# Surrogate-Channel Design of Universal LDPC Codes

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***Abstract*— A universal code is a code that may be used across a number of different channel types or conditions with little degradation relative to a good single-channel code. The explicit design of universal codes, which simultaneously seeks to solve a multitude of optimization problems, is a daunting task. This letter shows that a single channel may be used as a surrogate for an entire set of channels to produce good universal LDPC codes. This result suggests that sometimes a channel for which LDPC code design is simple may be used as a surrogate for a channel for which LDPC code design is complex.**

**We explore here the universality of LDPC codes over the BEC, AWGN, and flat Rayleigh fading channels in terms of decoding threshold performance. Using excess mutual information as a performance metric, we present design results which support the contention that an LDPC code designed for a single channel can be universally good across the three channels.**

***Index Terms*— low-density parity-check (LDPC) codes, universal codes, eIRA codes, density evolution, code design.**

## I. Introduction

LOW-density parity-check (LDPC) codes have been shown to be capacity-approaching on many different channels, including the binary erasure channel (BEC) [1], the AWGN [2] channel, and Rayleigh fading channels [3]. Due to the versatility and robustness of LDPC codes over different channels, many consider a given LDPC code as potentially being universal [4]-[14]. A universal code is a code that may be used across a number of different channel types or conditions with little degradation relative to a good single-channel code. We explore universal LDPC codes in this letter and show that a single channel may be used as a surrogate in the design of good universal LDPC codes for a set of channels. This result also suggests that it may be possible to use as a surrogate a channel for which LDPC code design is simple to design a code for a channel for which design is complex. For example, we show that the binary erasure channel (BEC) may in some cases be used as a surrogate in the design of LDPC codes for the Rayleigh fading channel.

An initial look at the surrogate-channel approach was presented in our previous work [6]. We showed that extended irregular repeat-accumulate (eIRA) codes [5], [6], are "universally" good on the burst-erasure channel (BuEC), the burst-erasure channel with Gaussian noise (BuEC-G), and the flat Rayleigh fading channel. In this paper, we present a more thorough investigation of the design of codes via surrogates for code rates 1/4, 1/2, and 3/4. We compare decoding thresholds of unconstrained LDPC (u-LDPC) and eIRA-constrained degree distributions resulting from surrogate code designs to optimal decoding thresholds for the target channels. The ability to design codes via a surrogate channel for use on other channels provides a practical path to universal code design.

The decoding thresholds are found numerically via density evolution. Except for special cases (e.g., the BEC), density evolution for the sum-product algorithm (SPA) decoder involves keeping track of an array of $N$ quantized samples of the probability density function, which is an $N$-dimensional problem, and can be highly complex and time-consuming. Several approximation techniques have been devised to transform the $N$-dimensional problem into a one- or two-dimensional problem, most notably the Gaussian approximation (GA) [7].

Our approach might be called the surrogate-channel approximation. In this letter we examine the decoding threshold behavior of u-LDPC and eIRA LDPC codes over the BEC, the binary-input AWGN channel, and the binary-input flat Rayleigh fading channel under different design criteria, that is, for different surrogate channels. Our main results are: (1) an LDPC code can be designed to be universally good across all three channels, (2) the Rayleigh channel is a particularly good surrogate in the design of LDPC codes for the three channels, and (3) with the Rayleigh channel as the target, the BEC may be used as a faithful surrogate in the design of eIRA codes of rate greater than or equal to 1/2, and there is a throughput loss of less than 6% if the BEC is used as a surrogate to design u-LDPC codes.

## II. Code Designs under Various Criteria

In this section, we take a look at the sum-product algorithm decoding threshold performance [8] of LDPC codes of rate 1/4, 1/2, and 3/4 designed under different criteria. Discretized density evolution [2] is used for the AWGN and Rayleigh channels. In all cases, the maximum number of decoder iterations, $N = 1000$, and the stopping threshold is $P_b = 10^{-6}$. We study the 10 LDPC code design criteria listed in Table I whose resulting degree distributions may be found in [15].

We consider both a large and a small maximum variable-node degree ($d_v = 50$ and $d_v = 8$), the former to approach theoretical limits and the latter to accommodate low-complexity encoding and decoding. For each design criterion and for each target channel, we compare the threshold-to-capacity gaps for each degree distribution pair obtained. For the AWGN and Rayleigh channels these gaps are called "excess SNR" and for the BEC these gaps are called "excess $\alpha$", where $\alpha$ is the BEC erasure probability. With AWGN and Rayleigh as the target channels, Fig. 1 presents the excess SNR results for the 10 design criteria for all three code rates. This figure will be discussed shortly.

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Table I. Design Criteria.

| Entry | Type | Surrogate channel | $\mathbf{d}_v$ |
|---|---|---|---|
| 1 | LDPC | BEC | 50 |
| 2 | LDPC | AWGN-GA | 50 |
| 3 | LDPC | AWGN | 50 |
| 4 | LDPC | Rayleigh | 50 |
| 5 | LDPC | BEC | 8 |
| 6 | LDPC | AWGN-GA | 8 |
| 7 | LDPC | AWGN | 8 |
| 8 | LDPC | Rayleigh | 8 |
| 9 | eIRA | BEC | 8 |
| 10 | eIRA | Rayleigh | 8 |

We can repeat this for the case where the BEC is the target channel and excess $\alpha$ is the performance metric, but we will find that such a plot would be redundant in view of a unifying performance metric we now consider. Specifically, it is convenient (in fact, proper) to present all of our results in a single plot using as the performance metric excess mutual information (MI) ([4], [9]), defined as

$$\text{excess MI} = I(\rho^*) - R.$$

In this expression $R$ is the design code rate and $I(\rho^*)$ is the mutual information for channel parameter $\rho^*$ at the threshold. $\rho$ is erasure probability for the BEC and signal-to-noise ratio ($E_b/N_0$) for the AWGN and Rayleigh channels. Note that when the BEC is the target channel, excess MI is equal to excess $\alpha$, obviating the need for an excess-$\alpha$ plot. We remark that, for the binary-input channels we consider, MI equals capacity, but we maintain the terminology "excess MI" for consistency with the literature [4], [9].

Fig. 2 presents the results of Fig. 1, recast in the context of excess MI, together with the BEC target channel results. We note in Fig. 2 that the excess MI is minimized for all 30 cases when the target channel matches the design criterion (and similarly for Fig. 1). We divide the discussions of universality and surrogate-channel design in Figs. 1 and 2 as follows: (a) $d_v = 50$ u-LDPC codes, (b) $d_v = 8$ u-LDPC codes, (c) $d_v = 8$ eIRA codes, and (d) design via surrogate channels.

**$\mathbf{d_v = 50}$ u-LDPC codes.** Starting with the excess-SNR metric in Fig. 1 ($d_v = 50$), we observe that, for each code rate, the Rayleigh design criterion (entry 4) leads to codes that are universally good on both the AWGN and Rayleigh channels. Specifically, the worst-case excess SNR is only 0.21 dB for rate 1/2 codes on the AWGN channel. At the other extreme, for a code rate of 3/4, the BEC design criterion (entry 1) leads to a worst-case excess SNR of 0.9 dB on the Rayleigh channel. While using the Rayleigh channel as a surrogate leads to the best universal codes, it is at the expense of much greater algorithm complexity. Using the AWGN channel as a surrogate (entry 3) might be preferred since it yields results that are nearly as good. In fact, using the AWGN-GA design criterion (entry 2) also appears to lead to universal codes that are quite good.

Similar comments can be made ($d_v = 50$) when using excess MI as the performance metric as in Fig. 2. We can add, however, that the BEC design criterion does not look quite as bad in this context. Consider that for the BEC surrogate channel (entry 1) the worst-case excess MI for rates 1/4,

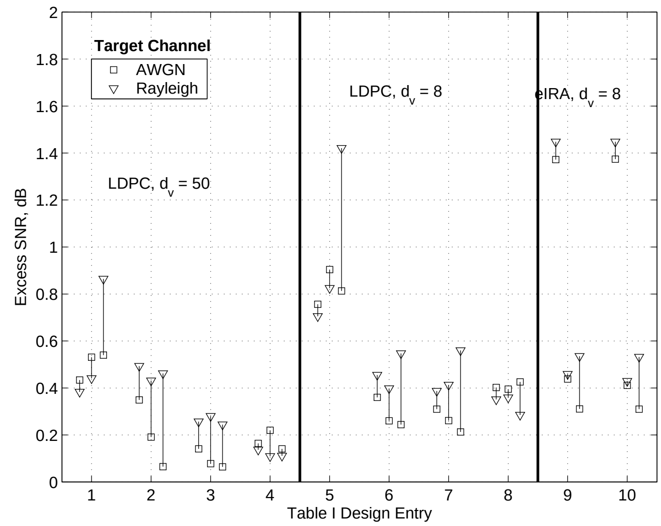


Fig. 1. Excess SNR ($E_b/N_0$) for codes designed under the criteria of Table I. The markers aligned with the entry numbers correspond to rate 1/2, those to the left correspond to rate 1/4, and those to the right correspond to rate 3/4.

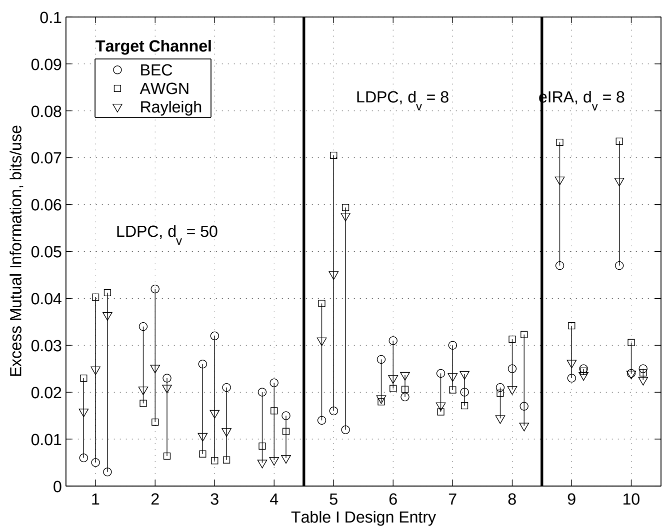


Fig. 2. Excess MI for codes on all three target channels designed under the criteria of Table I. The markers aligned with the entry numbers correspond to rate 1/2, those to the left correspond to rate 1/4, and those to the right correspond to rate 3/4.

1/2, and 3/4 are 0.023, 0.04, and 0.04, respectively. These correspond to worst-case throughput losses of $0.023/0.25 = 9.2\%$, $0.04/0.5 = 8\%$, and $0.04/0.75 = 5.3\%$, respectively. These are similar to the worst-case throughput losses of $8\%$, $4.4\%$, and $2\%$ for the much more complex Rayleigh design criterion (entry 4).

**$\mathbf{d_v = 8}$ u-LDPC codes.** For $d_v = 8$, in both Figs. 1 and 2, the BEC criterion (entry 5) leads to clearly inferior codes in terms of universality. For example, the worst-case excess SNR is 1.43 dB, which occurs for a rate-3/4 code on the Rayleigh channel. The corresponding excess MI value is 0.057, which corresponds to a throughput loss of $7.6\%$. On the other hand, the AWGN and Rayleigh criteria both lead to very good universal codes of nearly equal quality. The AWGN-GA criterion (entry 6) also results in codes that are good on all three channels.

$\mathbf{d_v = 8}$ **eIRA codes.** The parity-check matrix for eIRA codes [5], [6] possess a "dual-diagonal" $(n-k)\times(n-k)$ submatrix which permits efficient encoding via the parity-check matrix. Even though the structure of eIRA codes forces additional constraints on the degree distributions of a code's Tanner graph [6], as shown in the appendix, the infinite-length assumption is still valid in the density evolution process.

The empirical results in [6] indicate that the BEC design criterion, may be used to design eIRA codes for the Rayleigh fading channel with negligible performance difference. Here, we reconsider this issue from the perspective adopted in this paper. As seen in the figures, there is negligible difference between the eIRA codes designed using the BEC criterion (entry 9) and those designed using the Rayleigh criterion (entry 10). Thus the BEC design technique should be used in the case of eIRA codes for all three target channels. We note that, for rates 1/2 and 3/4, there are small excess MI losses (and occasionally gains) in going from $d_v = 8$ u-LDPC codes (entry 8) to $d_v = 8$ eIRA codes (entry 9). However, the excess MI loss is substantial for rate 1/4. This is because eIRA codes are constrained to $n-k-1$ degree-2 variable nodes, which is substantially more than the number required for the optimal threshold for rate 1/4. For example, with $d_v = 8$, for rate 1/4, $\lambda_2 \approx 0.66$ for eIRA codes, but $\lambda_2 \approx 0.43$ for an optimum u-LDPC code.

**Design via surrogate channels.** The previous paragraph argued that the BEC may be used as a surrogate with negligible penalty when the designer is interested only in rate-1/2 and -3/4 eIRA codes. For BEC-designed u-LDPC codes ($d_v = 50$) on the Rayleigh channel, the throughput loss compared to Rayleigh-designed codes is quite small for all three rates, with the worst-case loss occurring for rate 1/4: $0.012/0.25 = 4.8\%$ (entries 1 and 4). Additionally, the GA is a good "surrogate" when the target is AWGN, as is well known. As an example, the throughput loss compared to the AWGN criterion at $d_v = 50$ and rate 1/2 is $0.015/0.5 = 3\%$.

## III. Concluding Remarks

We have shown that an LDPC code can be designed to be universally good across the binary-input erasure, AWGN, and Rayleigh channels, with the Rayleigh channel a particular good surrogate at the expense of design complexity and run-time. We note that our universality results apply to codes selected from an optimum ensemble, not necessarily structured designs. We have also shown that BEC is a reliable surrogate in the design of both u-LDPC codes (all three rates) and eIRA codes (rates 1/2 and 3/4) for the Rayleigh channel.

## Appendix

For u-LDPC codes, Tanner graph variable-node and check-node degree distributions $\lambda(x)$ and $\rho(x)$ must satisfy

$$\begin{cases} \sum_{i=1}^{d_v} \lambda_i = 1 \\ \sum_{i=2}^{d_c} \rho_i = 1 \\ \sum_{j=2}^{d_c} \frac{\rho_j}{j} = (1-R)\sum_{i=1}^{d_v} \frac{\lambda_i}{i} \end{cases} \tag{1}$$

where $R$ is the code rate. Here, $\lambda(x) = \sum_i \lambda_i x^{i-1}$, where the coefficient $\lambda_i$ equals the fraction of edges connecting to variable nodes of degree $i$ and $\rho(x) = \sum_i \rho_i x^{i-1}$, where the coefficient $\rho_i$ equals the fraction of edges connecting to check nodes of degree $i$. For eIRA codes, since the structure of the code requires $n-k-1$ degree-2 nodes and a single degree-1 node, we have $\lambda_2 \cdot e = 2(n-k-1)$ and $\lambda_1 \cdot e = 1$, where $e$ is the number of edges. Combining these two equations with (1), we have

$$\begin{cases} \frac{\lambda_2 R}{2(1-R)-\frac{2}{n}} = \sum_{i=3}^{d_v} \frac{\lambda_i}{i} \\ \frac{\lambda_2(1-R)}{2(1-R)-\frac{2}{n}} = \sum_j \frac{\rho_j}{j} \\ \sum_{i=1}^{d_v} \lambda_i = 1 \\ \sum_{i=2}^{d_c} \rho_i = 1 \end{cases}.$$

For $n > 200$, this is approximately equivalent to

$$\begin{cases} \sum_{i=1}^{d_v} \lambda_i = 1 \\ \sum_{i=2}^{d_c} \rho_i = 1 \\ \sum_j \frac{\rho_j}{j} = \frac{1-R}{R} \sum_{i=3}^{d_v} \frac{\lambda_i}{i} \end{cases}. \tag{2}$$

Thus for eIRA codes of practical length, the infinite length assumption of density evolution can still be applied.

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