A Rate-Compatible Sphere-Packing Analysis of Feedback Coding with Limited Retransmissions

Adam Williamson, Tsung-Yi Chen, and Rick Wesel

UCLA Communication Systems Laboratory

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Variable-Length Feedback with Termination

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  $\Rightarrow$ General results with numerical examples for BSC and BEC.

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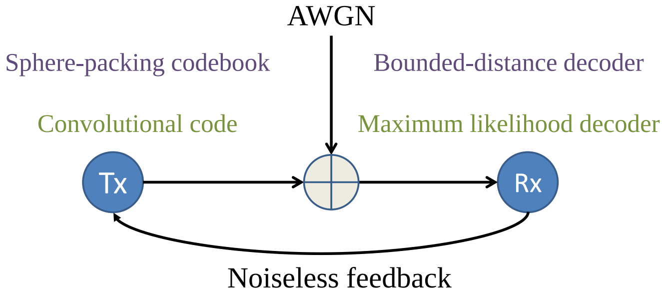
  - ⇒ Rate-compatible tail-biting convolutional code.
  - ⇒ Focused on the AWGN channel.

Incremental Redundancy Scheme Overview



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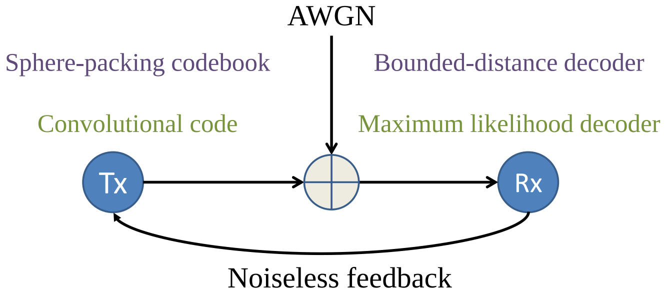

- Forward channel is **AWGN** with known SNR, η.
- The receiver attempts to decode after each incremental transmission, based on all received symbols.

Transmission Scheme Details (1st transmission)

- $k = \log_2 M$ = information bits.

- 1st transmission:
  - Send I_1 , decode with $N_1 = I_1$.
  - $R_1 = k/N_1$ = initial code rate.

Transmission Scheme Details (2nd transmission)

- $k = \log_2 M =$ information bits.

- 2nd transmission:
  - Send I_2 , decode with $N_2 = I_1 + I_2$.
  - $R_2 = k/N_2$ = code rate.

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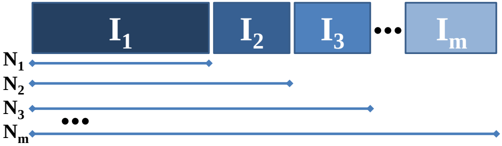

- ith transmission: $(i = 2, \ldots, m)$
  - Send I_i , decode with $N_i = N_{i-1} + I_i$.
  - I_i = incremental step size, N_i = block length at ith transmission.
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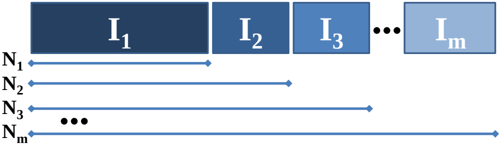

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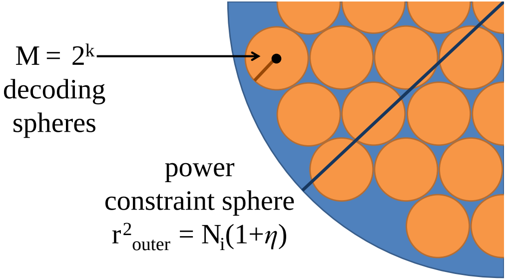
- m = maximum number of transmissions (before repetition).

Start Over if Failure after m Transmissions

- If decoding is unsuccessful after m transmissions, start over by sending I_1 bits, then I_2 bits, etc. (similar to ARQ).
- This is a practical limitation.
- Simplifies analysis.

Decoding Error Probability for Sphere-Packing



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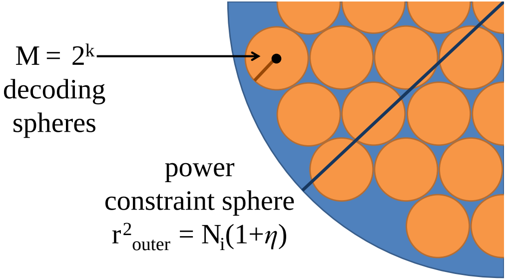

- P[error with block length N_i]

$$= \mathrm{P}(\zeta_i) = \mathrm{P}\left(\sum_{\ell=1}^{N_i} z_\ell^2 > r_i^2\right) = 1 - F_{\chi^2_{N_i}}(r_i^2),$$

  - $r_i^2 = \frac{N_i(1+\eta)}{2^{2k/N_i}}$ is the sphere-packing radius (squared),
  - $z_\ell \sim \mathcal{N}(0,1)$ are the noise samples.

Sphere-Packing: Myth or Reality?

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- ... but we will see that a convolutional code can achieve sphere-packing performance.

Marginal vs. Joint Decoding Error Probability

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- P[error after j transmissions] $= \mathrm{P}(\zeta_1, \zeta_2, \ldots, \zeta_j)$ **(joint)**

$$= \mathrm{P}\left(\sum_{\ell=1}^{N_1} z_\ell^2 > r_1^2, \sum_{\ell=1}^{N_2} z_\ell^2 > r_2^2, \ldots, \sum_{\ell=1}^{N_j} z_\ell^2 > r_j^2\right)$$

$$= \int_{r_1^2}^{\infty} \int_{r_2^2 - t_1}^{\infty} \cdots \int_{r_{j-1}^2 - \sum_{i=1}^{j-2} t_i}^{\infty} f_{\chi^2_{I_1}}(t_1) \ldots f_{\chi^2_{I_{j-1}}}(t_{j-1}) \times \left(1 - F_{\chi^2_{I_j}}\left(r_j^2 - \sum_{i=1}^{j-1} t_i\right)\right) dt_{j-1} \ldots dt_1.$$

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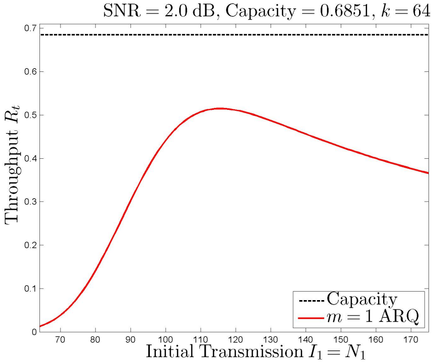

What About $m > 1$?

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$$\lambda = \frac{I_1 + \sum_{i=2}^{m} I_i P\left(\bigcap_{j=1}^{i-1} \zeta_j\right)}{1 - P\left(\bigcap_{j=1}^{m} \zeta_j\right)}$$

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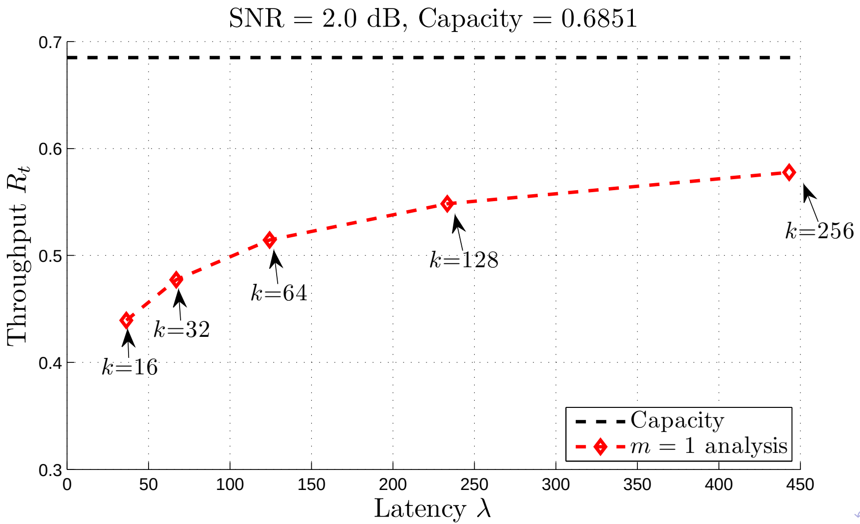
- **Throughput**

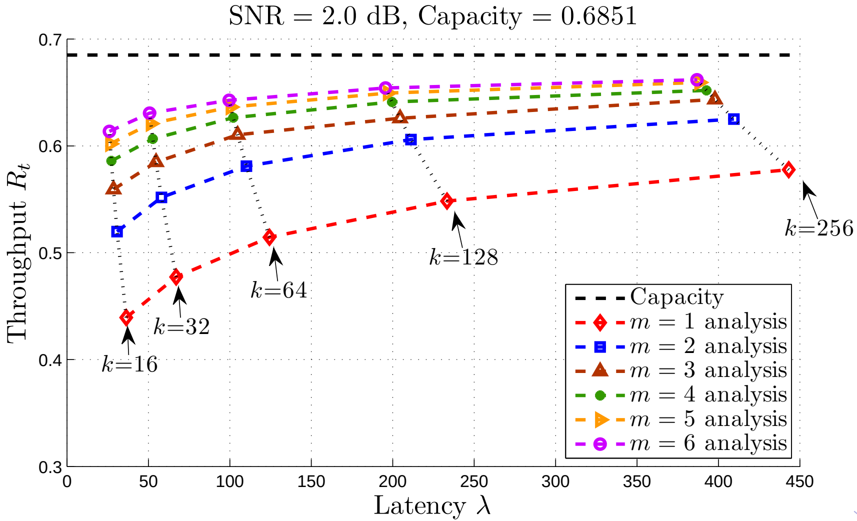
$$R_t = k/\lambda$$

- Select $\{I_1, I_2, \ldots, I_m\}$ to maximize R_t.

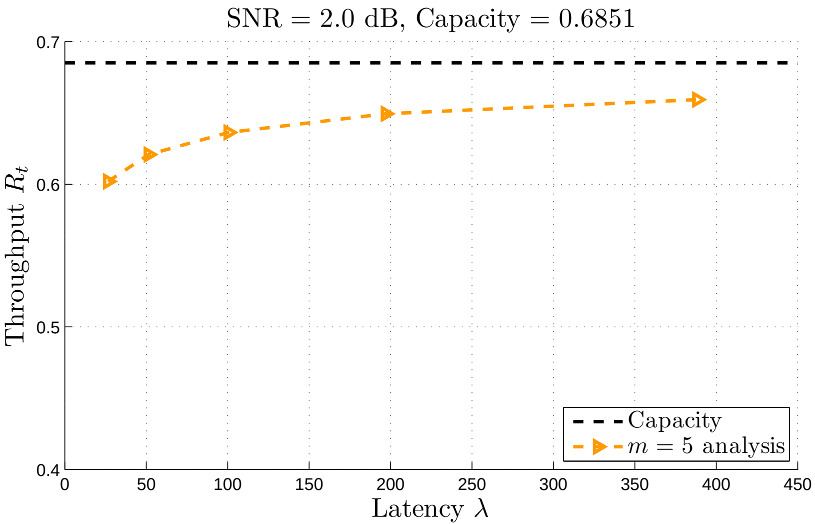
RCSP: Latency vs. Throughput for $m = 1$ (ARQ) Using Optimal Step Size I_1



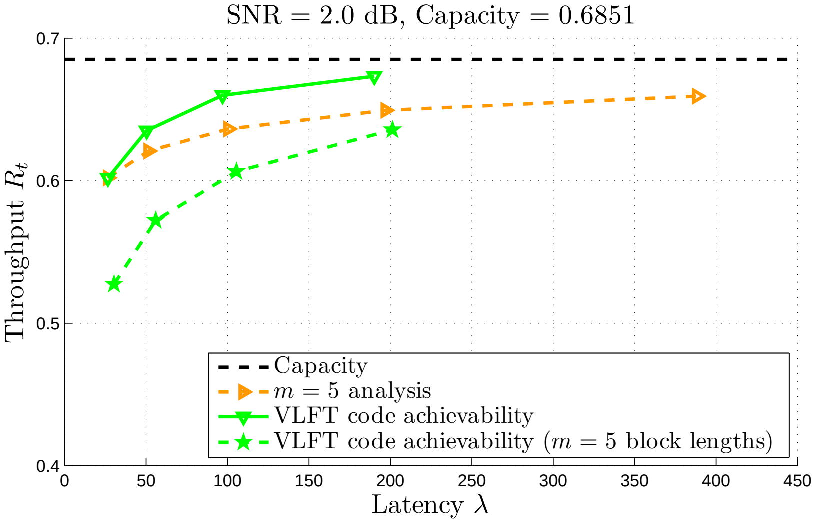
RCSP: Latency vs. Throughput for $m = 1$ to $m = 6$, Using Optimal Step Sizes I_i



RCSP: Latency vs. Throughput for $m = 5$, Using Optimal Step Sizes I_i



Comparison with [Polyanskiy et al. 2011]



Convolutional Code Simulations for $m = 5$

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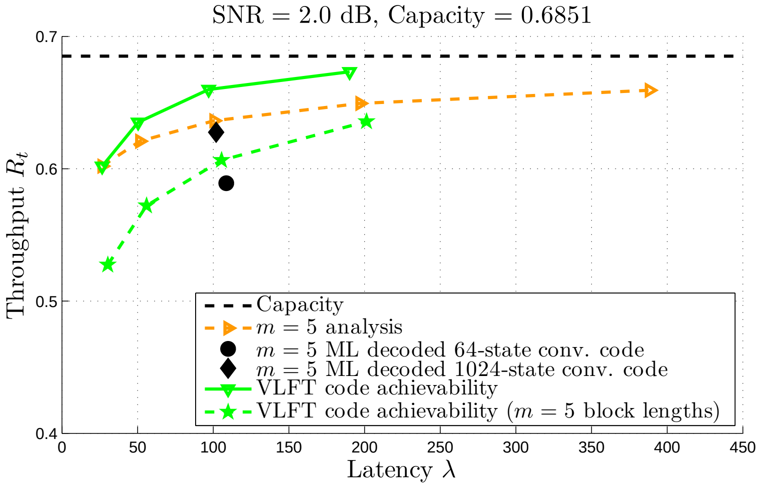
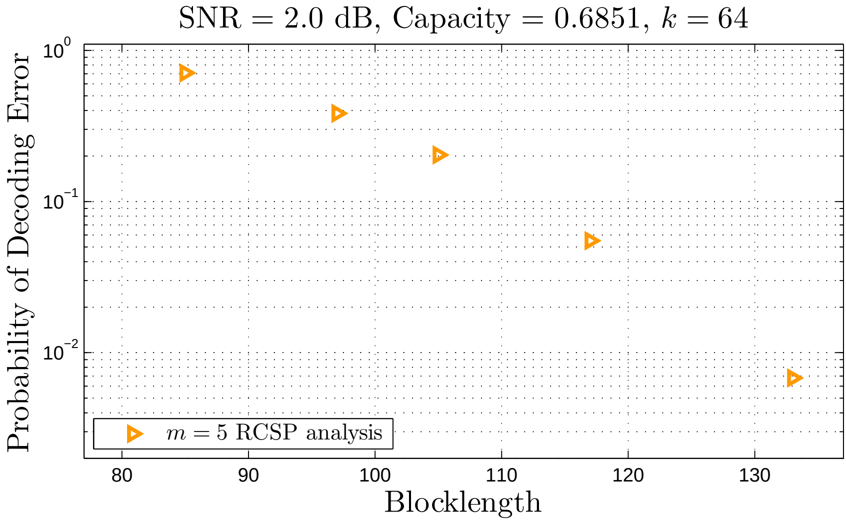
- High-rate codes obtained by pseudo-random puncturing of mother codes.
- **Maximum likelihood (ML)** decoding.
  - ML decoding regions completely fill the power constraint sphere.
- Tail-biting implementations used for throughput efficiency.

Convolutional Code Achievability, $m = 5$



- 90% of AWGN capacity in ~100 symbols.

Decoding Error Trajectory



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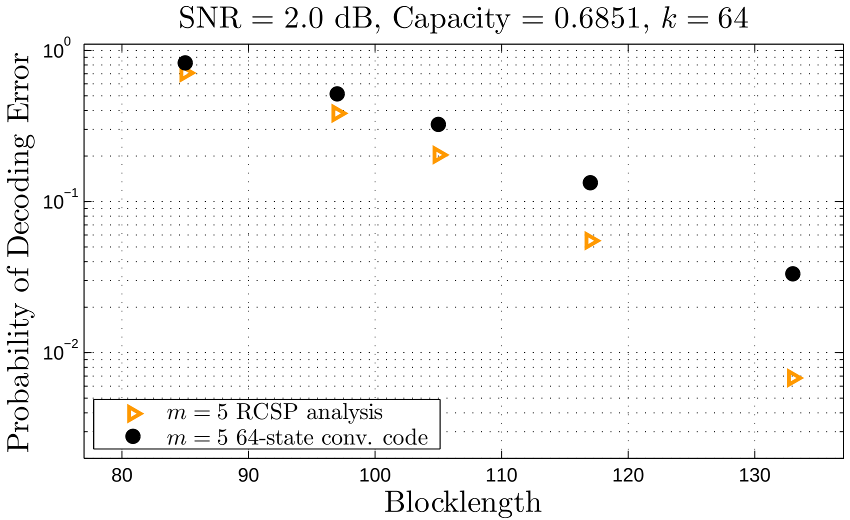

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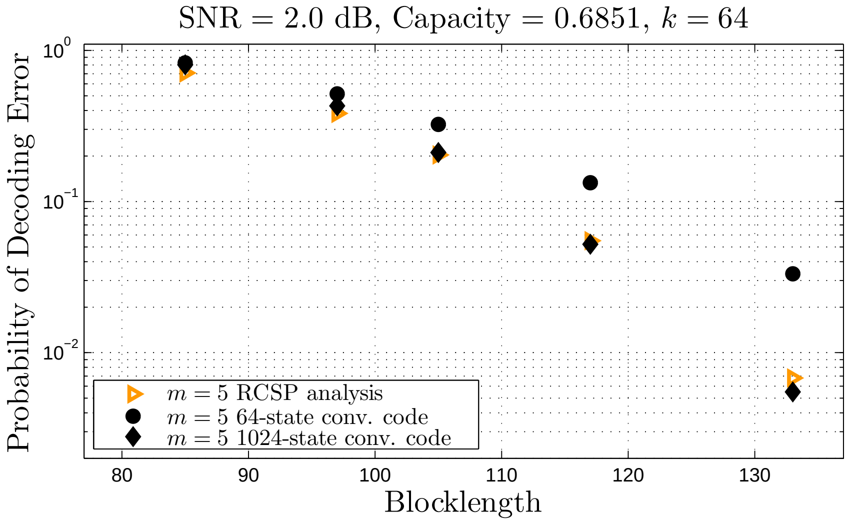

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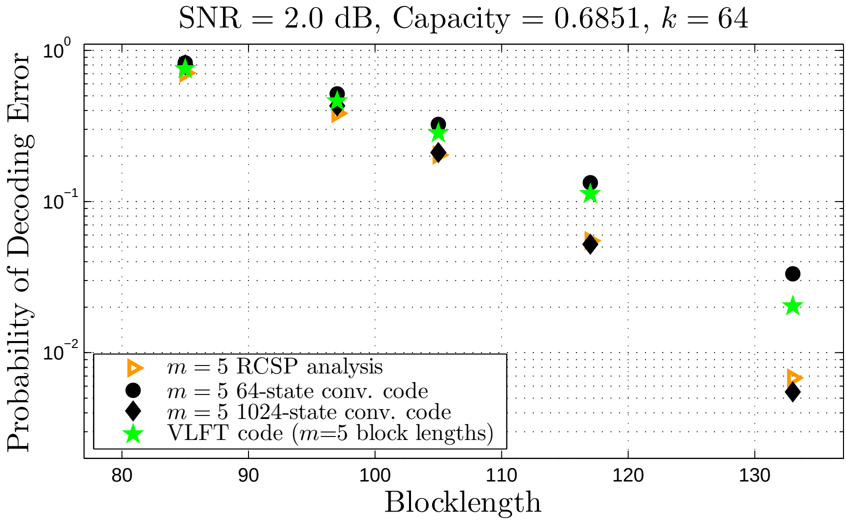

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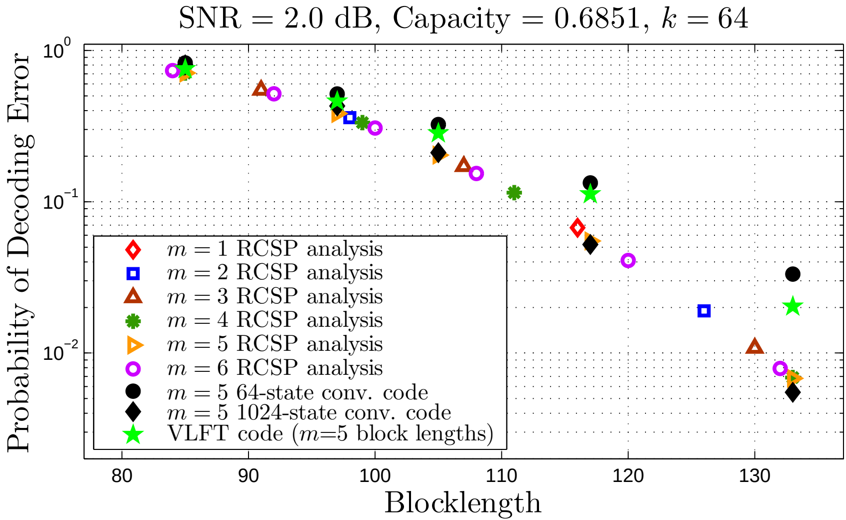
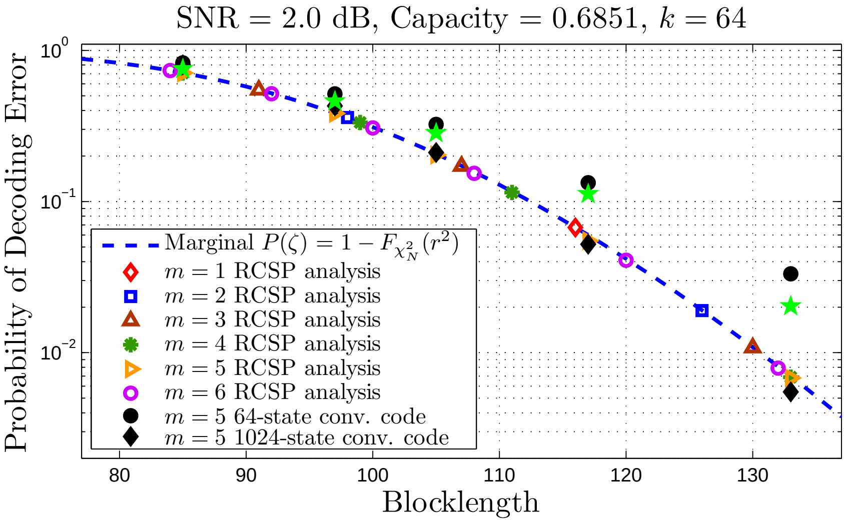
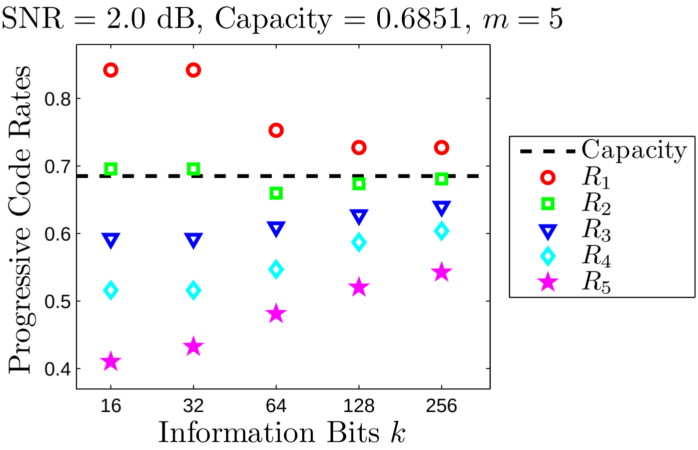


Marginal Chi-Square: A Design Objective



Optimal Rates



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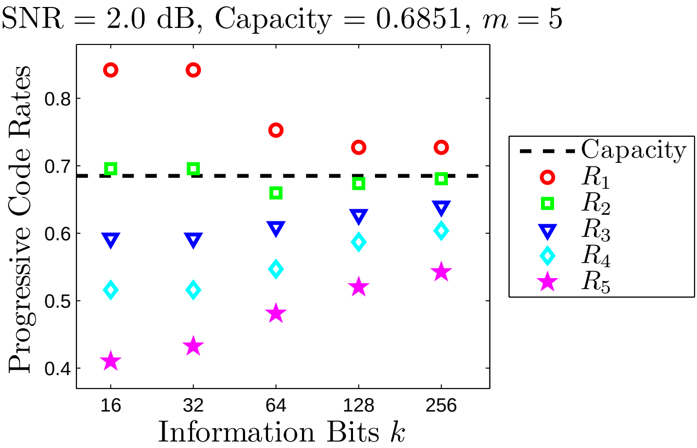

- $R_1 > C$

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- Questions?