Multiterminal Source Coding under Logarithmic Loss

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***Abstract*—We consider the two-encoder multiterminal source coding problem subject to distortion constraints computed under logarithmic loss. We provide a single-letter description of the achievable rate distortion region for arbitrarily correlated sources with finite alphabets. In doing so, we also give the rate distortion region for the CEO problem under logarithmic loss. Notably, the Berger-Tung inner bound is tight in both settings.**

I. Introduction

Characterizing the rate distortion region for the two-encoder lossy source coding problem is perhaps the most well-known, long-standing open problem in the field of multiterminal source coding. Indeed, it is commonly referred to as *the* multiterminal source coding problem (a tradition to which we adhere in the present paper). Although this problem was posed nearly four decades ago, a description of the rate distortion region eluded researchers for any nontrivial choice of source distribution and distortion measure until the seminal work [1] by Wagner et al. in 2008.

In [1], the authors characterized the rate distortion region for jointly Gaussian sources subject to quadratic distortion constraints. Notably, [1] showed that the extension of the single-encoder vector quantization scheme to two encoders (commonly referred to as the Berger-Tung achievability scheme) suffices to attain any point in the achievable rate distortion region. However, due to the reliance of [1] on the peculiarities of the Gaussian distribution, it was still not clear whether the Berger-Tung achievability scheme would be optimal in other settings of interest.

In the present paper, we answer this point in the affirmative for the two-encoder setting. Specifically, we show that the Berger-Tung achievability scheme is optimal for all finite-alphabet sources when distortion is measured under logarithmic loss. To our knowledge, this constitutes the first time that the entire rate distortion region has been described for the multiterminal source coding problem with nontrivial finite-alphabet sources and nontrivial distortion constraints.

Organization

This paper is organized as follows. In Section II we formally define the logarithmic loss function and the multiterminal source coding problem we consider. In Section III we define the CEO problem and give the rate distortion region under logarithmic loss. In Section IV we return to the multiterminal source coding problem and derive the rate distortion region for the two-encoder setting. Section V delivers concluding remarks.

II. Problem Definition

Let $\{Y_1(j), Y_2(j)\}_{j=1}^n$ be a sequence of independent, identically distributed random variables with finite alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, and joint pmf $p(y_1, y_2)$.

In this paper, we take the reproduction alphabet $\hat{\mathcal{Y}}_i$ to be equal to the set of probability distributions over the source alphabet $\mathcal{Y}_i$ for $i = 1, 2$. Thus, for a vector $\hat{Y}_i^n \in \hat{\mathcal{Y}}_i^n$, we will use the notation $\hat{Y}_i[y_i](j)$ to mean the j^{th} coordinate ($1 \leq j \leq n$) of $\hat{Y}_i^n$ (which is a probability distribution on $\mathcal{Y}_i$) evaluated for the outcome $y_i \in \mathcal{Y}_i$. In other words, the decoder generates "soft" estimates of the source sequences.

We consider the logarithmic loss distortion measure defined as follows:

$$d(y_i, \hat{y}_i) = \log\left(\frac{1}{\hat{y}_i[y_i]}\right) \text{ for } i = 1, 2.$$

Using this definition for symbol-wise distortion, it is standard to define the distortion between sequences as

$$d(y_i^n, \hat{y}_i^n) = \frac{1}{n}\sum_{j=1}^{n} d(y_i(j), \hat{y}_i(j)) \text{ for } i = 1, 2.$$

We remark that logarithmic loss is a widely used penalty function in the theory of learning and has natural interpretations and applications in gambling and portfolio theory (cf. [2, Chapter 9]). Several applications of logarithmic loss in the context of source coding are discussed in the full manuscript [3]. To the best of our knowledge, logarithmic loss first appeared explicitly as a distortion measure in the context of multiterminal source coding in [4].

A rate distortion code (of blocklength n) consists of encoding functions:

$$g_i^{(n)} : \mathcal{Y}_i^n \to \left\{1, \ldots, M_i^{(n)}\right\} \text{ for } i = 1, 2 \tag{1}$$

and decoding functions

$$\psi_i^{(n)} : \left\{1, \ldots, M_1^{(n)}\right\} \times \left\{1, \ldots, M_2^{(n)}\right\} \to \hat{\mathcal{Y}}_i^n \text{ for } i = 1, 2.$$

A rate distortion vector (R_1, R_2, D_1, D_2) is strict-sense achievable if there exists a blocklength n, encoding functions $g_1^{(n)}, g_2^{(n)}$, and a decoder $(\psi_1^{(n)}, \psi_2^{(n)})$ such that

$$R_i \geq \frac{1}{n} \log M_i^{(n)} \text{ for } i = 1, 2$$
$$D_i \geq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} d(Y_i(j), \hat{Y}_i(j)) \text{ for } i = 1, 2.$$

Where

$$\hat{Y}_i^n = \psi_i^{(n)}(g_1^{(n)}(Y_1^n), g_2^{(n)}(Y_2^n)) \text{ for } i = 1, 2.$$

Definition 1. *Let $\mathcal{RD}^\star$ denote the set of strict-sense achievable rate distortion vectors and define the set of achievable rate distortion vectors to be its closure, $\overline{\mathcal{RD}}^\star$.*

Our ultimate goal in the present paper is to give a single-letter characterization of the region $\overline{\mathcal{RD}}^\star$. However, in order to do this, we first consider an associated CEO problem. In this sense, the roadmap for our argument is similar to that of [1]. Specifically, both arguments couple the multiterminal source coding problem to a parametrized family of CEO problems. Then, the parameter in the CEO problem is "tuned" to yield the converse result. Despite this apparent similarity, the proofs are quite different since the results in [1] depend heavily on the properties of the Gaussian sources.

III. The CEO problem

In order to attack the general multiterminal problem, we begin by studying the CEO problem (See [5] for an introduction.). To this end, let $\{X(j), Y_1(j), Y_2(j)\}_{j=1}^n$ be a sequence of independent, identically distributed random variables with joint pmf $p(x, y_1, y_2) = p(x)p(y_1|x)p(y_2|x)$. That is, $Y_1 \leftrightarrow X \leftrightarrow Y_2$ form a Markov chain.

In this section, we consider the reproduction alphabet $\hat{\mathcal{X}}$ to be equal to the set of probability distributions over the source alphabet $\mathcal{X}$. As before, for a vector $\hat{X}^n \in \hat{\mathcal{X}}^n$, we will use the notation $\hat{X}[x](j)$ to mean the j^{th} coordinate of $\hat{X}^n$ (which is a probability distribution on $\mathcal{X}$) evaluated for the outcome $x \in \mathcal{X}$. As in the rest of this paper, $d(\cdot, \cdot)$ is the logarithmic loss distortion measure.

A rate distortion CEO code (of blocklength n) consists of encoding functions $g_1^{(n)}, g_2^{(n)}$ as in (1), and a decoding function

$$\psi^{(n)} : \left\{1, \ldots, M_1^{(n)}\right\} \times \left\{1, \ldots, M_2^{(n)}\right\} \to \hat{\mathcal{X}}^n.$$

A rate distortion vector (R_1, R_2, D) is strict-sense achievable for the CEO problem if there exists a blocklength n, encoding functions $g_1^{(n)}, g_2^{(n)}$ and a decoder $\psi^{(n)}$ such that

$$R_i \geq \frac{1}{n} \log M_i^{(n)} \text{ for } i = 1, 2$$
$$D \geq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} d(X(j), \hat{X}(j)).$$

Where

$$\hat{X}^n = \psi^{(n)}(g_1^{(n)}(Y_1^n), g_2^{(n)}(Y_2^n)).$$

Definition 2. *Let $\mathcal{RD}_{CEO}^\star$ denote the set of strict-sense achievable rate distortion vectors and define the set of achievable rate distortion vectors to be its closure, $\overline{\mathcal{RD}}_{CEO}^\star$.*

A. Inner Bound

Definition 3. *Let $(R_1, R_2, D) \in \mathcal{RD}_{CEO}^i$ if and only if there exists a joint distribution of the form*

$$p(x)p(y_1|x)p(y_2|x)p(u_1|y_1, q)p(u_2|y_2, q)p(q)$$

where $|\mathcal{U}_j| \leq |\mathcal{Y}_j|$ and $|\mathcal{Q}| \leq 4$, which satisfies

$$R_1 \geq I(Y_1; U_1|U_2, Q)$$
$$R_2 \geq I(Y_2; U_2|U_1, Q)$$
$$R_1 + R_2 \geq I(U_1, U_2; Y_1, Y_2|Q)$$
$$D \geq H(X|U_1, U_2, Q).$$

Theorem 1. $\mathcal{RD}_{CEO}^i \subseteq \overline{\mathcal{RD}}_{CEO}^\star$.

Before proceeding with the proof, we cite the following variant of a well-known inner bound:

Proposition 1 (Berger-Tung Inner Bound [6])**.** *The rate distortion vector (R_1, R_2, D) is achievable if*

$$R_1 \geq I(U_1; Y_1|U_2, Q)$$
$$R_2 \geq I(U_2; Y_2|U_1, Q)$$
$$R_1 + R_2 \geq I(U_1, U_2; Y_1, Y_2|Q)$$
$$D \geq \mathbb{E}\left[d(X, f(U_1, U_2, Q)\right]$$

for a joint distribution

$$p(x)p(y_1|x)p(y_2|x)p(u_1|y_1, q)p(u_2|y_2, q)p(q)$$

and reproduction function

$$f : \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Q} \to \hat{\mathcal{X}}.$$

Proof of Theorem 1: Apply Proposition 1 with the reproduction function $f(U_1, U_2, Q) := \Pr\left[X = x|U_1, U_2, Q\right]$. Then simply note that $\mathbb{E}\left[d(X, f(U_1, U_2, Q)\right] = H(X|U_1, U_2, Q)$, which yields the desired result. ■

Thus, we note that our inner bound is merely the Berger-Tung inner bound specialized to the case of logarithmic loss.

B. A Matching Outer Bound

A particularly useful property of the logarithmic loss distortion measure is that the expected distortion is lower-bounded by a conditional entropy, a property also enjoyed by quadratic distortion for Gaussian random variables. The following lemma is a key tool in the proof of the converse.

Lemma 1. *Let $Z = (g_1^{(n)}(Y_1^n), g_2^{(n)}(Y_2^n))$ be the argument of the reproduction function $\psi^{(n)}$. Then $n\mathbb{E}d(X^n, \hat{X}^n) \geq H(X^n|Z)$.*

Proof: By definition of the reproduction alphabet, we can consider the reproduction $\hat{X}^n$ to be a probability distribution on $\mathcal{X}^n$ conditioned on the argument Z. In particular, if $\hat{x}^n = \psi^{(n)}(z)$, define $s(x^n|z) = \prod_{j=1}^n \hat{x}[x(j)](j)$, which is a probability measure on $\mathcal{X}^n$. Then, we obtain the following lower bound on the expected distortion, conditioned on $Z = z$:

$$\mathbb{E}\left[d(X^n, \hat{X}^n)|Z = z\right] = \frac{1}{n}\sum_{j=1}^{n}\sum_{x^n \in \mathcal{X}^n} p(x^n|z)\log\frac{1}{\hat{x}[x(j)](j)}$$
$$= \frac{1}{n}D\left(p(x^n|z)\|s(x^n|z)\right) + \frac{1}{n}H(X^n|Z = z)$$
$$\geq \frac{1}{n}H(X^n|Z = z),$$

where $p(x^n|z) = \Pr\left(X^n = x^n|Z = z\right)$ is the true conditional distribution. Averaging both sides over all values of Z, we obtain the desired result. ∎

Definition 4. *Let $(R_1, R_2, D) \in \mathcal{RD}^o_{CEO}$ if and only if there exists a joint distribution of the form*

$$p(x)p(y_1|x)p(y_2|x)p(u_1|y_1, q)p(u_2|y_2, q)p(q),$$

which satisfies

$$R_1 \geq I(Y_1; U_1|X, Q) + H(X|U_2, Q) - D \quad (2)$$
$$R_2 \geq I(Y_2; U_2|X, Q) + H(X|U_1, Q) - D$$
$$R_1 + R_2 \geq I(U_1; Y_1|X, Q) + I(U_2; Y_2|X, Q) + H(X) - D$$
$$D \geq H(X|U_1, U_2, Q). \quad (3)$$

Theorem 2. *If (R_1, R_2, D) is strict-sense achievable for the CEO problem, then $(R_1, R_2, D) \in \mathcal{RD}^o_{CEO}$.*

Proof: Suppose the triple (R_1, R_2, D) is strict-sense achievable. Let A be a nonempty subset of $\{1, 2\}$, and let $F_j = g_j^{(n)}(Y_j^n)$ be the message sent by encoder j. Define $U_j(i) = (F_j, Y_j(1:i-1))$ and $Q(i) = (X(1:i-1), X(i+1:n))$ and observe that:

$$n\sum_{k\in A} R_k \geq \sum_{k\in A} H(F_k) \geq I(Y_A^n; F_A|F_{A^c})$$
$$= I(X^n, Y_A^n; F_A|F_{A^c}) \quad (4)$$
$$= I(X^n; F_A|F_{A^c}) + \sum_{k\in A} I(F_k; Y_k^n|X^n) \quad (5)$$
$$= H(X^n|F_{A^c}) - H(X^n|F_1, F_2)$$
$$+ \sum_{k\in A}\sum_{i=1}^{n} I(Y_k(i); F_k|X^n, Y_k(1:i-1))$$
$$\geq H(X^n|F_{A^c})$$
$$+ \sum_{k\in A}\sum_{i=1}^{n} I(Y_k(i); F_k|X^n, Y_k(1:i-1)) - nD \quad (6)$$
$$= \sum_{i=1}^{n} H(X(i)|F_{A^c}, X(1:i-1))$$
$$+ \sum_{k\in A}\sum_{i=1}^{n} I(Y_k(i); U_k(i)|X(i), Q(i)) - nD \quad (7)$$
$$\geq \sum_{i=1}^{n} H(X(i)|U_{A^c}(i), Q(i))$$
$$+ \sum_{k\in A}\sum_{i=1}^{n} I(Y_k(i); U_k(i)|X(i), Q(i)) - nD. \quad (8)$$

The non trivial steps above can be justified as follows:

- (4) follows since F_A is a function of Y_A^n.
- (5) follows since F_k is a function of Y_k^n and hence $F_1 \leftrightarrow X^n \leftrightarrow F_2$ form a Markov chain (since $Y_1^n \leftrightarrow X^n \leftrightarrow Y_2^n$ form a Markov chain).
- (6) follows since $nD \geq H(X^n|F_1, F_2)$ by Lemma 1.
- (7) follows by the chain rule and also from the Markov condition $Y_k(i) \leftrightarrow X^n \leftrightarrow Y_k(1:i-1)$ resulting from the i.i.d. nature of the source sequences.
- (8) follows since conditioning reduces entropy.

Therefore, dividing both sides by n, we have:

$$\sum_{k\in A} R_k \geq \frac{1}{n}\sum_{i=1}^{n} H(X(i)|U_{A^c}(i), Q(i))$$
$$+ \sum_{k\in A}\frac{1}{n}\sum_{i=1}^{n} I(Y_k(i); U_k(i)|X(i), Q(i)) - D.$$

Also, using Lemma 1 and the fact that conditioning reduces entropy, we have:

$$D \geq \frac{1}{n}H(X^n|F_1, F_2) \geq \frac{1}{n}\sum_{i=1}^{n} H(X(i)|U_1(i), U_2(i), Q(i)).$$

Observe that $Q(i)$ is independent of $(X(i), Y_1(i), Y_2(i))$ and, conditioned on $Q(i)$, we have the Markov chain $U_1(i) \leftrightarrow Y_1(i) \leftrightarrow X(i) \leftrightarrow Y_2(i) \leftrightarrow U_2(i)$. Thus, a standard time-sharing argument proves the theorem. ∎

Theorem 3. $\mathcal{RD}^o_{CEO} = \mathcal{RD}^i_{CEO} = \overline{\mathcal{RD}^\star_{CEO}}$.

Proof: We first remark that the cardinality bounds in the definition of $\mathcal{RD}^i_{CEO}$ can be imposed without any loss of generality. This is a consequence of [7, Lemma 2.2] and is discussed in detail in the full manuscript [3].

Fix $p(q)$, $p(u_1|y_1, q)$, and $p(u_2|y_2, q)$ and consider the extreme points of polytope defined by the inequalities (2)-(3):

$$P_1 = (0, 0, I(Y_1; U_1|X, Q) + I(Y_2; U_2|X, Q) + H(X))$$
$$P_2 = (I(Y_1; U_1|Q), 0, I(U_2; Y_2|X, Q) + H(X|U_1, Q))$$
$$P_3 = (0, I(Y_2; U_2|Q), I(U_1; Y_1|X, Q) + H(X|U_2, Q))$$
$$P_4 = (I(Y_1; U_1|Q), I(Y_2; U_2|U_1, Q), H(X|U_1, U_2, Q))$$
$$P_5 = (I(Y_1; U_1|U_2, Q), I(Y_2; U_2|Q), H(X|U_1, U_2, Q)),$$

where the point P_j is a triple $(R_1^{(j)}, R_2^{(j)}, D^{(j)})$. We say a point $(R_1^{(j)}, R_2^{(j)}, D^{(j)})$ is *dominated by* a point in $\mathcal{RD}^i_{CEO}$ if there exists some $(R_1, R_2, D) \in \mathcal{RD}^i_{CEO}$ for which $R_1 \leq R_1^{(j)}$, $R_2 \leq R_2^{(j)}$, and $D \leq D^{(j)}$. Observe that each of these extreme points is dominated by a point in $\mathcal{RD}^i_{CEO}$:

- First, observe that $(R_1^{(4)}, R_2^{(4)}, D^{(4)})$ and $(R_1^{(5)}, R_2^{(5)}, D^{(5)})$ are both in $\mathcal{RD}^i_{CEO}$, so these points are not problematic.
- Next, observe that the point $(0, 0, H(X))$ is in $\mathcal{RD}^i_{CEO}$, which can be seen by setting all auxiliary random variables to be constant. This point dominates $(R_1^{(1)}, R_2^{(1)}, D^{(1)})$.

- By using auxiliary random variables $(\hat{U}_1, \hat{U}_2, Q) = (U_1, \emptyset, Q)$, the point $(I(Y_1; U_1|Q), 0, H(X|U_1, Q))$ is in $\mathcal{RD}^i_{CEO}$, and dominates the point $(R_1^{(2)}, R_2^{(2)}, D^{(2)})$. By a symmetric argument, the point $(R_1^{(3)}, R_2^{(3)}, D^{(3)})$ is also dominated by a point in $\mathcal{RD}^i_{CEO}$.

Since $\mathcal{RD}^o_{CEO}$ is the convex hull of all such extreme points (i.e., the convex hull of the union of extreme points over all appropriate joint distributions), the theorem is proved. ■

Remark 1. *Theorem 3 can be extended to the general case of m-encoders. Moreover, the converse of the theorem continues to hold when the reproduction alphabet $\hat{\mathcal{X}}^n$ is not restricted to the set of product distributions. The key observation is that Lemma 1 continues to hold. The reader is directed to the complete manuscript [3] for details.*

IV. Multiterminal Source Coding

With Theorem 3 in hand, we are now in a position to characterize the achievable rate distortion region for the multiterminal source coding problem under logarithmic loss.

A. Inner Bound

Definition 5. *Let $(R_1, R_2, D_1, D_2) \in \mathcal{RD}^i$ if and only if there exists a joint distribution of the form*

$$p(y_1, y_2)p(u_1|y_1, q)p(u_2|y_2, q)p(q)$$

where $|\mathcal{U}_j| \le |\mathcal{Y}_j|$ and $|\mathcal{Q}| \le 5$, which satisfies

$$\begin{aligned} R_1 &\ge I(Y_1; U_1|U_2, Q) \\ R_2 &\ge I(Y_2; U_2|U_1, Q) \\ R_1 + R_2 &\ge I(U_1, U_2; Y_1, Y_2|Q) \\ D_1 &\ge H(Y_1|U_1, U_2, Q) \\ D_2 &\ge H(Y_2|U_1, U_2, Q). \end{aligned}$$

Theorem 4. $\mathcal{RD}^i \subseteq \overline{\mathcal{RD}}^{\star}$.

Again, we require an appropriate version of the Berger-Tung inner bound:

Proposition 2 (Berger-Tung Inner Bound [6])**.** *The rate distortion vector (R_1, R_2, D_1, D_2) is achievable if*

$$\begin{aligned} R_1 &\ge I(U_1; Y_1|U_2, Q) \\ R_2 &\ge I(U_2; Y_2|U_1, Q) \\ R_1 + R_2 &\ge I(U_1, U_2; Y_1, Y_2|Q) \\ D_1 &\ge \mathbb{E}\left[d(Y_1, f_1(U_1, U_2, Q)\right] \\ D_2 &\ge \mathbb{E}\left[d(Y_2, f_2(U_1, U_2, Q)\right]. \end{aligned}$$

for a joint distribution

$$p(y_1, y_2)p(u_1|y_1, q)p(u_2|y_2, q)p(q)$$

and reproduction functions

$$f_i : \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Q} \to \hat{\mathcal{Y}}_i, \text{ for } i = 1, 2.$$

Proof of Theorem 4: Similar to the proof of Theorem 1, apply Proposition 2 with the reproduction functions $f_i(U_1, U_2, Q) := \Pr\left[Y_i = y_i|U_1, U_2, Q\right]$. ■

B. A Matching Outer Bound

Theorem 5. $\mathcal{RD}^i = \overline{\mathcal{RD}}^{\star}$.

Proof: Assume (R_1, R_2, D_1, D_2) is strict-sense achievable. We first note that the cardinality bounds in the definition of $\mathcal{RD}^i$ can be imposed without any loss of generality. This is a consequence of [7, Lemma 2.2] and is discussed in detail in the full manuscript [3]. Thus, it suffices to show that $(R_1, R_2, D_1, D_2) \in \mathcal{RD}^i$, ignoring the cardinality constraints.

With foresight, define a new random variable X as:

$$X = \begin{cases} (Y_1, 1) & \text{with probability } t \\ (Y_2, 2) & \text{with probability } 1 - t. \end{cases} \tag{9}$$

In other words, $X = (Y_B, B)$, where B is a Bernoulli random variable independent of Y_1, Y_2. Observe that we have the Markov chain $Y_1 \leftrightarrow X \leftrightarrow Y_2$, and thus, we are able to apply Theorem 3.

Since (R_1, R_2, D_1, D_2) is strict-sense achievable, there exist reproductions $\hat{Y}_i^n$ satisfying

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}d(Y_i(j), \hat{Y}_i(j)) \le D_i \text{ for } i = 1, 2.$$

Fix the encoding operations and set $\hat{X}[(y_1, 1)](j) = t\hat{Y}_1[y_1](j)$ and $\hat{X}[(y_2, 2)](j) = (1-t)\hat{Y}_2[y_2](j)$. Then for the CEO problem defined by (X, Y_1, Y_2):

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}d(X(j), \hat{X}(j)) &= h_2(t) + \frac{t}{n}\sum_{j=1}^{n} \mathbb{E}d(Y_1(j), \hat{Y}_1(j)) \\ &\quad + \frac{1-t}{n}\sum_{j=1}^{n} \mathbb{E}d(Y_2(j), \hat{Y}_2(j)) \\ &\le h_2(t) + tD_1 + (1-t)D_2, \end{aligned}$$

where $h_2(t)$ is the binary entropy function. Hence, for this CEO problem, distortion $h_2(t) + tD_1 + (1-t)D_2$ is achievable and Theorem 3 yields a joint distribution[1] $p(y_1, y_2)p_t(u_1|y_1, q)p_t(u_2|y_2, q)p_t(q)$ satisfying

$$\left.\begin{aligned} R_1 &\ge I(U_1^{(t)}; Y_1|U_2^{(t)}, Q^{(t)}) \\ R_2 &\ge I(U_2^{(t)}; Y_2|U_1^{(t)}, Q^{(t)}) \\ R_1 + R_2 &\ge I(U_1^{(t)}, U_2^{(t)}; Y_1, Y_2|Q^{(t)}) \end{aligned}\right\} \tag{10}$$

$$\begin{aligned} tD_1 + (1-t)D_2 &\ge tH(Y_1|U_1^{(t)}, U_2^{(t)}, Q^{(t)}) \\ &\quad + (1-t)H(Y_2|U_1^{(t)}, U_2^{(t)}, Q^{(t)}), \end{aligned} \tag{11}$$

where the distortion constraint (11) follows since

$$\begin{aligned} H(X|U_1^{(t)}, U_2^{(t)}, Q^{(t)}) &= h_2(t) + tH(Y_1|U_1^{(t)}, U_2^{(t)}, Q^{(t)}) \\ &\quad + (1-t)H(Y_2|U_1^{(t)}, U_2^{(t)}, Q^{(t)}). \end{aligned}$$

Next, fix $\epsilon > 0$, and partition the interval $[0, 1]$ as $0 = t_1 < t_2 < \cdots < t_m = 1$, such that $|t_{j+1} - t_j| < \frac{\epsilon}{H(Y_1, Y_2)}$. We have just proven that, for each t_j in the partition, there exist distributions $p_{t_j}(u_1|y_1, q), p_{t_j}(u_2|y_2, q), p_{t_j}(q)$ for which the corresponding joint distribution satisfies (10)-(11).

[1]Henceforth, we use the superscript (t) to explicitly denote the dependence of the auxiliary random variables on the distribution parametrized by t.

Now, suppose t satisfies $t_j < t < t_{j+1}$. Then we can express t as a convex combination $t = \theta t_j + (1-\theta)t_{j+1}$. By timesharing between the distributions $\{p_{t_j}(u_1|y_1,q),$ $p_{t_j}(u_2|y_2,q),\ p_{t_j}(q)\}$ with probability θ and the distributions $\{p_{t_{j+1}}(u_1|y_1,q),\ p_{t_{j+1}}(u_2|y_2,q),\ p_{t_{j+1}}(q)\}$ with probability $(1-\theta)$, we obtain a set of distributions[2] $\{p_t(u_1|y_1,q),$ $p_t(u_2|y_2,q),\ p_t(q)\}$ for which the corresponding joint distribution satisfies

$$\begin{aligned}
R_1 &\geq I(U_1^{(t)};Y_1|U_2^{(t)},Q^{(t)})\\
&= \theta I(U_1^{(t_j)};Y_1|U_2^{(t_j)},Q^{(t_j)})\\
&\quad + (1-\theta)I(U_1^{(t_{j+1})};Y_1|U_2^{(t_{j+1})},Q^{(t_{j+1})})\\
R_2 &\geq I(U_2^{(t)};Y_2|U_1^{(t)},Q^{(t)})\\
&= \theta I(U_2^{(t_j)};Y_2|U_1^{(t_j)},Q^{(t_j)})\\
&\quad + (1-\theta)I(U_2^{(t_{j+1})};Y_2|U_1^{(t_{j+1})},Q^{(t_{j+1})})\\
R_1+R_2 &\geq I(U_1^{(t)},U_2^{(t)};Y_1,Y_2|Q^{(t)})\\
&= \theta I(U_1^{(t_j)},U_2^{(t_j)};Y_1,Y_2|Q^{(t_j)})\\
&\quad + (1-\theta)I(U_1^{(t_{j+1})},U_2^{(t_{j+1})};Y_1,Y_2|Q^{(t_{j+1})}).
\end{aligned}$$

By repeating this procedure for each interval in the partition, we obtain a family of such distributions parametrized by $t \in [0,1]$. Next, we show that the following holds for any t:

$$\begin{aligned}
&tH(Y_1|U_1^{(t)},U_2^{(t)},Q^{(t)}) + (1-t)H(Y_2|U_1^{(t)},U_2^{(t)},Q^{(t)})\\
&\quad \leq tD_1 + (1-t)D_2 + \epsilon. \qquad (12)
\end{aligned}$$

To simplify notation, define $f_i(t) \triangleq H(Y_i|U_1^{(t)},U_2^{(t)},Q^{(t)})$ for $i=1,2$ and $t \in [0,1]$. By construction, we have that

$$tf_1(t) + (1-t)f_2(t) \leq tD_1 + (1-t)D_2 \qquad (13)$$

whenever $t = t_j$ for some t_j in the partition. Next, observe that if $t = \theta t_j + (1-\theta)t_{j+1}$, then $f_i(t) = \theta f_i(t_j) + (1-\theta)f_i(t_{j+1})$ by virtue of the time-sharing scheme. Furthermore, f_i is piecewise-linear (and therefore continuous) and bounded from above by $H(Y_1,Y_2)$. Now, suppose $t = \theta t_j + (1-\theta)t_{j+1}$ for some j and θ. Then some straightforward algebra yields:

$$\begin{aligned}
&tf_1(t) + (1-t)f_2(t)\\
&= (\theta t_j + (1-\theta)t_{j+1})\,(\theta f_1(t_j) + (1-\theta)f_1(t_{j+1}))\\
&\quad + (1-\theta t_j - (1-\theta)t_{j+1})\,(\theta f_2(t_j) + (1-\theta)f_2(t_{j+1}))\\
&\leq \theta^2\,[t_j f_1(t_j) + (1-t_j)f_2(t_j)]\\
&\quad + (1-\theta)^2\,[t_{j+1}f_1(t_{j+1}) + (1-t_{j+1})f_2(t_{j+1})]\\
&\quad + \theta(1-\theta)\,[(1-t_{j+1})f_2(t_{j+1}) + (1-t_j)f_2(t_j)\\
&\qquad\qquad + t_j f_1(t_j) + t_{j+1}f_1(t_{j+1})] + \epsilon \qquad (14)\\
&\leq \theta^2\,[t_j D_1 + (1-t_j)D_2]\\
&\quad + (1-\theta)^2\,[t_{j+1}D_1 + (1-t_{j+1})D_2]\\
&\quad + \theta(1-\theta)\,[(1-t_{j+1})D_2 + (1-t_j)D_2\\
&\qquad\qquad + t_j D_1 + t_{j+1}D_1] + \epsilon \qquad (15)\\
&= tD_1 + (1-t)D_2 + \epsilon, \qquad (16)
\end{aligned}$$

[2]We can embed the timesharing scheme in the auxiliary variable $Q^{(t)}$.

where (14) follows since $|t_{j+1} - t_j|$ is small, and (15) follows from (13).

This proves (12) and implies that it is impossible to have

$$f_1(t) > D_1 + \epsilon \text{ and } f_2(t) > D_2 + \epsilon$$

simultaneously for any $t \in [0,1]$, else we would contradict (16). Also, we have the following two inequalities at the endpoints of the interval $[0,1]$:

$$f_1(1) \leq D_1 \text{ and } f_2(0) \leq D_2$$

since $t_1 = 0$ and $t_m = 1$ are in the partition. Combining these observations with the fact that f_1 and f_2 are continuous, there must exist some $t^* \in [0,1]$ for which $f_1(t^*) \leq D_1 + \epsilon$ and $f_2(t^*) \leq D_2 + \epsilon$ simultaneously.

Therefore, distributions $\{p_{t^*}(u_1|y_1,q), p_{t^*}(u_2|y_2,q), p_{t^*}(q)\}$ corresponding to t^* yield a joint distribution which satisfies the rate constraints (10) and the distortion constraints

$$D_i \geq H(Y_i|U_1^{(t^*)},U_2^{(t^*)},Q^{(t^*)}) - \epsilon \quad \text{for } i = 1,2.$$

Since ϵ was arbitrary, this proves the converse. ∎

Remark 2. *Like Theorem 3, the converse of Theorem 5 continues to hold when the reproduction alphabets are not restricted to the set of product distributions. The key step is to consider the super-sources* $(\mathbf{X},\mathbf{Y}_1,\mathbf{Y}_2) = (X^n,Y_1^n,Y_2^n)$ *in the proof of Theorem 5 and apply the strengthened CEO converse to obtain the desired joint distribution satisfying* (10)-(11)*. Details are in the complete manuscript [3].*

V. Concluding Remarks

Generalizing Theorem 5 to three or more encoders represents a formidable challenge. Indeed, an extension of the converse alone would not be sufficient since this would imply optimality of the Berger-Tung inner bound for more than two encoders. This is known to be false since the Berger-Tung achievability scheme is suboptimal for the lossless modulo-sum problem studied by Körner and Marton in [8].

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