Efficient Binomial Channel Capacity Computation with an Application to Molecular Communication

Richard D. Wesel, Emily E. Wesel,
Lieven Vandenberghe, Christos Komninakis, and
Muriel Medard

2018 Information Theory and its Applications Workshop
February 13, 2018

Outline

- The Binomial Channel
- Capacity-Achieving Distribution
- Csiszar's Min-Max Capacity Theorem
- Dynamic Assignment Blahut-Arimoto
- Application to a Molecular Channel

The Binomial Channel

Channel input X is probability of success in a Bernoulli trial.

Channel output Y is number of successes in n Bernoulli trials.

UCLA

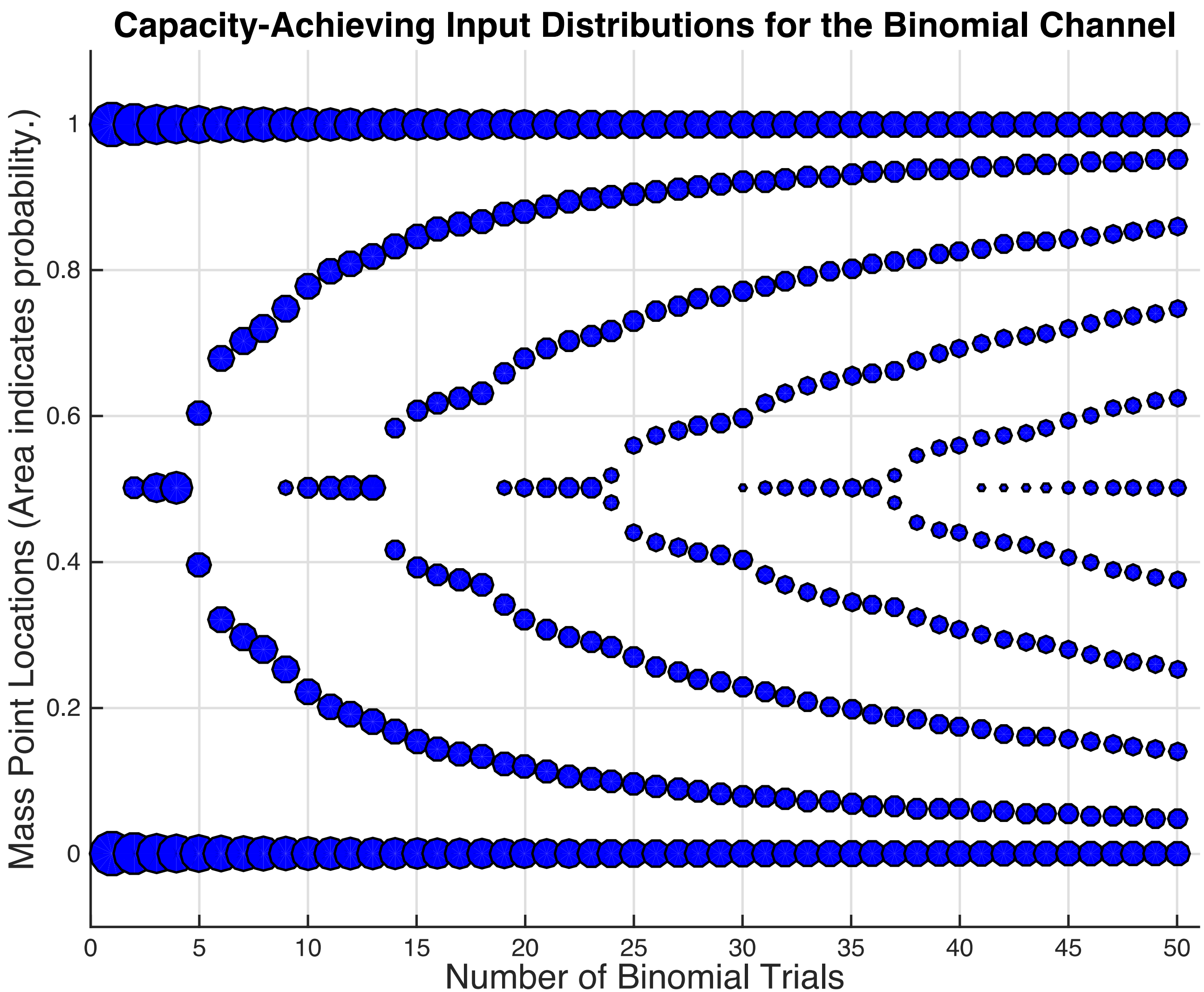
Input Alphabet, Capacity Achieving Support

- The input alphabet is the unit interval, so it is uncountably infinite.
- …so Blahut-Arimoto is awkward.
- However, the capacity-achieving input distribution has at most n+1 mass points from [Witsenhausen, Dubins].

H. S. Witsenhausen, “Some aspects of convexity useful in information theory,” *IEEE Transactions on Information Theory*, vol. 26, no. 3, pp. 265–271, May 1980.

L. E. Dubins, “On extreme points of convex sets,” *Journal of Mathematical Analysis and Applications*, vol. 5, no. 2, pp. 237–244, 1962.

Capacity Achieving Distributions



UCLA

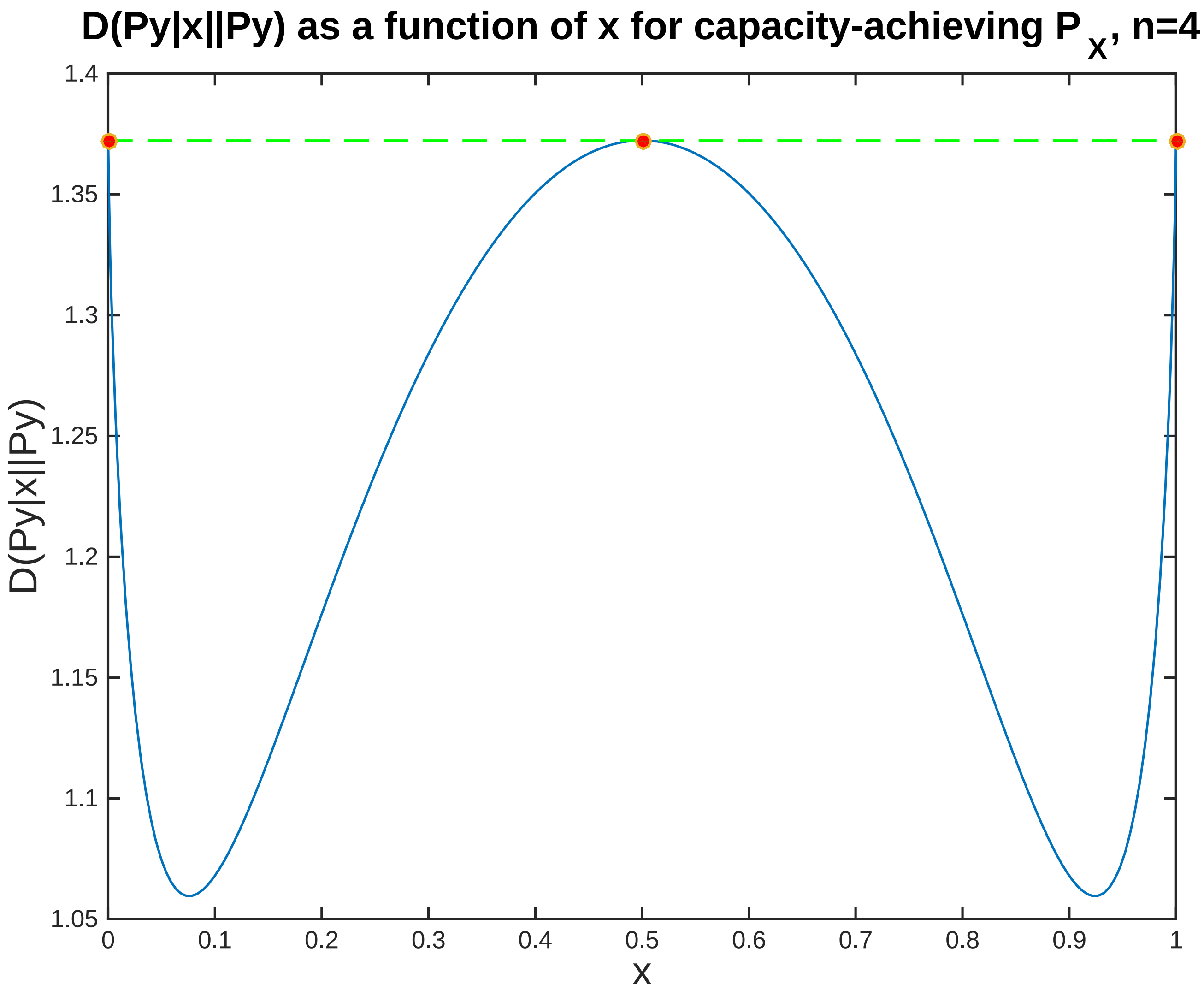
Csiszar's Min-Max Capacity Theorem

$$C = \min_{P_Y} \max_{x} D\left(P_{Y|X=x} \| P_Y\right)$$

This theorem instructs us to find the capacity-achieving *output* distribution.

UCLA

$$C = \min_{P_Y} \max_x D\left(P_{Y|X=x} \| P_Y\right)$$



UCLA

A Stopping Criterion

$$C = \min_{P_Y} \max_x D\left(P_{Y|X=x} \| P_Y\right)$$

For any P_X and corresponding P_Y:

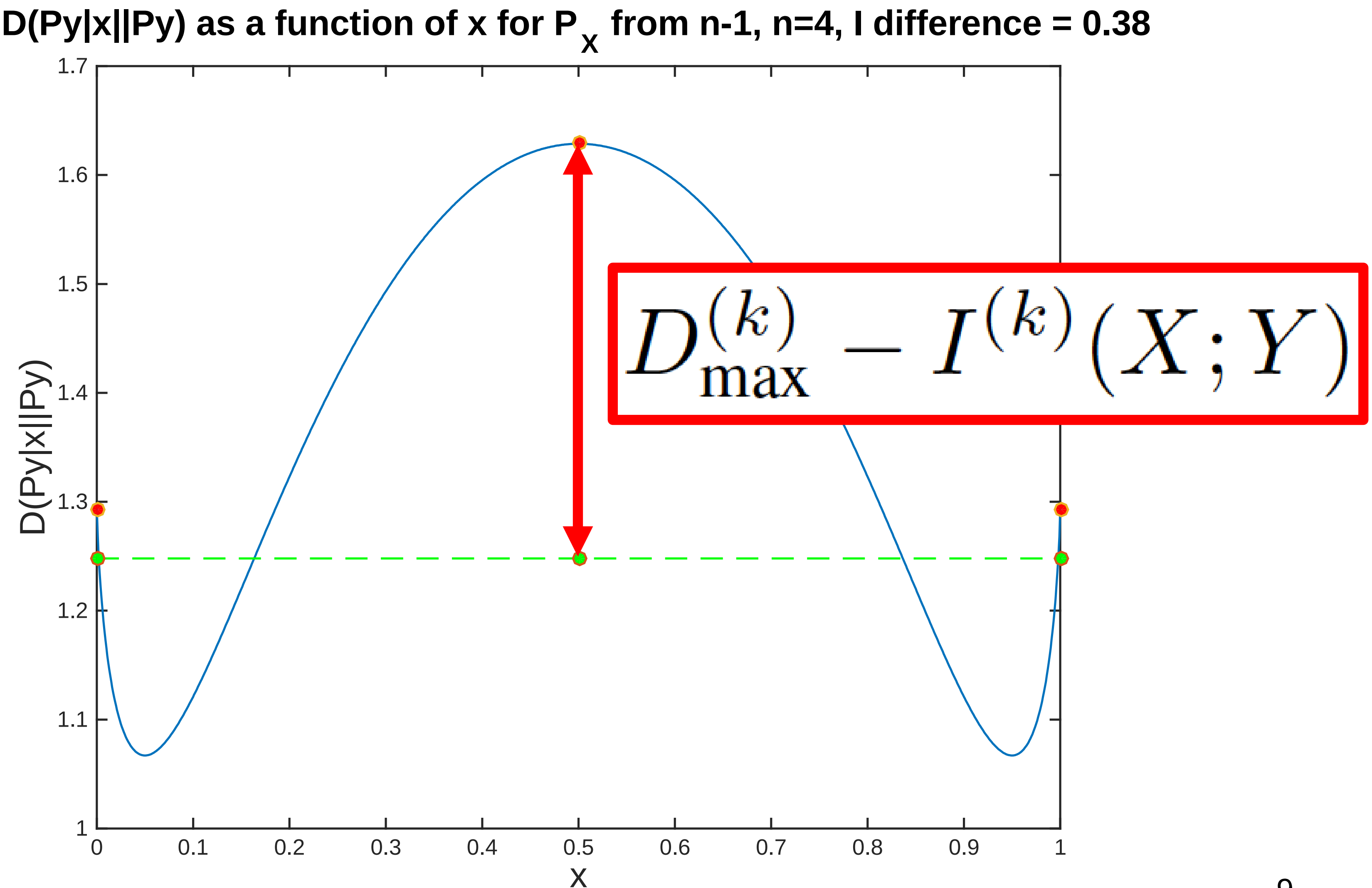
$\max_x D\left(P_{Y|X=x} \| P_Y\right)$ is an upper bound.

$I(X;Y)$ is a lower bound.

When $\max_x D\left(P_{Y|X=x} \| P_Y\right) - I(X;Y)$ is small enough, declare victory.

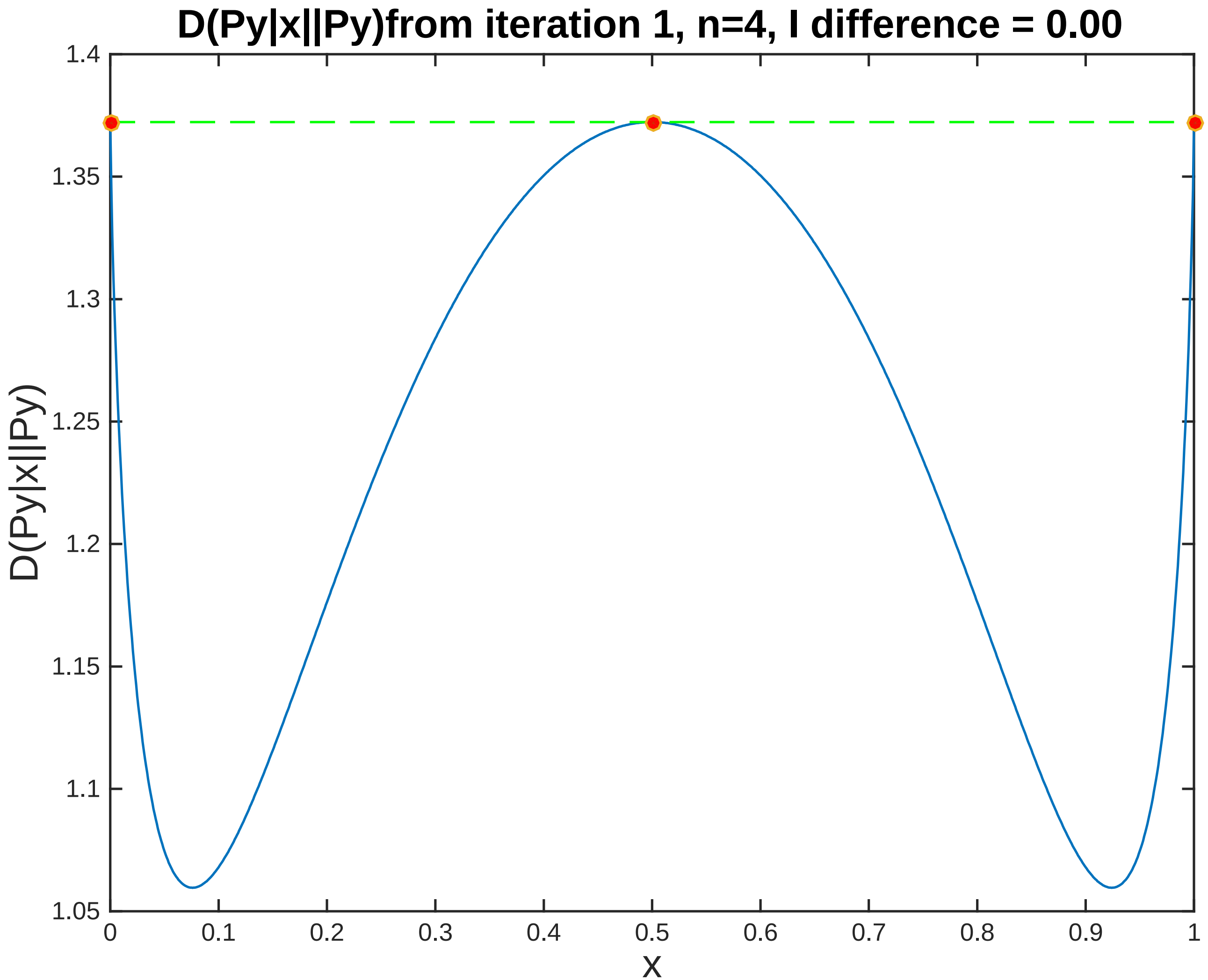
UCLA

$$C = \min_{P_Y} \max_x D\left(P_{Y|X=x} \| P_Y\right)$$

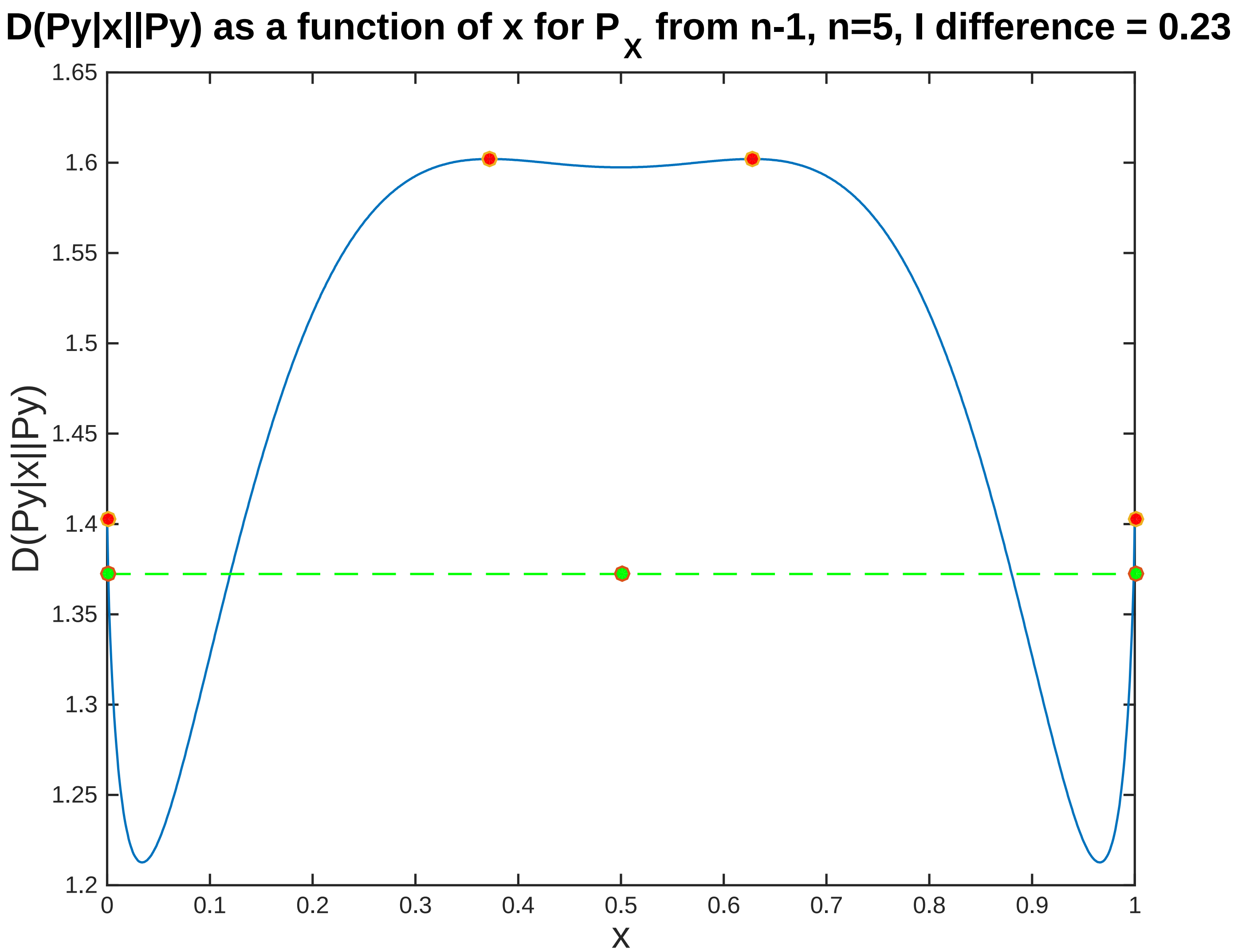


UCLA

Blahut-Arimoto found $n=4$ capacity

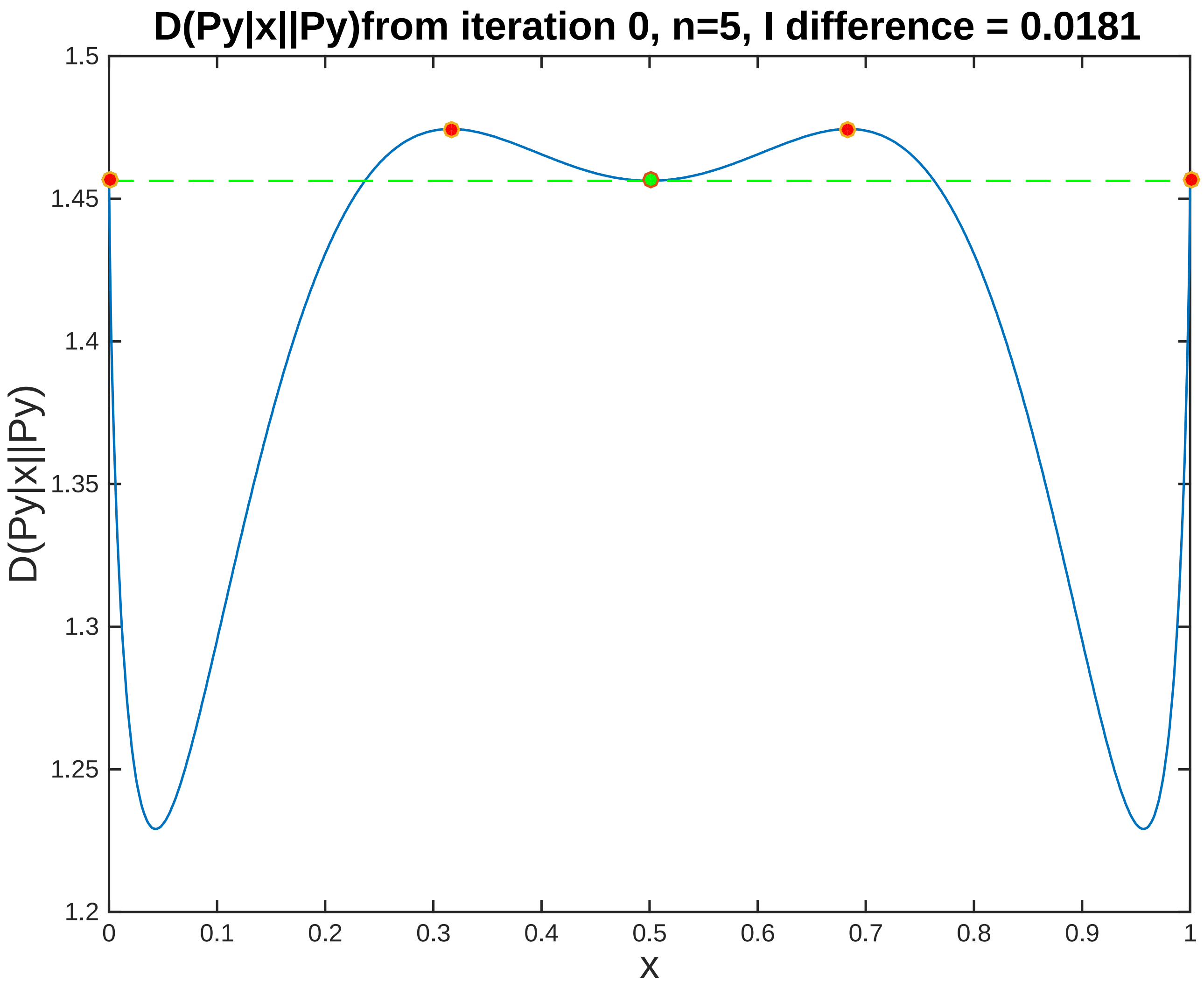


Use the $n=4$ solution for $n=5$



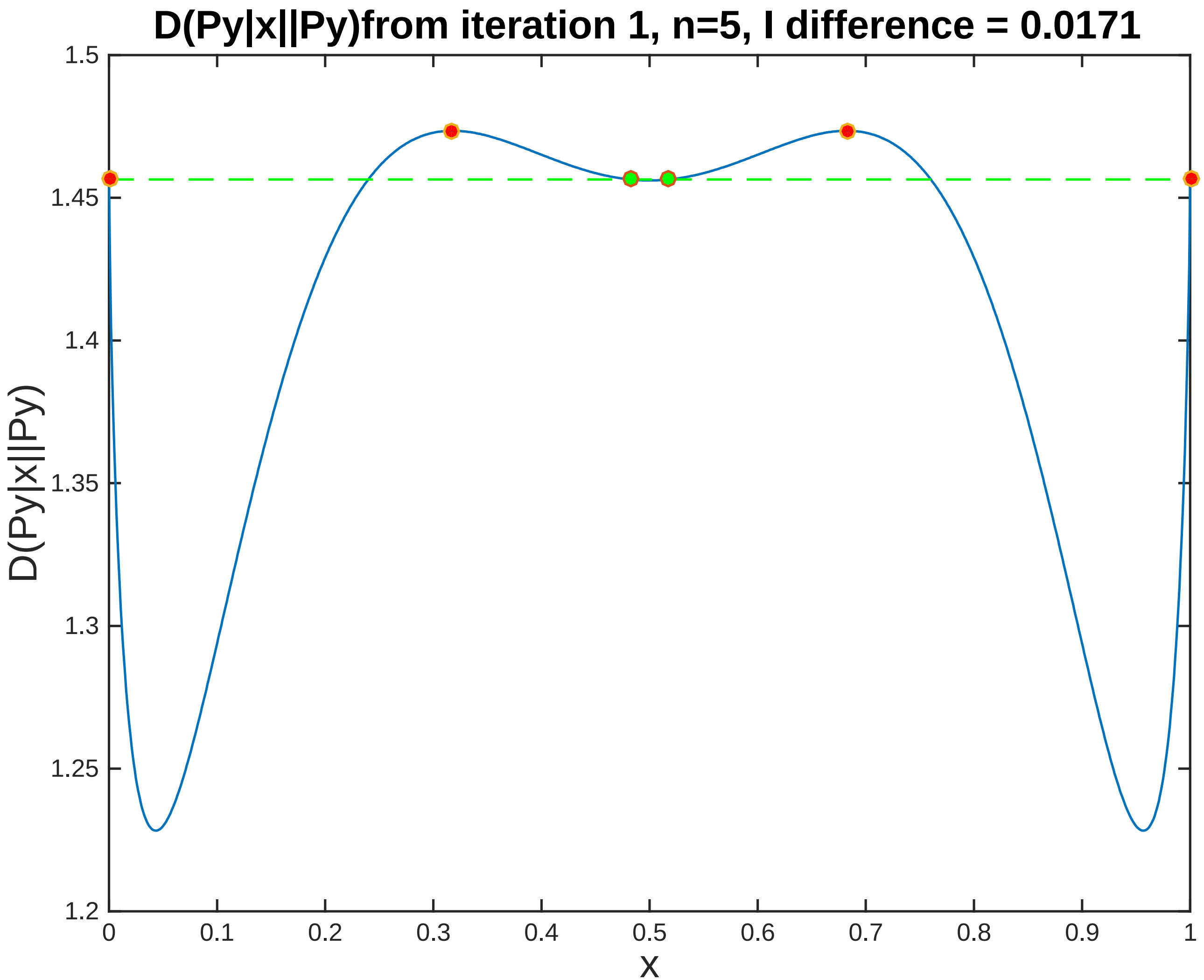
UCLA

$n=3$ solution for $n=4$, after Blahut-Arimoto



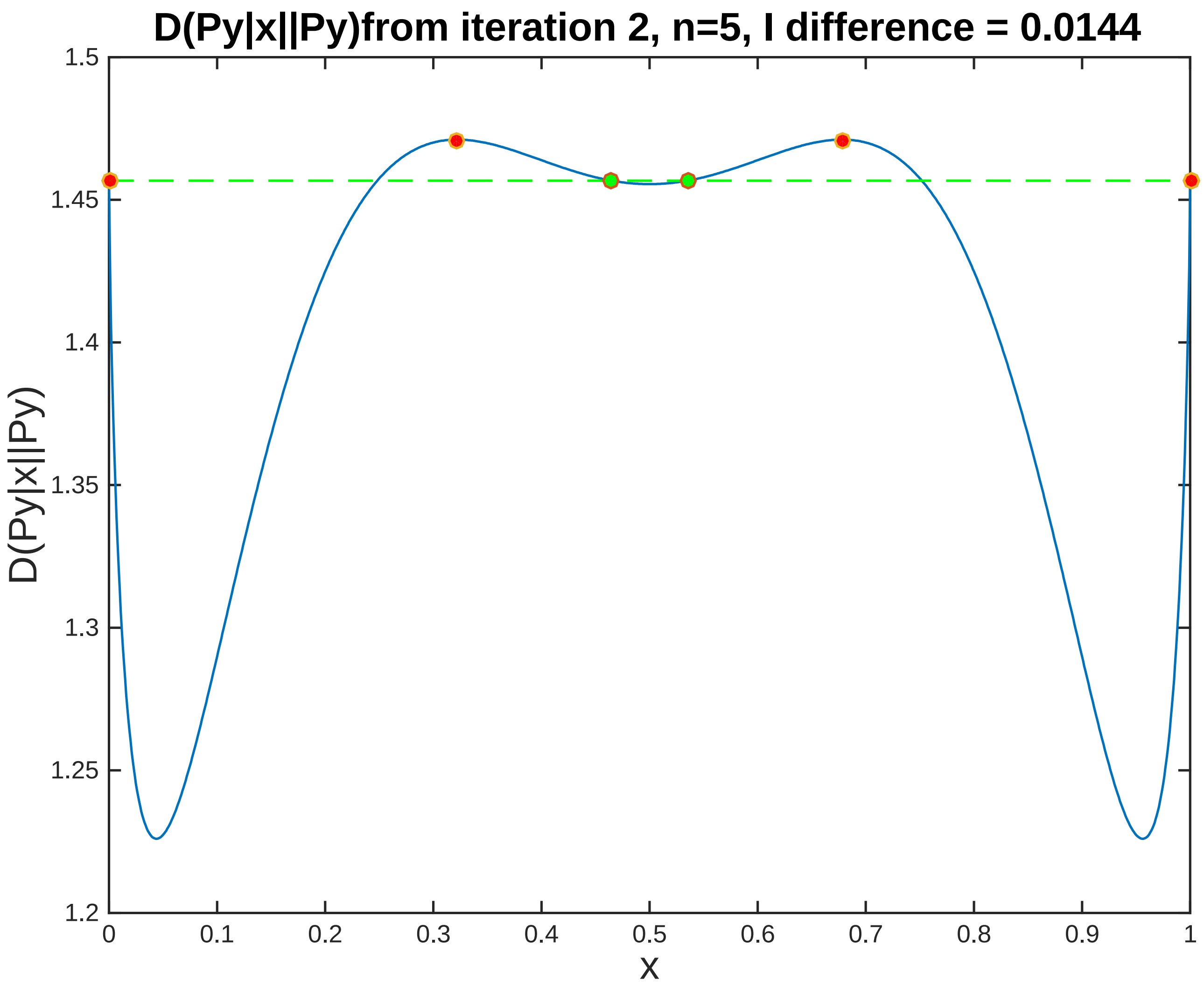
UCLA

Split the center, shift towards maxima

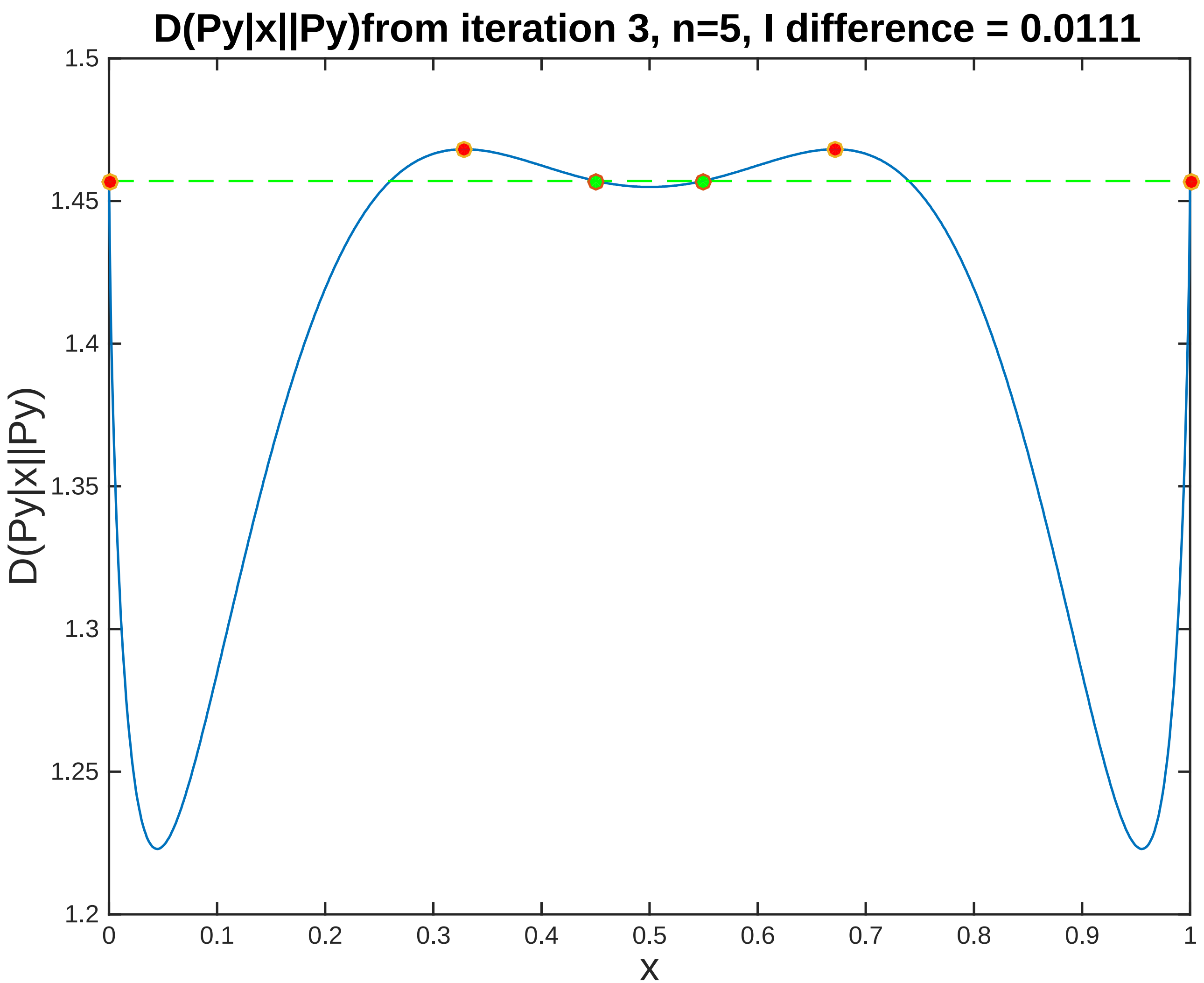


UCLA

Shift towards maxima

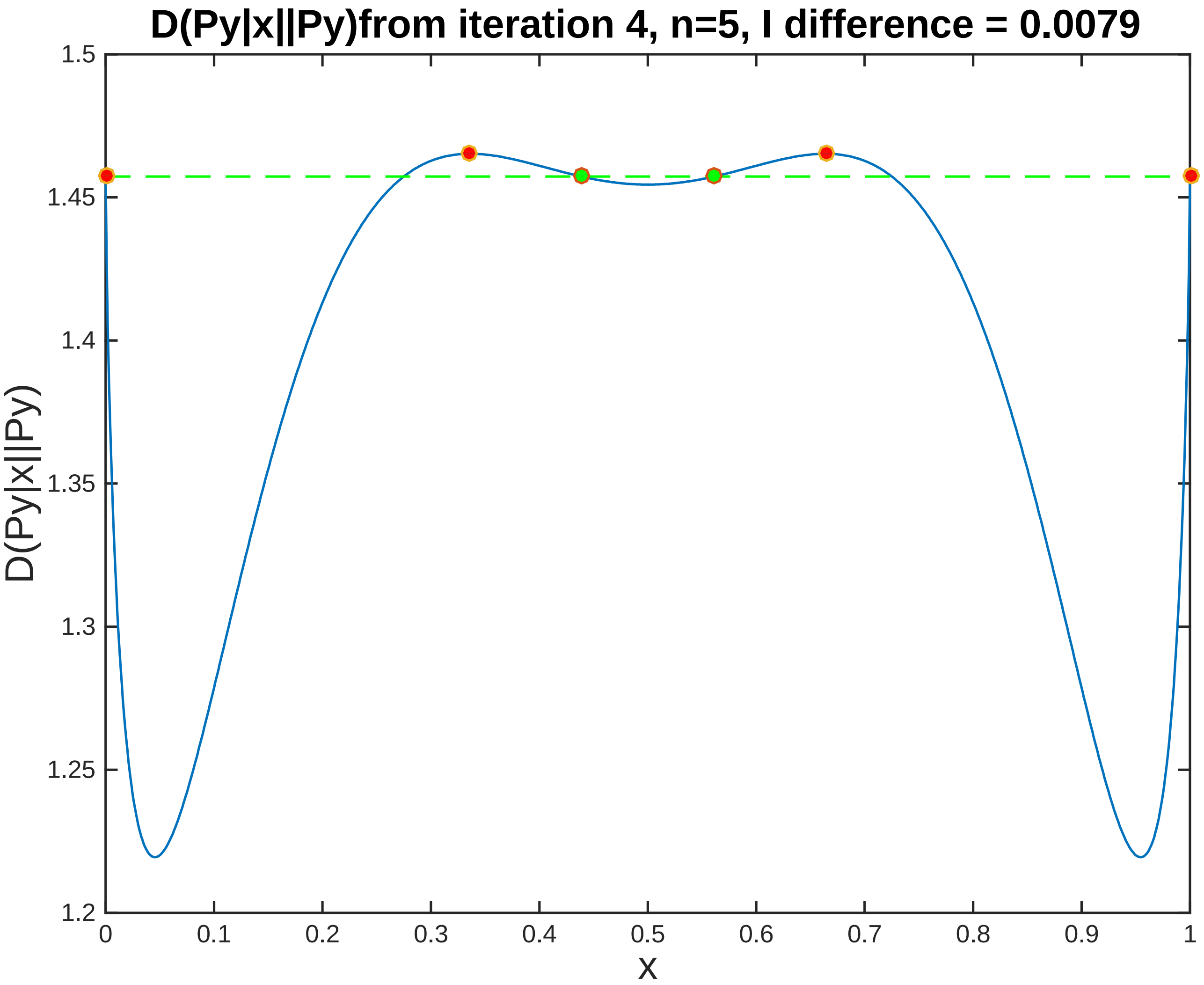


Shift towards maxima



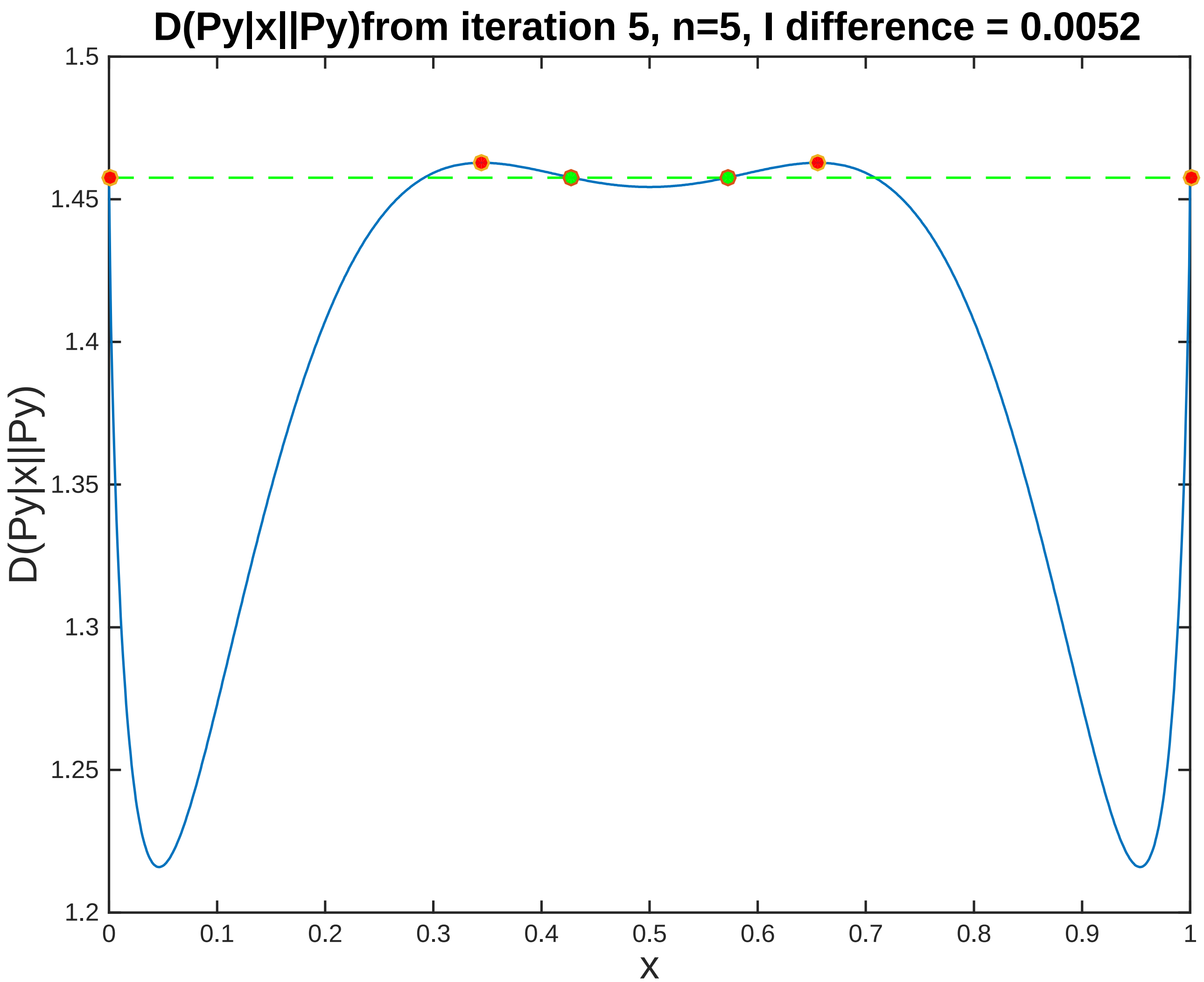
UCLA

Shift towards maxima

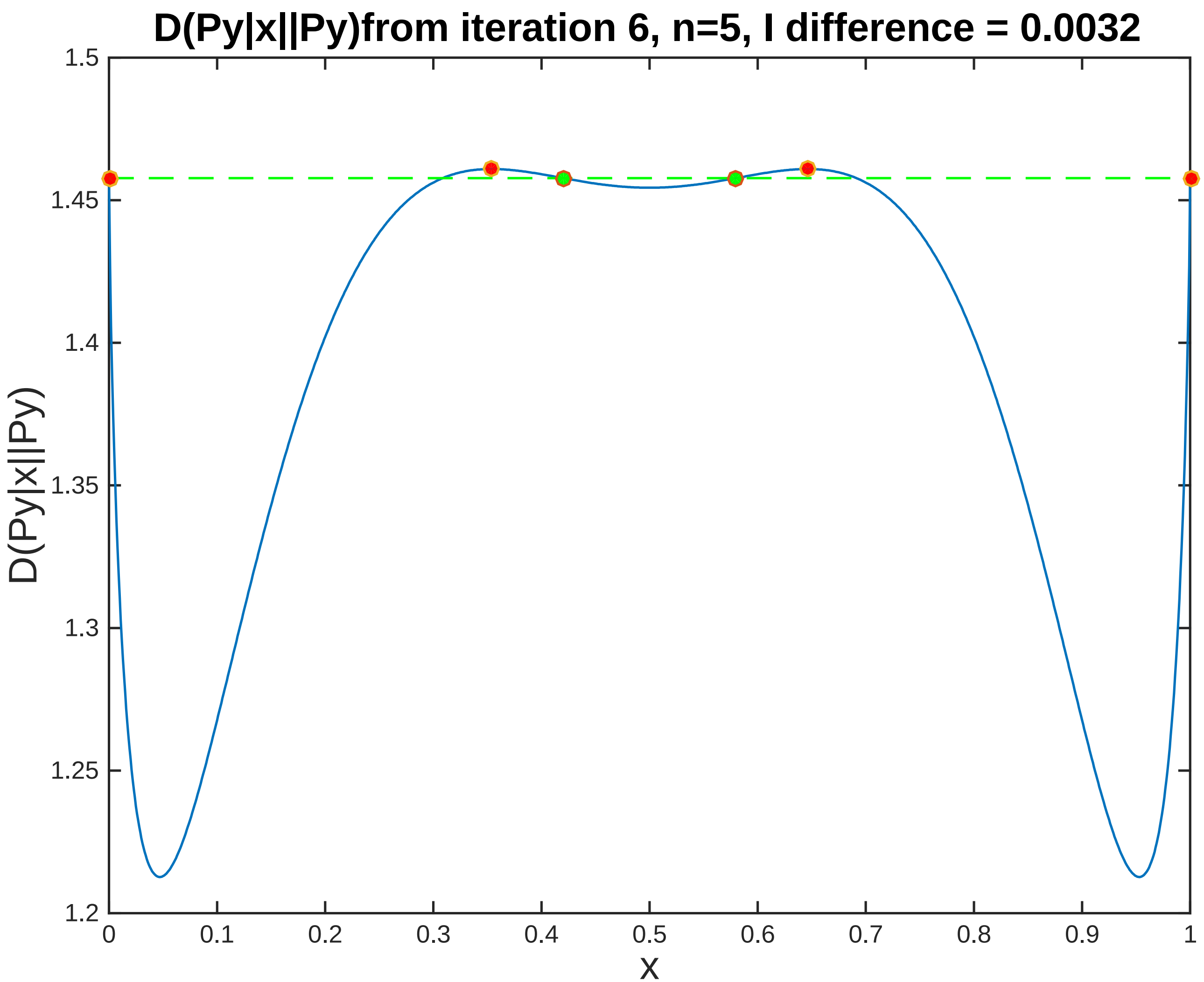


UCLA

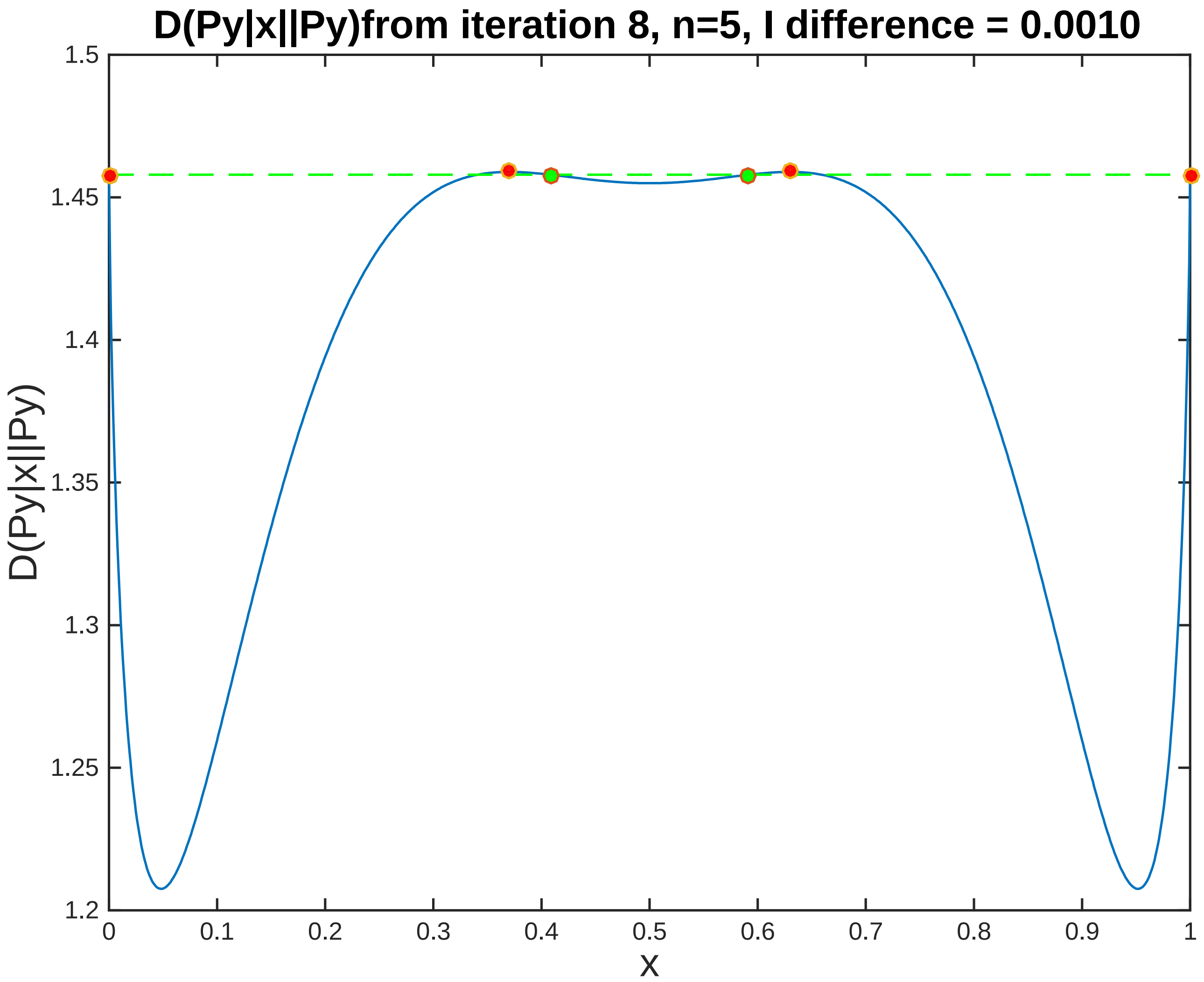
Shift towards maxima



Shift towards maxima

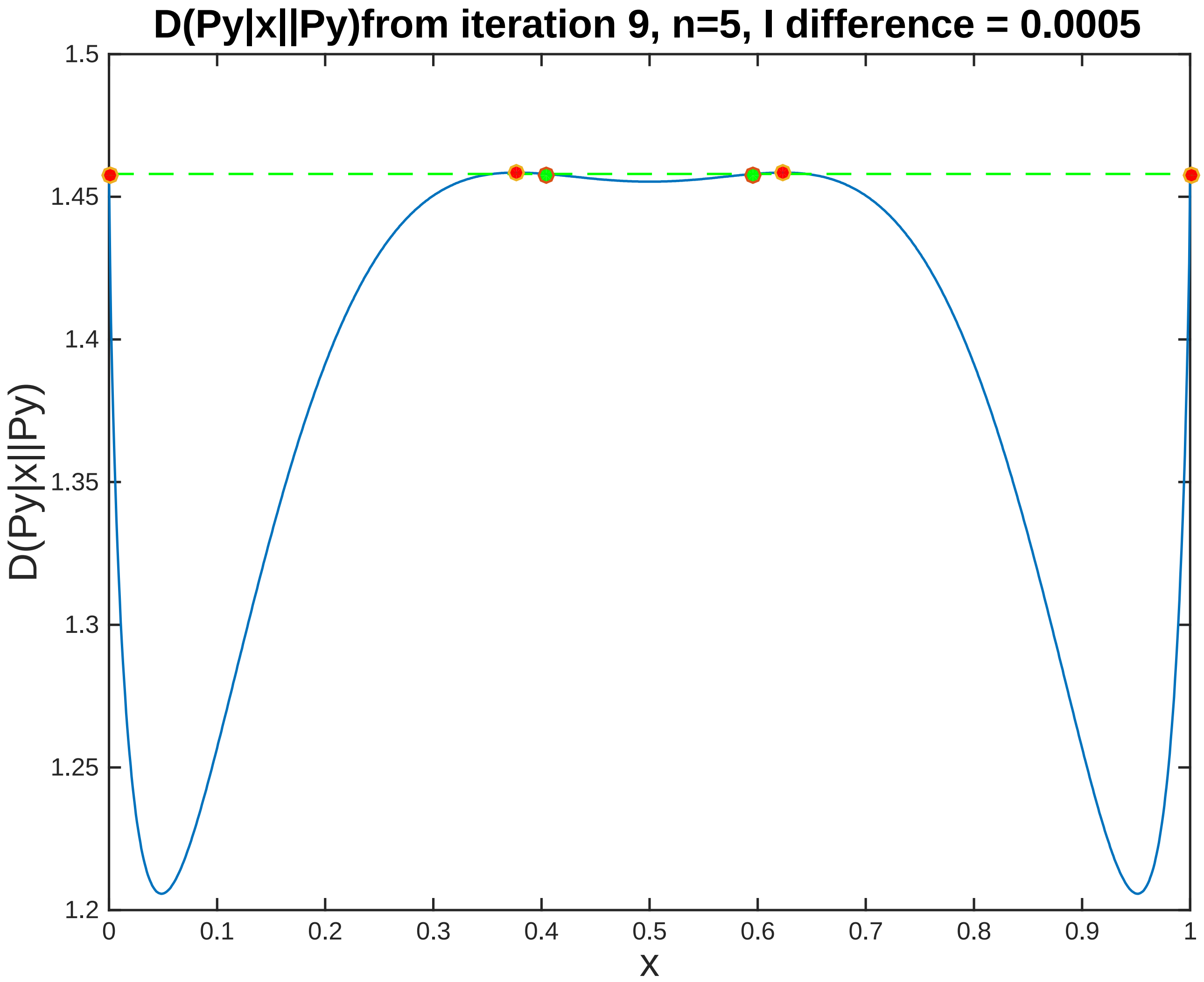


Shift towards maxima

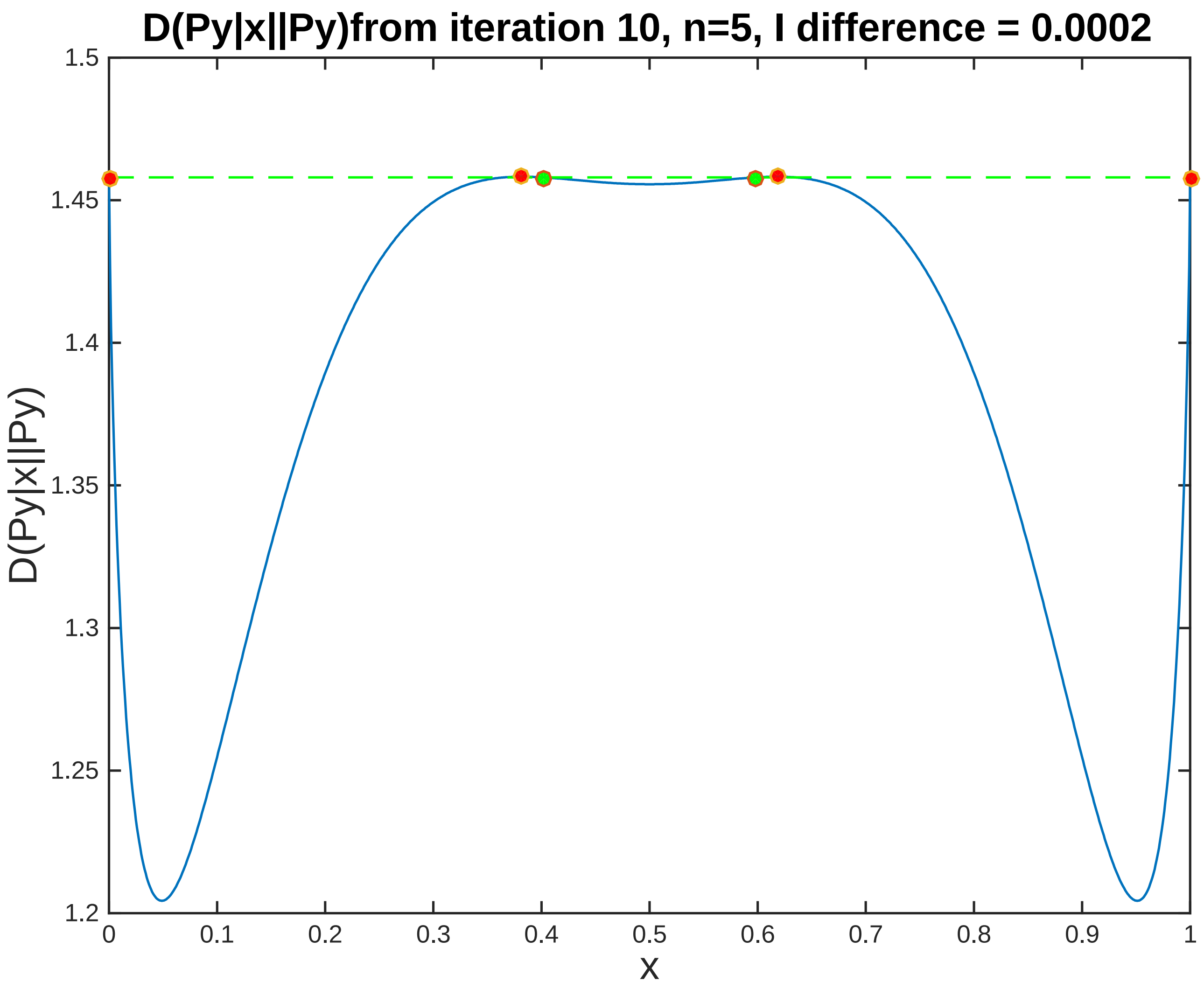


UCLA

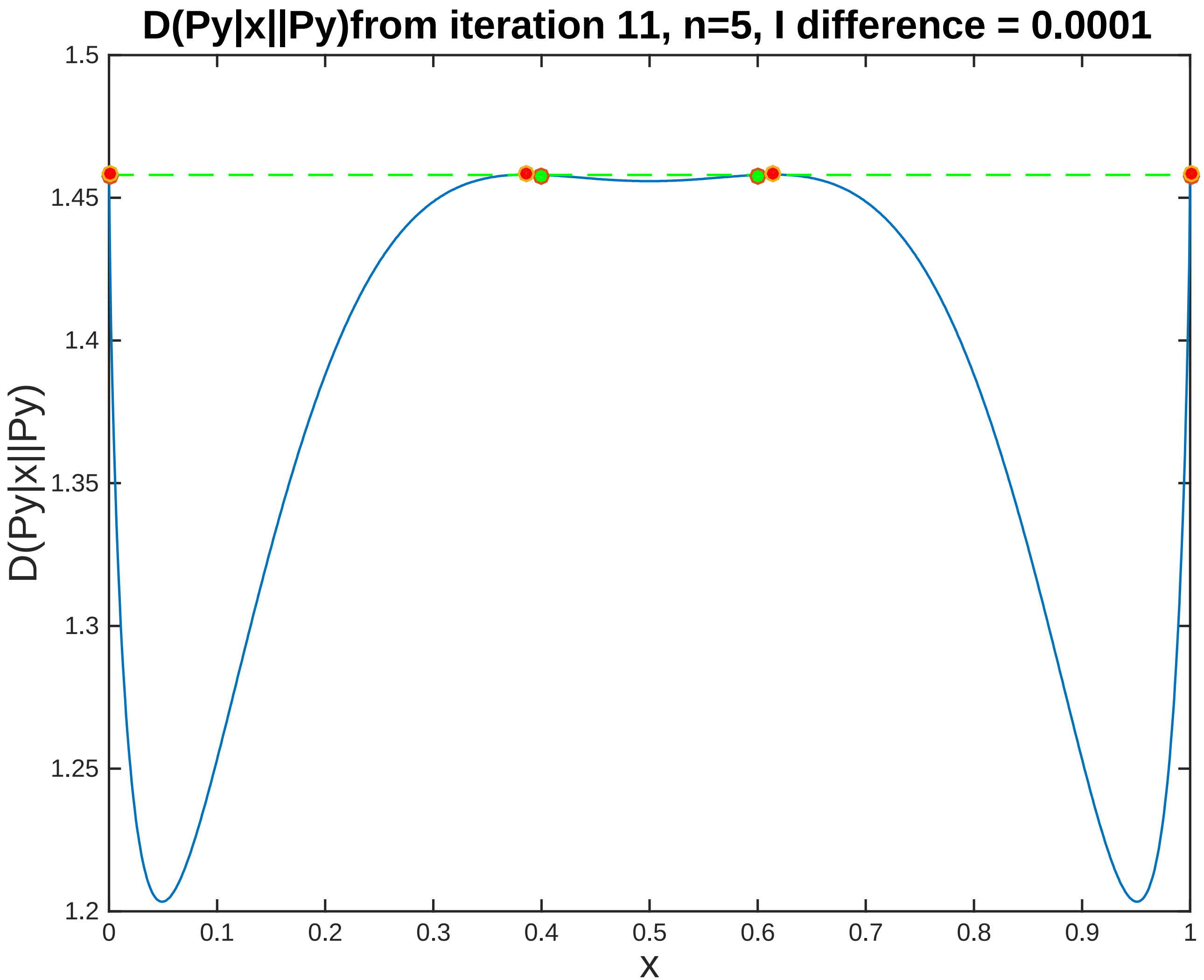
Shift towards maxima



Shift towards maxima

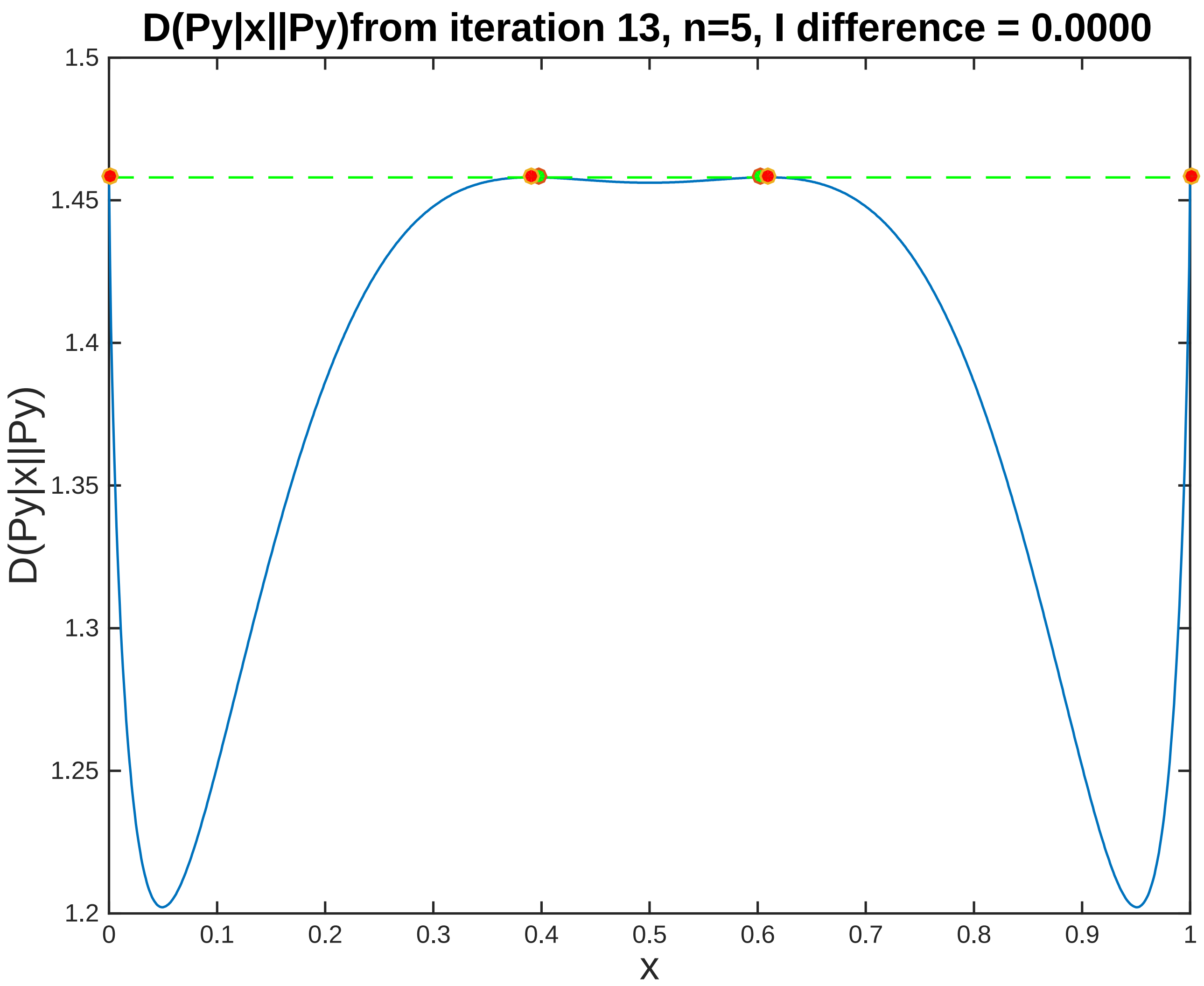


Shift towards maxima



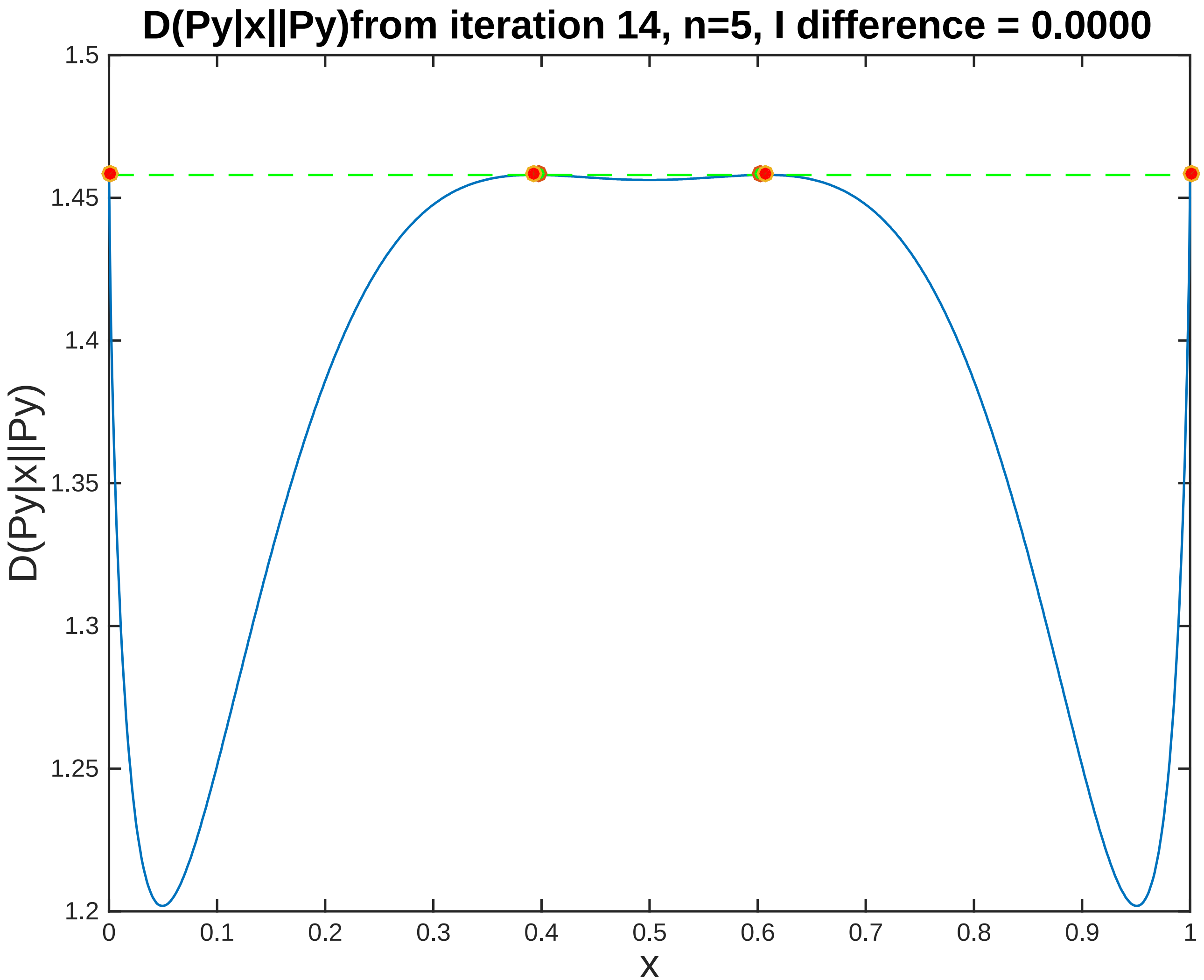
UCLA

Shift towards maxima



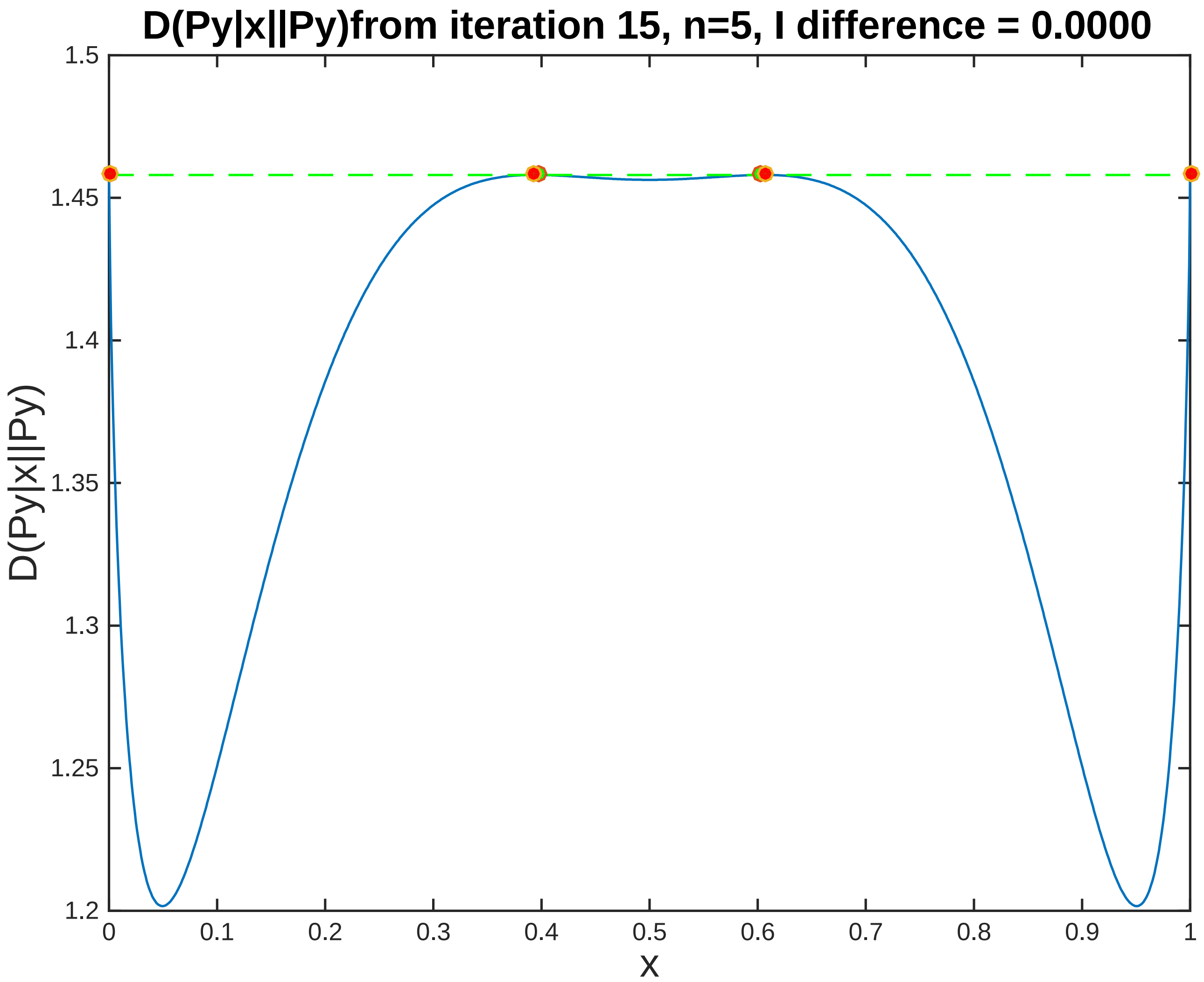
UCLA

Shift towards maxima



UCLA

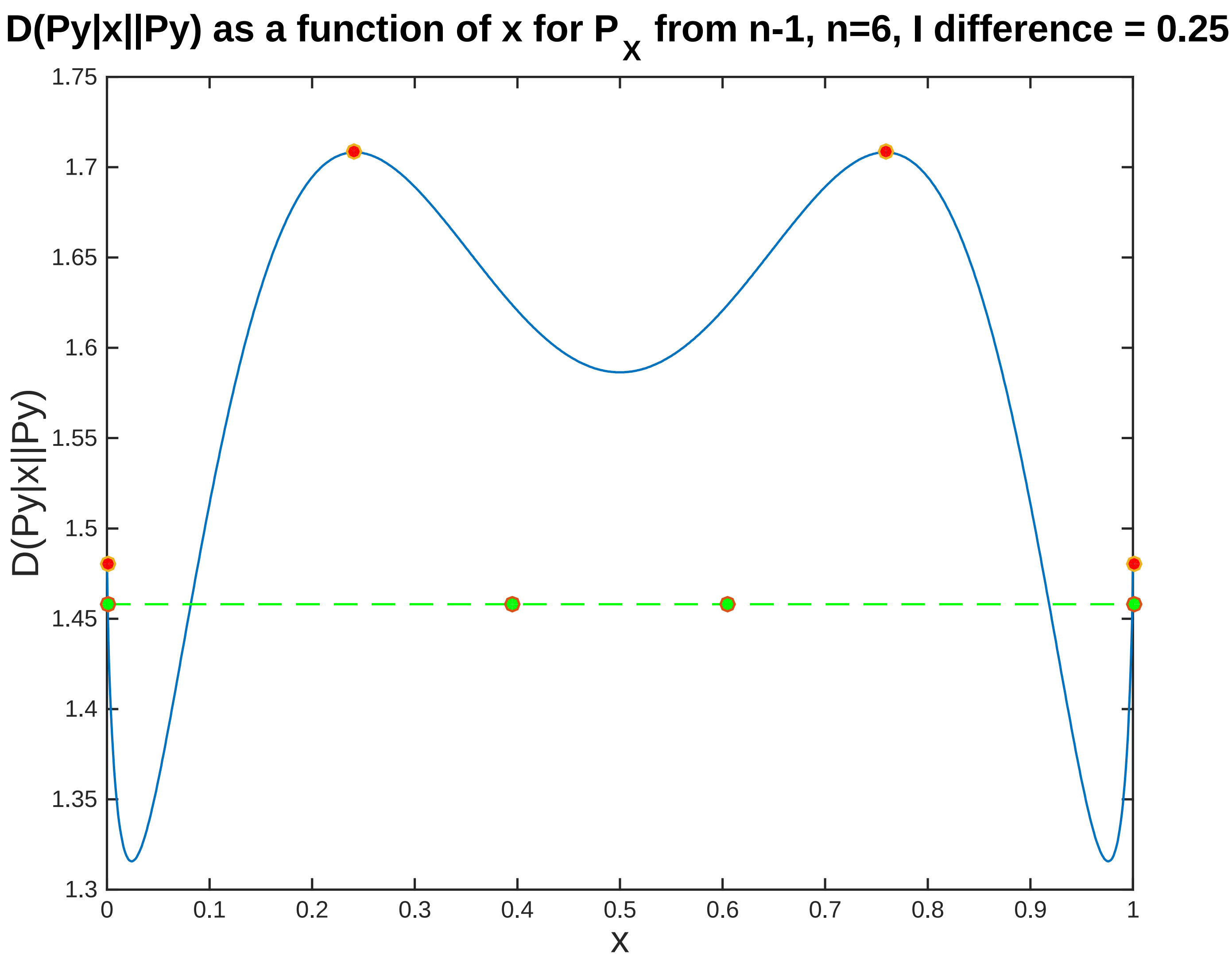
Done!



D(Py|x||Py)from iteration 15, n=5, I difference = 0.0000
D(Py|x||Py)
x
1.5
1.45
1.4
1.35
1.3
1.25
1.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

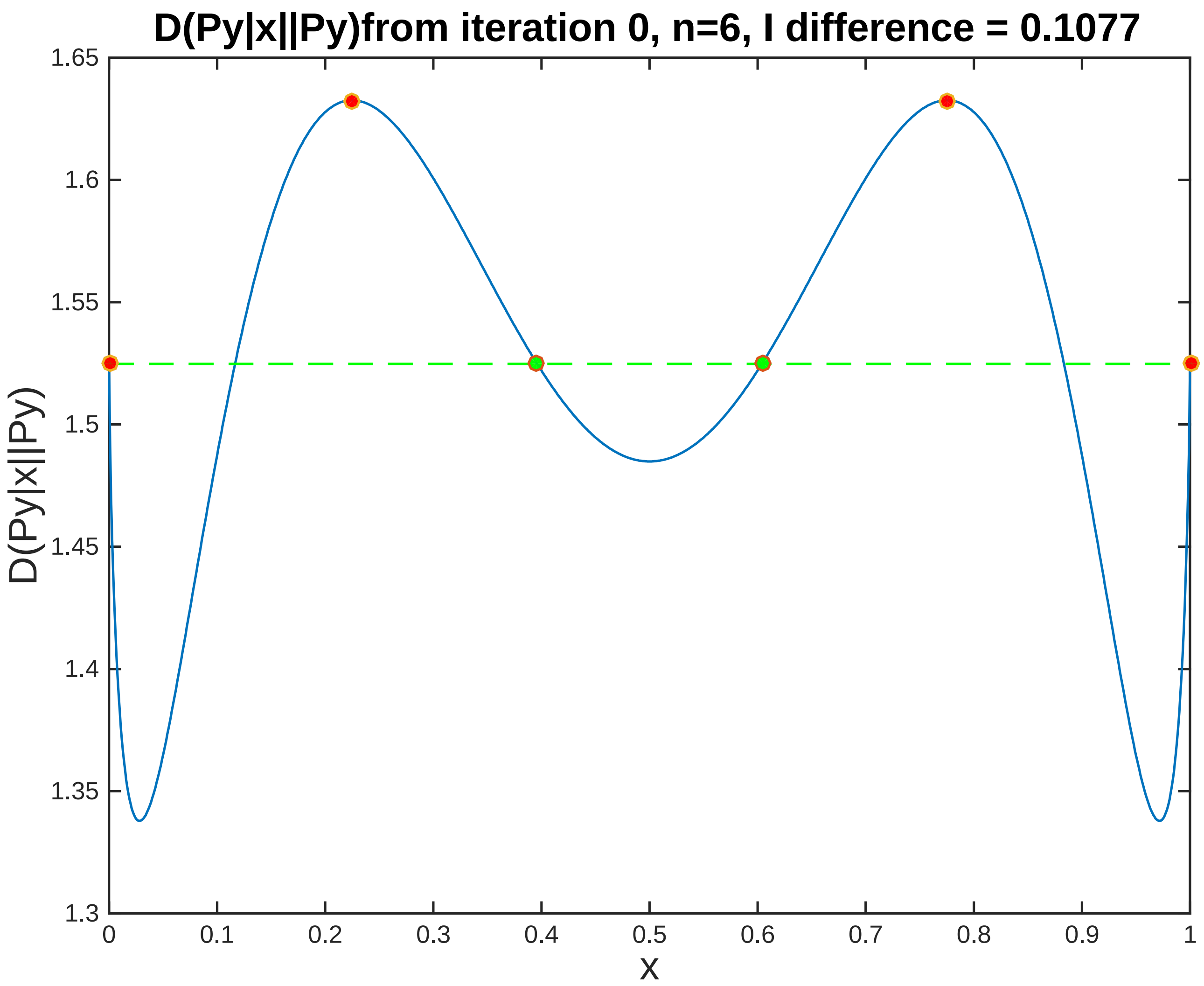
UCLA

Now for $n=6$, start with $n=5$ solution

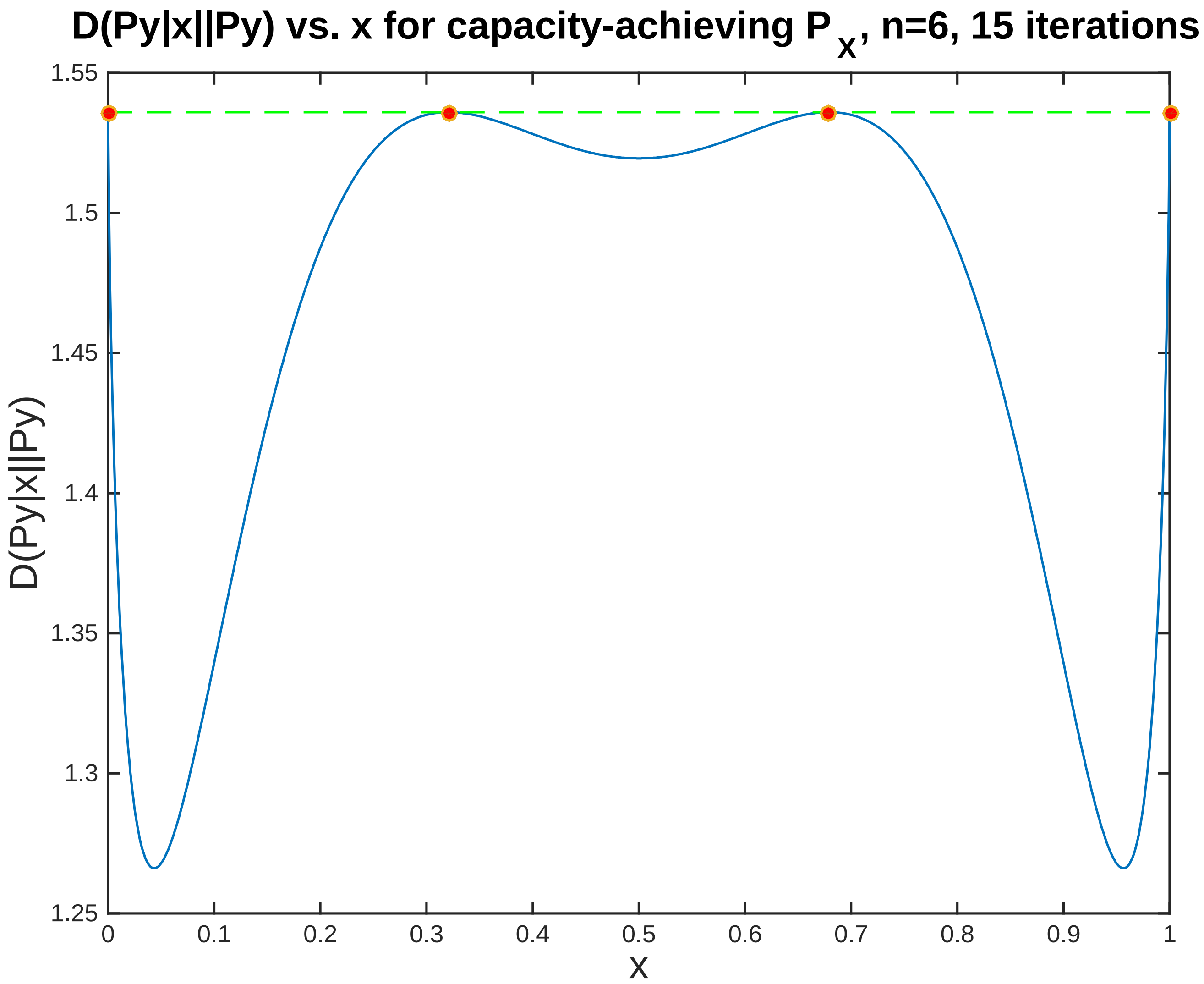


UCLA

$n=5$ solution for $n=6$, after Blahut-Arimoto

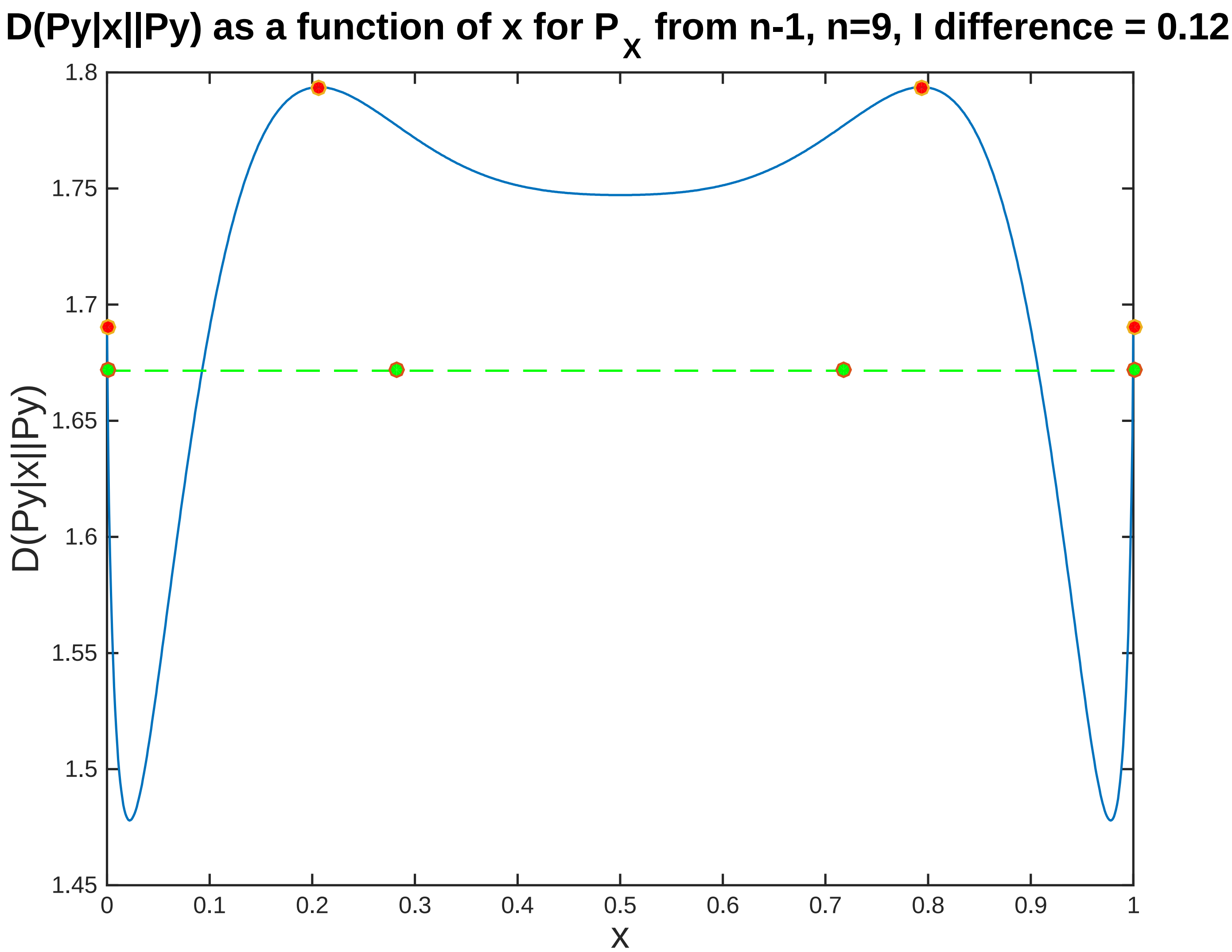


15 iterations later



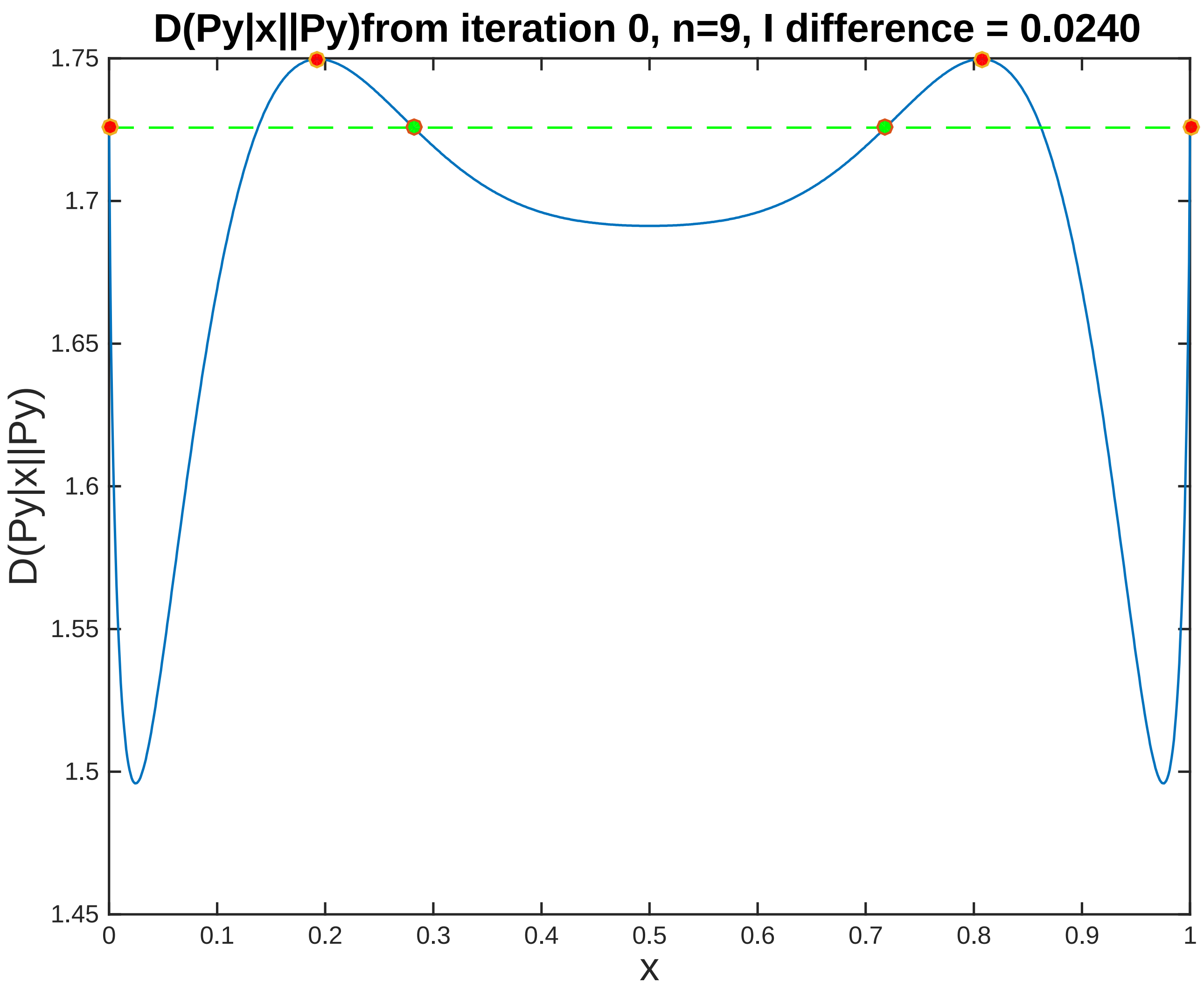
UCLA

Now for $n=9$, start with $n=8$ solution



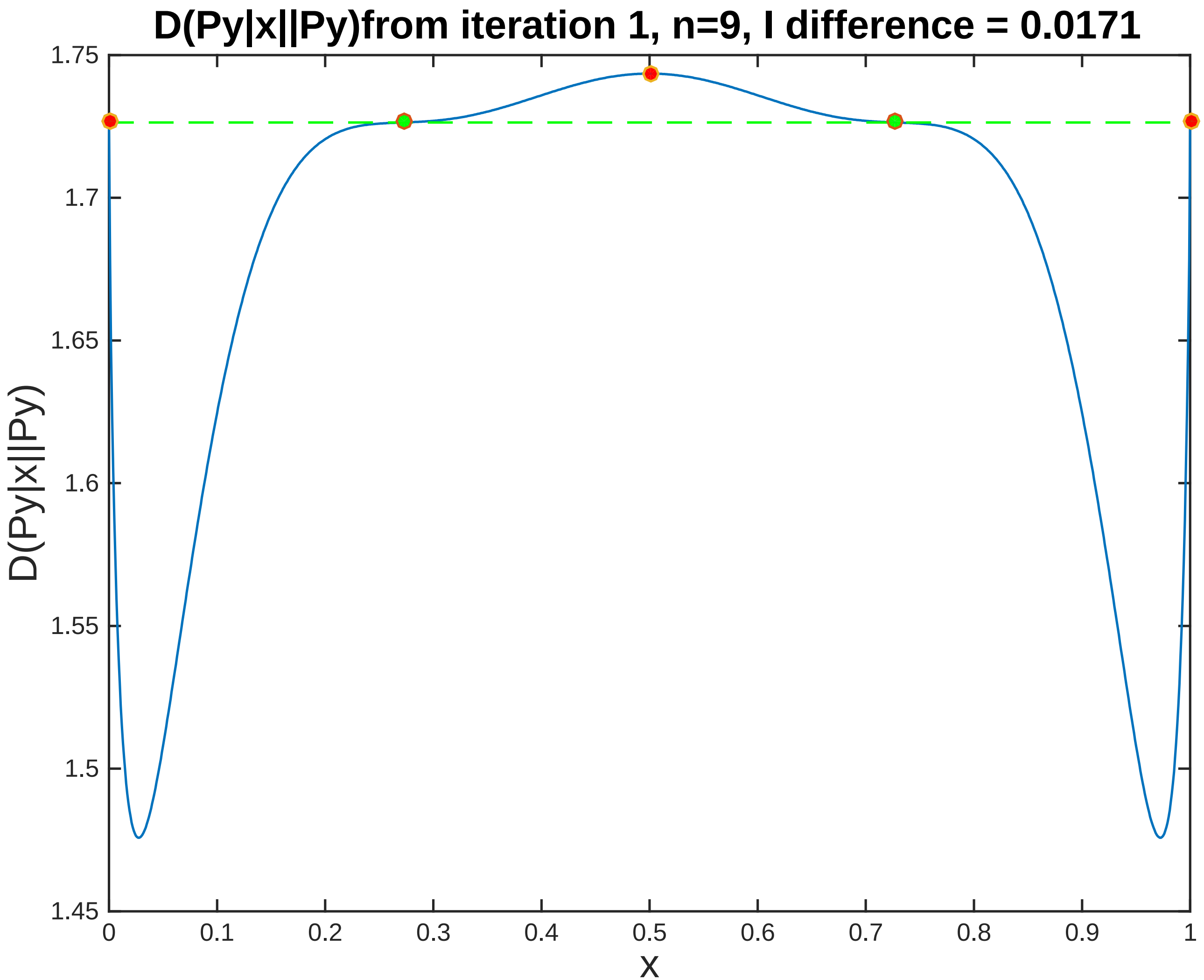
UCLA

$n=8$ solution for $n=9$, after Blahut-Arimoto

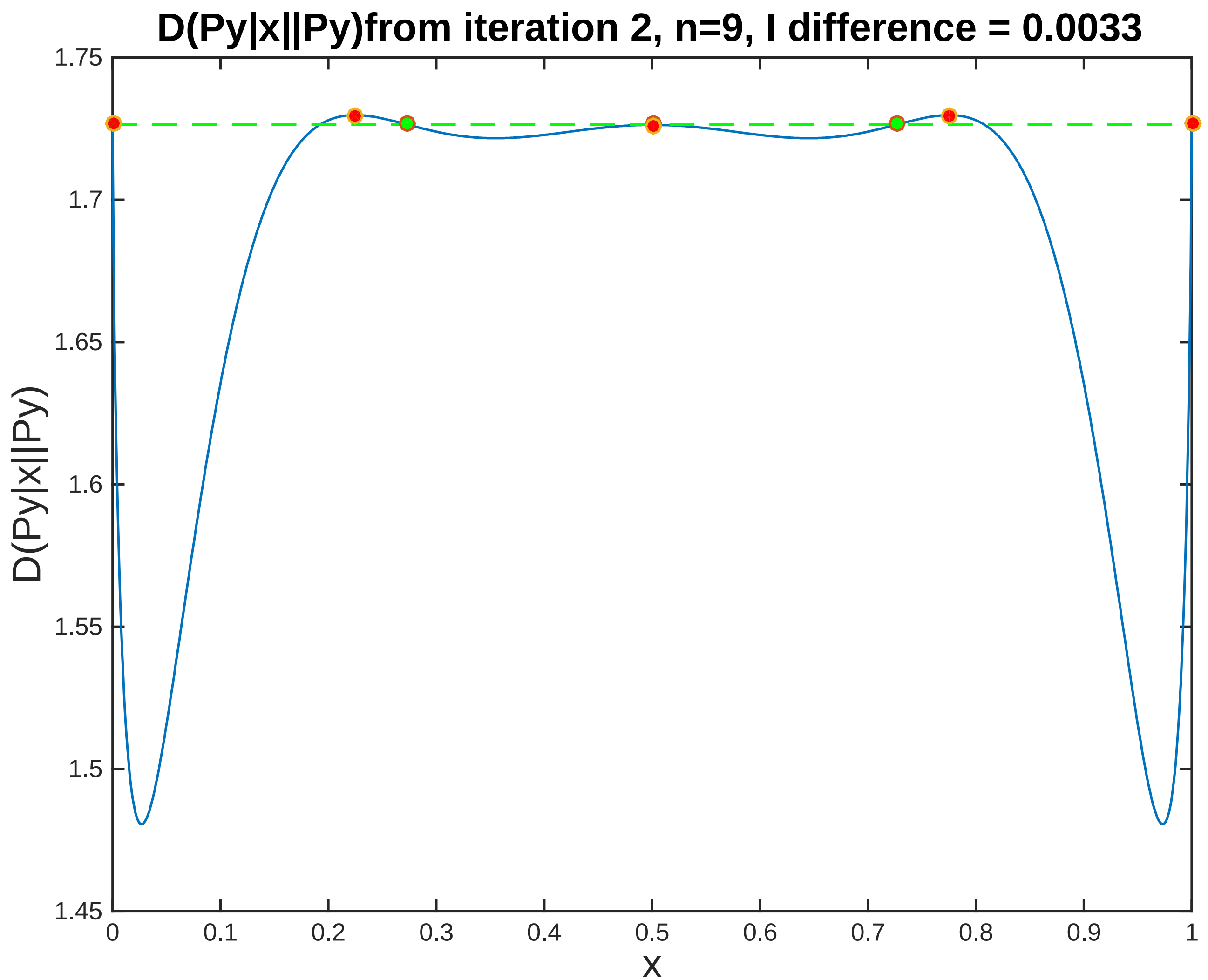


UCLA

Bump in the middle. Need new mass point.

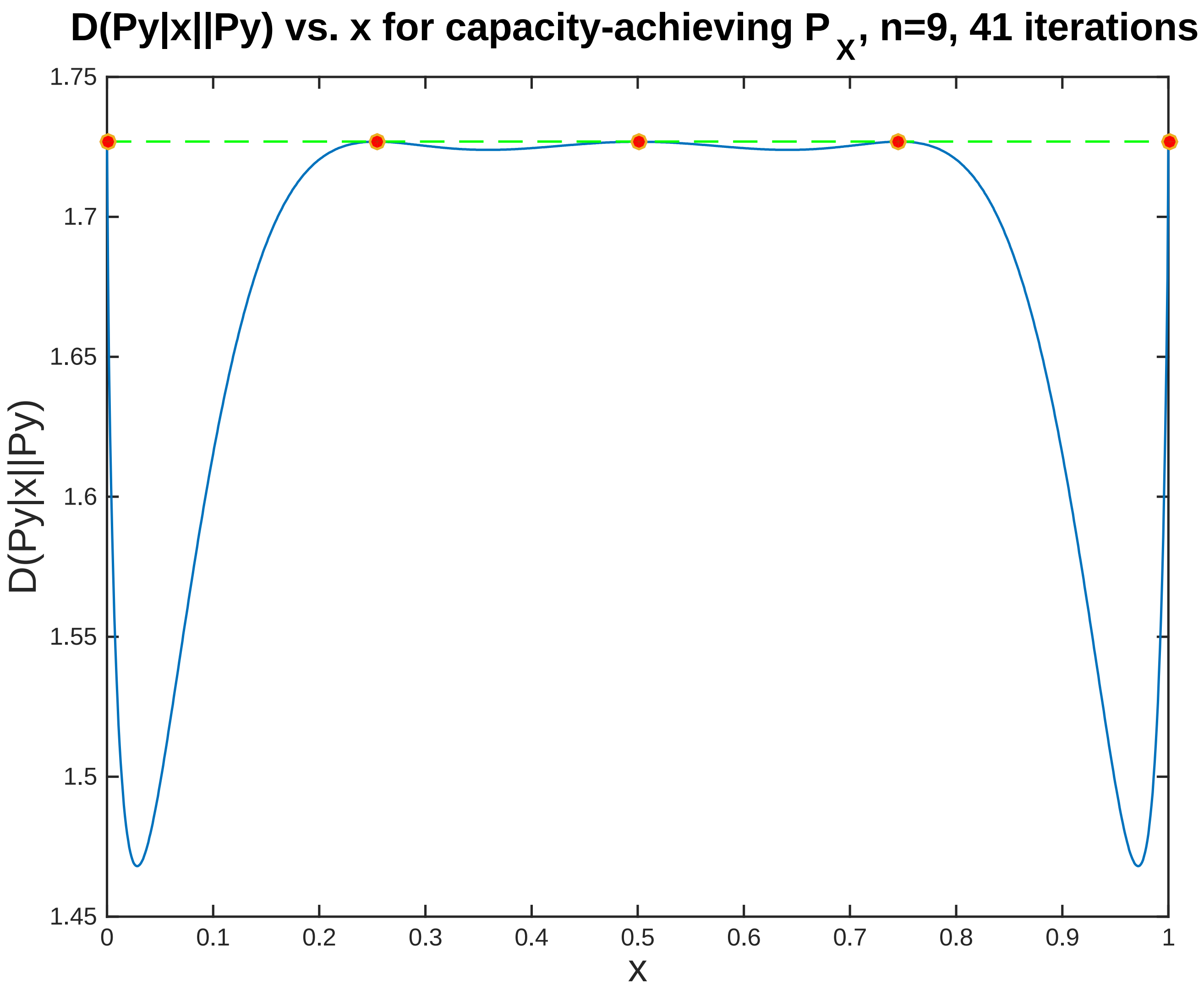


Bump in the middle. Need new mass point.



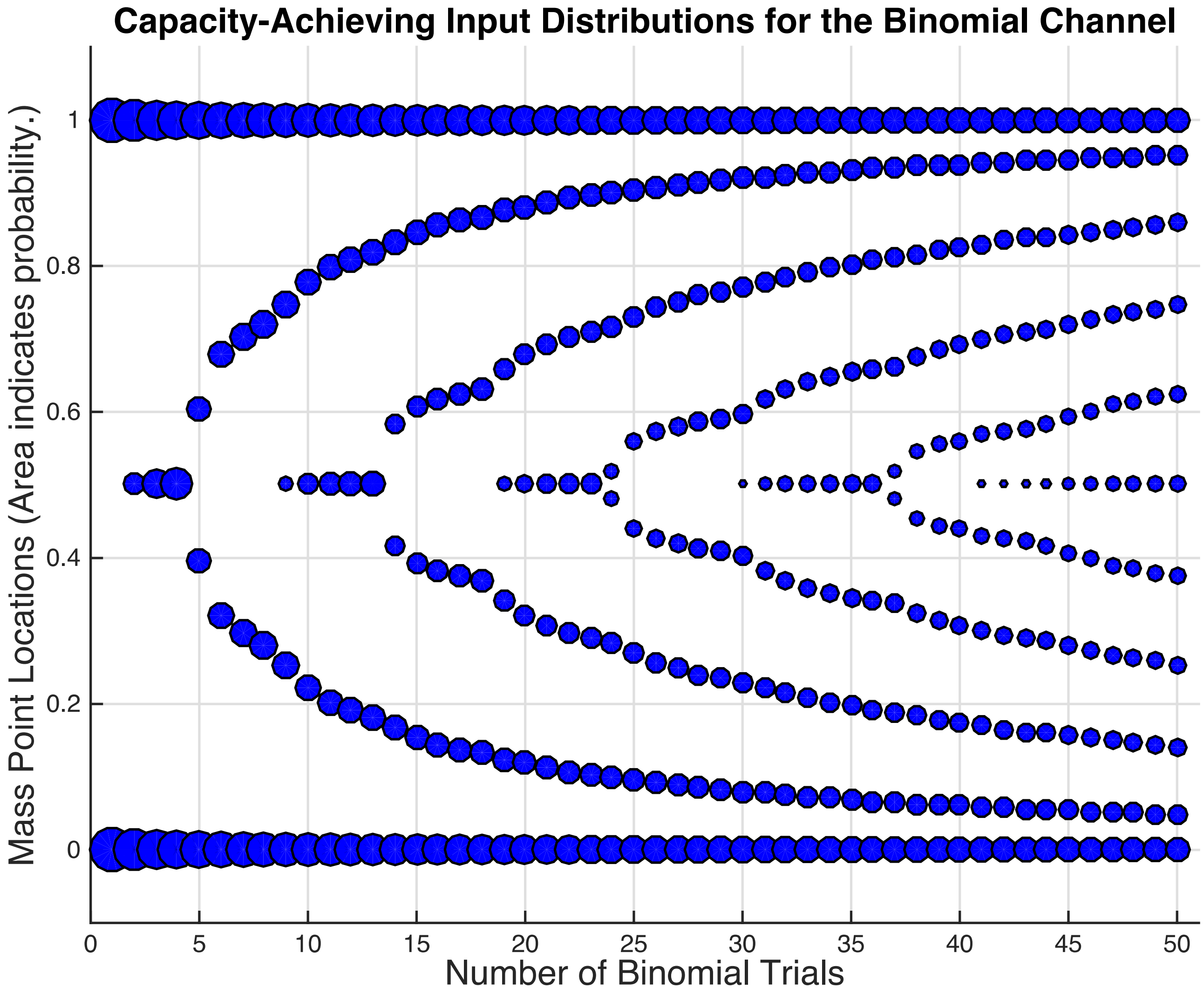
UCLA

All is well, with 5 mass points for $n=9$.



UCLA

Capacity Achieving Distributions



UCLA

Dynamic Assignment Blahut-Arimoto (DAB)

If the termination condition is not met, the mass point locations need to be adjusted so that $D\left(P_{Y|X=x_{\max}^{(k)}} \| P_Y\right)$ is reduced. There are three possible adjustments as follows:

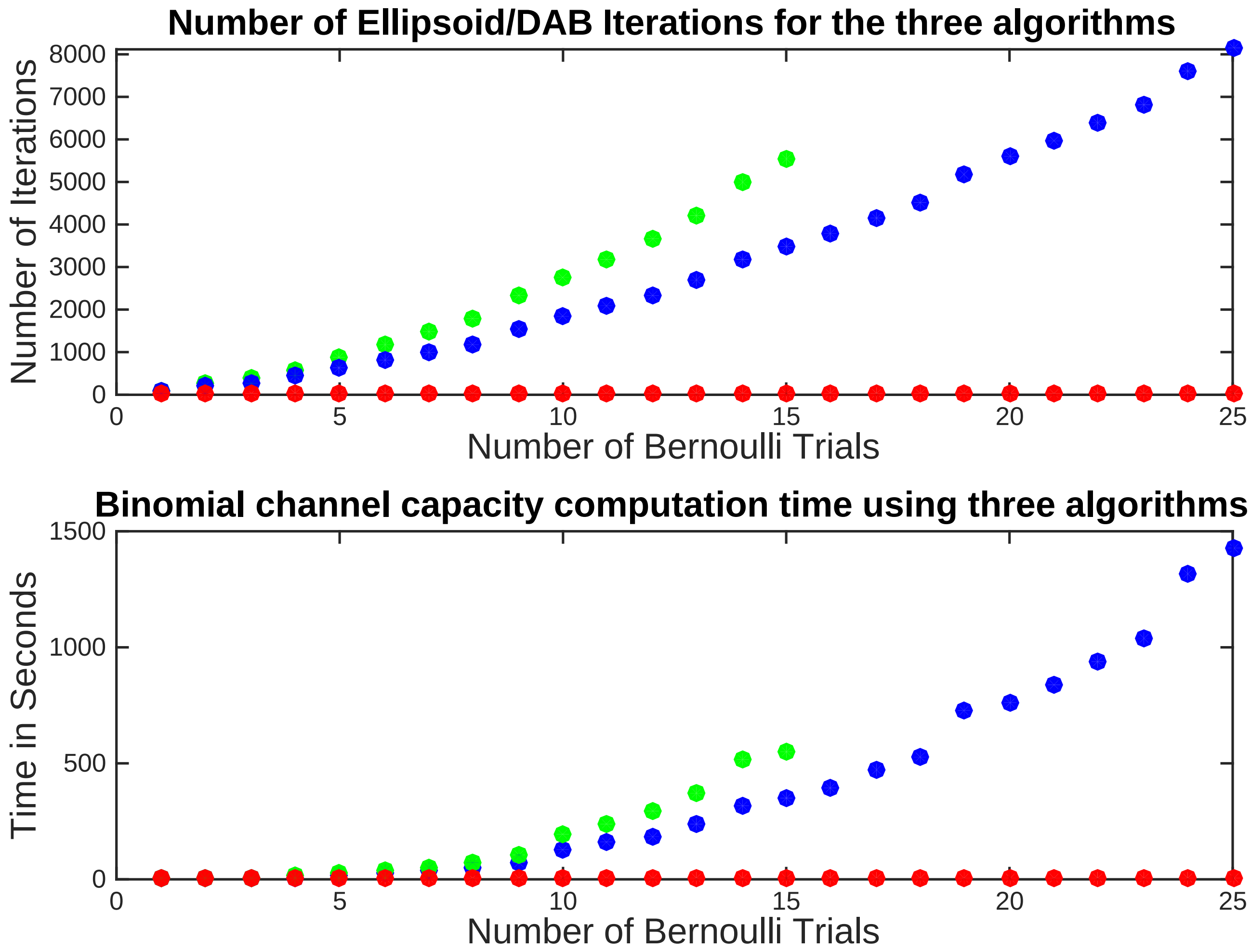
1) If the current number of mass points is even and $x_{\max}^{(k)}$ is closer to 0.5 than any of the mass points, a new mass point is introduced at 0.5.
2) If the current number of mass points is odd and $x_{\max}^{(k)}$ is closest to the mass point at 0.5, then this mass point splits into two mass points $x = 0.5 \pm \delta(x_{\max}^{(k)} - 0.5)$.
3) If neither of the above two conditions is met, then DAB identifies the mass point location x_{closest} that is closest to $x_{\max}^{(k)}$, not including the mass points at zero and one, which never move. This mass point is moved to

$$x_{\text{new}} = x_{\text{closest}} + \delta\left(x_{\max}^{(k)} - x_{\text{closest}}\right). \tag{23}$$

Also, the point at location $1 - x_{\text{closest}}$ is moved to $1 - x_{\text{new}}$ preserving symmetry.

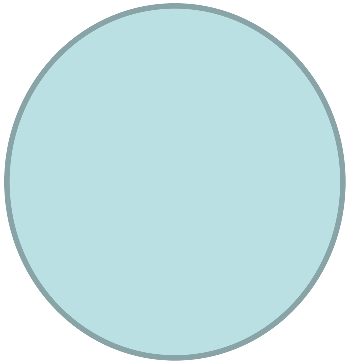
UCLA

DAB rocks!



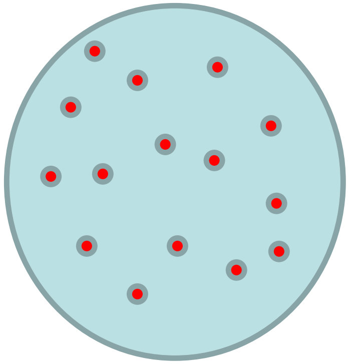
Number of Ellipsoid/DAB Iterations for the three algorithms
Number of Iterations
8000
7000
6000
5000
4000
3000
2000
1000
0
0
5
10
15
20
25
Number of Bernoulli Trials
Binomial channel capacity computation time using three algorithms
Time in Seconds
1500
1000
500
0
0
5
10
15
20
25
Number of Bernoulli Trials

A Molecular Communication Channel



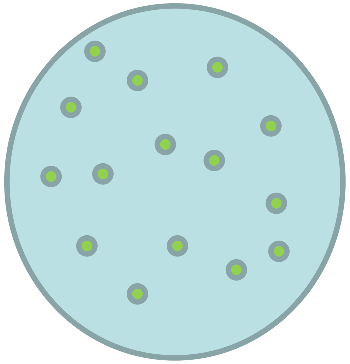
Transmitter

A Molecular Communication Channel



Generates n particles in time τ.

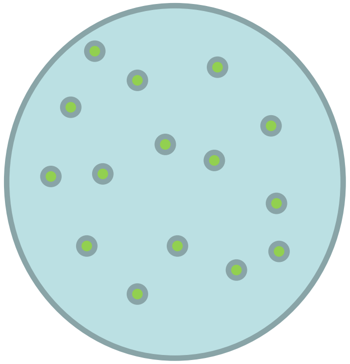
A Molecular Communication Channel



Generates n particles in time τ.

Particles selected for release with probability σ.

A Molecular Communication Channel



Generates n particles in time τ.

Particles selected for release with probability σ.

Selected particles *actually* released with probability α.

A Molecular Communication Channel

Generates n particles in time τ.

Particles selected for release with probability σ.

Selected particles *actually* released with probability α.

Particles make it to receiver with probability ρ.

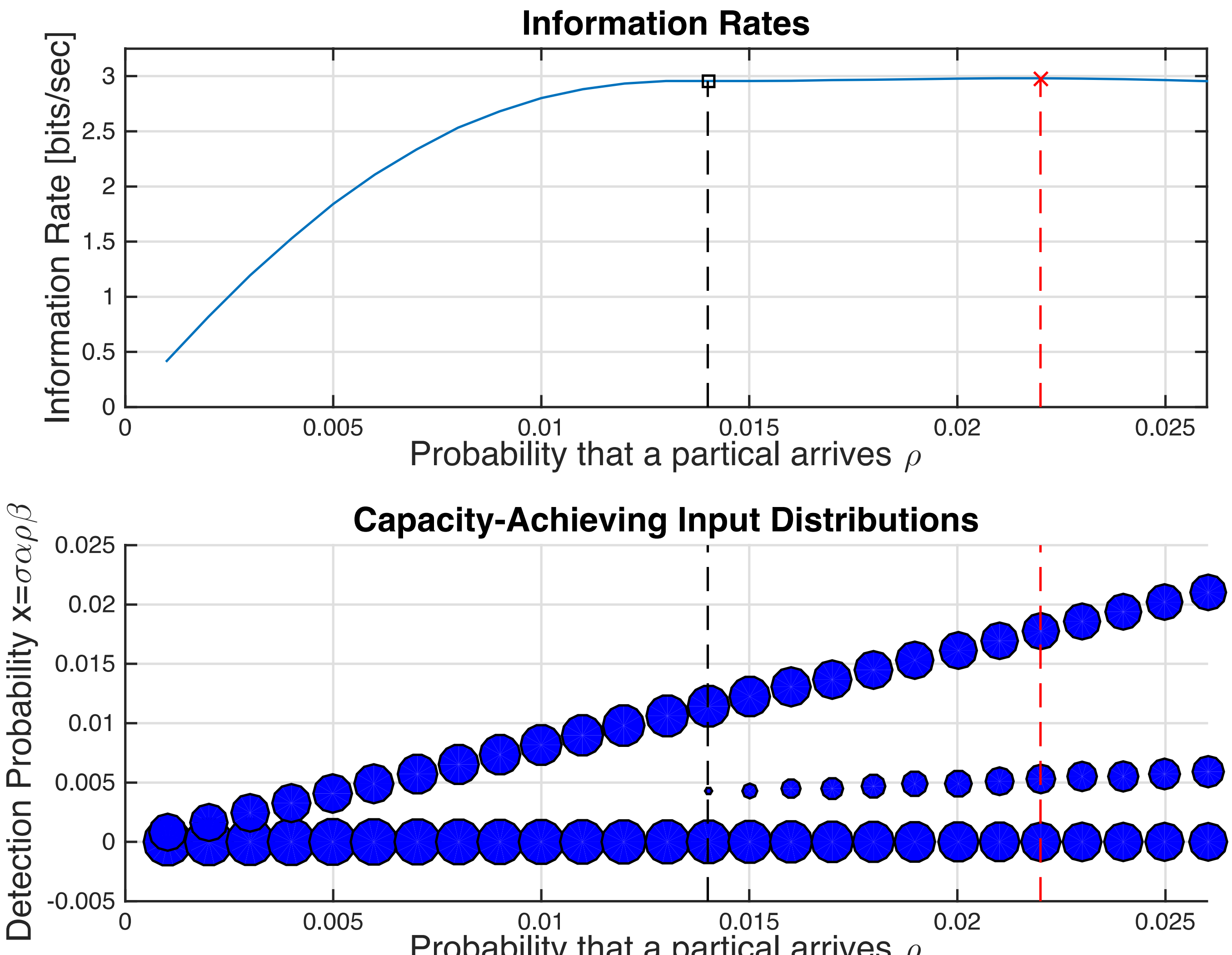
Particles detected by receiver with probability β.

A Binomial Channel!

Channel input X is probability of Success at every step. Lies on $[0, \alpha\rho\beta]$

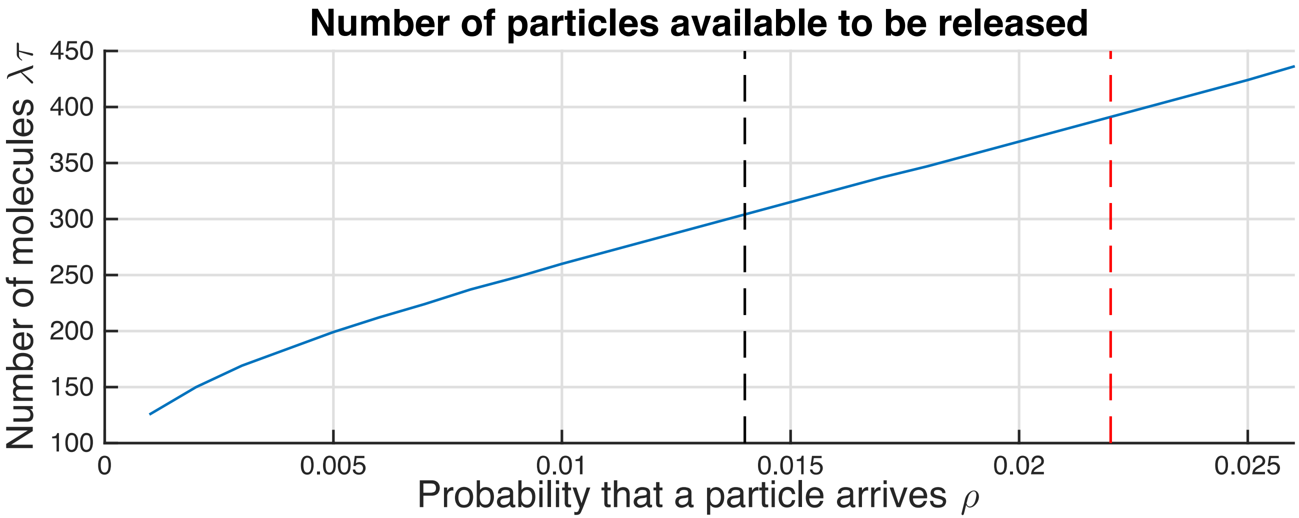
Channel output Y is number of detected particles of n generated.

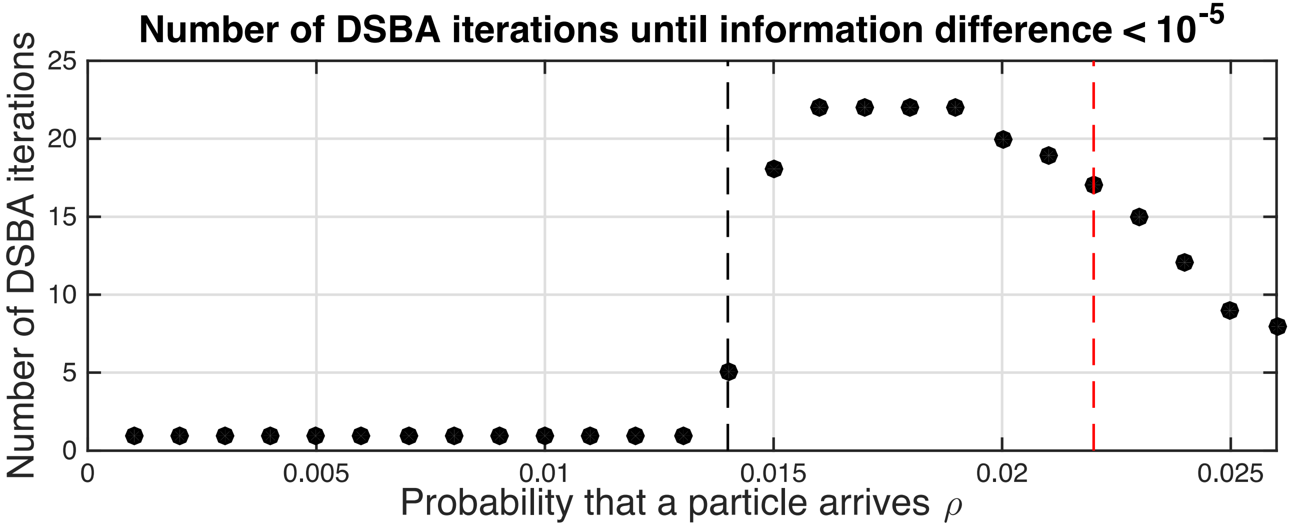
Molecular Communication Capacity Results



UCLA

Molecular Communication Capacity Results





UCLA

Conclusions

- Dynamic Assignment Blahut-Arimoto uses Csiszar's Min-Max Capacity Theorem to compute capacity in cases where the alphabet is uncountable but capacity is achieved by a (small) finite support.
- It's much faster (and easier) than the ellipsoid algorithm.
- We used this method to study a molecular communication channel.

UCLA