad (97)$$

$$\|\pi_k v_m\|_\infty \leq \|\tilde{g}_m\|_\infty \frac{1}{2} + \|H_m\|_1 \left((n\varrho)^{\frac{1}{m}} (\bar{z}_f^x + \bar{z}_f^u \|A\|_1) \|\pi_k v_m\|_\infty + \bar{d} \right) \quad (98)$$

where we use $\|\pi_k v_m\|_\infty = \max\{\|v_m(0)\|_\infty, \ldots, \|v_m(k)\|_\infty\}$ and $\tilde{g}_m(k) = A_{cl,m}^k \vec{1}$. By means of lemma 6.2, the formula (96) is obtained by isolating $\|\pi_k v_m\|_\infty$ in (98) and substituting it back in (97).□

F. Sufficiency for the case $\bar{z}_a = 0$

If $\bar{z}_a = 0$ then $A_{cl,m}$ of Theorem 6.3 can be expressed as:

$$A_{cl,m} = \begin{bmatrix} 2^{\log|a|} 2^{-C_1+\alpha_1(m)} & \ldots & \\ 0 & \ddots & \vdots \\ 0 & 0 & 2^{\log|a|} 2^{-C_n+\alpha_n(m)} \end{bmatrix} \quad (99)$$

Accordingly, condition (C1) of Theorem 6.3 can be written as $C_i > R + \alpha_i(m)$, with $R = \log|a|$. Also, from condition (C2), we find that by increasing the difference $C_i - (R + \alpha_i(m))$ we reduce $\|H_m\|_1$ and that creates robustness to uncertainty as measured by $(\bar{z}_f^x + \|A\|_1 \bar{z}_f^u)$.

G. Sufficiency for the deterministic/time-invariant case

Accordingly, the following theorem establishes the multi-dimensional analog to theorem 3.2. We omit the proof as it is a direct adaptation of the proof of Theorem 6.3.

Theorem 6.4: (**Sufficiency conditions for Robust Stability**) Let A be the dynamic matrix of (80), $\varrho \in \mathbb{N}_+ \bigcup \{\infty\}$, $\bar{z}_f^x, \bar{z}_f^u \in [0,1)$, $\bar{z}_a \in [0,1)$ and $\bar{d} \geq 0$ be given and $H(k)$ be defined as

$$H(k) = A_{cl}^k, \ k \geq 0$$

where $A_{cl} = |A| \left(diag(2^{-C_1}, \ldots, 2^{-C_n}) + \bar{z}_a \vec{1}\vec{1}^T \right)$ and $[|A|]_{ij} = |[A]_{ij}|$. Consider that $x(k)$ is the solution of (80) under the feedback scheme of (88)-(89) as well as the following conditions:

- **(C 1)** $\max_i \lambda_i(A_{cl}) < 1$
- **(C 2)** $(\bar{z}_f^x + \|A\|_1 \bar{z}_f^u)\|H\|_1 < 1$

If conditions **(C 1)** and **(C 2)** are satisfied then the following holds:

$$\|\bar{x}(k)\|_\infty \leq \|H\|_1 \left((\bar{z}_f^x + \|A\|_1 \bar{z}_f^u) \frac{\|H\|_1 \bar{d} + \|\tilde{g}\|_\infty \frac{1}{2}}{1 - (\bar{z}_f^x + \|A\|_1 \bar{z}_f^u)\|H\|_1} + \bar{d} \right) + \|\tilde{g}(k)\|_\infty \frac{1}{2} \tag{100}$$

where $\tilde{g}(k) = A_{cl}^k \vec{1}$.

H. Solving the Allocation Problem

Take the scheme of section as a starting point. Assume also that $\mathbf{r}(l)$ is always a multiple of n, i.e., $r(l) \in \{0, n, 2n, \ldots, \bar{r}\}$. In order to satisfy this assumption, we only need to adapt the scheme described in the Appendix, by selecting packets whose size is a multiple of n. Consequently, we can include, in every packet, an equal number of bits to encode each component $[\mathbf{x}(l)]_i$. By including the most important bits in the highest priority packets, we guarantee that each $[\vec{\mathbf{r}}(l)]_i$ corresponds to the instantaneous rate of a truncation operator. As such, we adopt the following allocation:

$$[\vec{\mathbf{r}}(l)]_i = \frac{\mathbf{r}(l)}{n} \tag{101}$$

where we also use C_i and define the zero mean i.i.d. random variable $[\vec{\mathbf{r}}^\delta(l)]_i$, satisfying:

$$[\vec{\mathbf{r}}(l)]_i = C_i - [\vec{\mathbf{r}}^\delta(l)]_i \tag{102}$$

From the definition of $\alpha_i(m)$ and $\alpha(m)$, the parameters characterizing the allocation (101) and $\mathbf{r}(l)$ are related through:

$$C_i = \frac{1}{n} C \text{ and } \alpha_i(m) = \frac{1}{m} \log \mathcal{E}[2^{-m[\vec{\mathbf{r}}^\delta(l)]_i}] = \frac{1}{n}\frac{n}{m} \log \mathcal{E}[2^{-\frac{m}{n}\mathbf{r}^\delta(l)}] = \frac{1}{n}\alpha(\frac{m}{n}) \tag{103}$$

The following Proposition shows that, under the previous assumptions, the necessary condition of Theorem 4.1 is not conservative.

Proposition 6.5: Let $\varrho \in \mathbb{N}_+$ be a given constant. If $C - \alpha(\frac{m}{n}) > nR$ then there exist constants $\bar{d} > 0$ and $\bar{z}_f^x, \bar{z}_f^u \in [0, 1)$ such that the state-space representation (80)-(81) can be robustly stabilized in the m-th moment sense.

Proof: We can use Theorem 6.3 to guarantee that the following is a sufficient condition for the existence of $\bar{d} > 0$ and $\bar{z}_f^x, \bar{z}_f^u \in [0, 1)$ such that (80)-(81) is robustly stabilizable:

$$C_i > \alpha_i(m) + R \tag{104}$$

On the other hand, by means of (103), the assumption $C - \alpha(\frac{m}{n}) > nR$ can be written as (104). □

APPENDIX

The purpose of this section[4] is to motivate the truncation operator of definition 2.1. In addition, details of the synchronization between the encoder and the decoder are discussed.

Consider that we want to use a wireless medium to transmit information between nodes A and B. In our formulation, node A represents a central station, which measures the state of the plant. The goal of the transmission system is to send information, about the state, from node A to node B, which represents a controller that has access to the plant. Notice that node A maybe a communication center which may communicate to several other nodes, but we assume that node B only communicates with node A. Accordingly, we will concentrate on the communication problem between nodes A and B only, without loss of generality.

Definition 1.1: (Basic Communication Scheme) We assume the existence of an external time-synchronization variable, denoted as k. The interval between k and $k+1$ is of T seconds, of which $T_T < T$ is reserved for transmission. We also denote the number of bits in each packet as Π, excluding headers. In order to submit an ordered set of packets for transmission, we consider the following basic communication protocol, at the media access control level:

(Initialization) A variable denoted by $c(k)$ is used to count how many packets are sent in the interval $t \in [kT, kT + T_T]$. We consider yet another counter p, which is used to count the number of periods for which no packet is sent. The variables are initialized as $k = 0$, $p = 0$ and $c(0) = 0$.

(For node A)

(Synchronization) If k changes to $k := k + 1$ then step 1 is activated.

- **Step1** The packets to be submitted for transmission are numbered according to their priority; 0 is the highest priority. The order of each packet is included in the header of the packet. The variable $c(k)$ is initialized to $c(k) = 0$ and p is incremented to $p := p + 1$. The first packet (packet number 0) has an extra header, comprising the pair $(c(k - p - 1), p)$. Move to step 2.

[4]This section is not essential for understanding the necessity and sufficiency theorems of sections III, IV and VI.

- **Step 2:** Stands by until it can send packet number $c(k)$. If such opportunity occurs, move to step 3.
- **Step 3:** Node A sends packet number $c(k)$ to node B and waits for an ACK signal from node B. If node A receives an ACK signal then $c(k) := c(k) + 1$, $p = 0$ and move back to step 2. If time-out then go back to Step 2.

The *time-out* decision may be derived from several events: a fixed waiting time; a random timer or a new opportunity to send a packet.

(For node B)

- **Step 1:** Node B stands by until it receives a packet from node A. Once a packet is received, check if it is a first packet: if so, extract $(c(k-p-1), p)$ and construct $\mathbf{r}_{dec}(i)$, with $i \in \{k-p-1, \dots, k-1\}$, according to:

$$\begin{cases} \mathbf{r}_{dec}(k-p-1) = \mathbf{c}(k-p-1)\Pi, \ \mathbf{r}_{dec}(i) = 0, i \in \{k-p, \dots, k-1\} & \text{if } p \geq 1 \\ \mathbf{r}_{dec}(k-1) = \mathbf{c}(k-1)\Pi & \text{otherwise} \end{cases}$$

  where Π is the size of the packets, excluding the header. If the packet is not duplicated then make the packet available to the controller. Move to step 2.
- **Step 2:** Wait until it can send an ACK signal to node A. Once ACK is sent, go to step 1.

The scheme of definition 1.1 is the simplest version of a class of media access control (MAC) protocols, denoted as Carrier Sense Multiple Access (*CSMA*). A recent discussion and source of references about CSMA is [8]. Such scheme also describes the MAC operation for a wireless communication network between two nodes. Also, we adopt the following strong assumptions:

- Every time node A sends a packet to node B: either it is sent without error or it is lost. This assumption means that we are not dealing with, what is commonly referred to as, a *noisy channel*.
- Every ACK signal sent by node B will reach node A before k changes to $k + 1$. This assumption is critical to guarantee that no packets are wasted. Notice that node B can use the whole interval $t \in (kT + T_T, (k + 1)T)$ to send the last ACK. During this period, the controller is not expecting new packets. The controller will generate $\mathbf{u}(k)$ using the packets that were sent in the interval $t \in [kT, kT + T_T]$. Consequently, such ACK is not important in the generation of $\mathbf{u}(k)$. It will be critical only for $\mathbf{u}(i)$ for $i > k$.

We adopt k, the discrete-time unit, as a reference. According to the usual framework of digital control, k will correspond to the discrete time unit obtained by partitioning the continuous-time in periods of duration T. Denote by $T_T < T$ the period allocated for transmission. Now, consider that the aim of a discrete-time controller is to control a continuous-time linear system, which admits[5] a discretization of the form $\mathbf{x}_c((k+1)T) = \mathbf{A}(k)\mathbf{x}_c(kT) + \mathbf{u}(k)$. The discretization is such that $u(k)$ represents the effect of the control action over $t \in (kT+T_T, (k+1)T)$. Information about $\mathbf{x}(k) = \mathbf{x}_c(kT)$, the state of the plant at the sampling instant $t = kT$, is transmitted during $t \in [kT, kT + T_T]$. Whenever k changes, we construct a new queue and assume that the cycle of definition 1.1 is reset to step 1.

1) Synchronization between the encoder and the decoder: Denote by $\mathbf{r}_{enc}(k)$ the total number of bits that the encoder has successfully sent between k and $k+1$, i.e., the number of bits for which the encoder has received an ACK. The variable $\mathbf{r}_{enc}(k)$ is used by the encoder to keep track of how many bits were sent. The corresponding variable at the decoder is represented as $\mathbf{r}_{dec}(k)$. From definition 1.1, we infer that $\mathbf{r}_{dec}(k-1)$ may not be available at all times. On the other hand, we emphasize that the following holds:

$$\mathbf{c}(k) \neq 0 \implies \mathbf{r}_{dec}(i) = \mathbf{r}_{enc}(i) \text{ for } i \in \{0, \dots, k-1\} \tag{105}$$

In section III, the stabilizing control is constructed in a way that: if no packet goes through between k and $k+1$, i.e., $c(k) = 0$ then $\mathbf{u}(k) = 0$. That shows that $\mathbf{r}_{dec}(k-1)$ is not available only when it is not needed. That motivated us to adopt the simplifying assumption that $\mathbf{r}(k-1) = \mathbf{r}_{enc}(k-1) = \mathbf{r}_{dec}(k-1)$. We denote by $\mathbf{r}(k)$ the random variable which represents the total number of bits that are transmitted in the time interval $t \in [kT, kT + T_T]$. The $\mathbf{r}(k)$ transmitted bits are used by the controller to generate $\mathbf{u}(k)$. Notice that our scheme does not pressupose an extra delay, because the control action will act, in continuous time, in the interval $t \in (kT + T_T, (k+1)T)$.

2) Encoding and Decoding for First Order Systems: Given the transmission scheme described above, the only remaining degrees of freedom are how to encode the measurement of the state and how to construct the queue. From the proofs of theorems 4.1, 3.2 and 3.4, we infer that a necessary and sufficient condition for stabilization is the ability to transmit, between nodes A

[5]A controllable linear and time-invariant system admits a discretization of the required form. If the system is stochastic an equivalent condition has to be imposed

and B, an estimate of the state $\hat{\mathbf{x}}(k)$ with an accuracy lower-bounded[6] by $\mathcal{E}[|\hat{\mathbf{x}}(k) - \mathbf{x}(k)|^m] < 2^{-R}$, where $R > 0$ is a given constant that depends on the state-space representation of the plant. Since the received packets preserve the original order of the queue, we infer that the *best* way to construct the queues, at each k, is to compute the binary expansion of $\mathbf{x}(k)$ and position the packets so that the bits corresponding to higher powers of 2 are sent first. The lost packets will always[7] be the *less important*. The abstraction of such procedure is given by the truncation operator of definition 2.1. The random behavior of $\mathbf{r}(k)$ arises from random time-out, the existence of collisions generated by other nodes trying to communicate with node A or from the fading that occurs if node B is moving. The fading phenomena may also occur from interference.

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