A Tighter Bhattacharyya Bound for Decoding Error Probability

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***Abstract*— The Bhattacharyya bound has been widely used to upper bound the pair-wise probability of error when transmitting over a noisy channel. However, the bound as it appears in most textbooks on channel coding can be improved by a factor of 1/2 when applied to the frame error probability. For the particular case of symmetric channels, the pairwise error probability can also be improved by a factor of 1/2. This letter provides a simple proof of these tighter bounds that has the same simplicity as the proof of the standard Bhattacharyya bound currently found in textbooks.**

***Index Terms*— Channel coding, Bhattacharyya bound, error probability.**

I. Introduction

The Bhattacharyya bound has been widely used to upper bound the pair-wise probability of error when transmitting over a noisy channel. The bound, as it appears in most textbooks on channel coding [1][2][3], is expressed as follows:

$$P(X \to \tilde{X}) \le \sum_{Y \in \mathcal{Y}} \sqrt{p(Y|\tilde{X})p(Y|X)}, \tag{1}$$

where X is the transmitted codeword, $\tilde{X}$ is any other possible codeword, $P(X \to \tilde{X})$ denotes the pairwise probability of decoding $\tilde{X}$ given that X was transmitted, $p(Y|X)$ is the probability of receiving Y given that X was transmitted and $\mathcal{Y}$ is the set of all possible received words. This letter shows that the Bhattacharyya bound for the frame error rate can be improved by a factor of 1/2. We also show that in the case of symmetric channels, the bound for the pairwise error probability can also be improved by a factor of 1/2. To our knowledge, this improved bound has been shown by Kailath in [4] in 1967, for the case of signal detection under two hypotheses, using Kolmogorov variational distance in a proof that is distinct (and more complex) than what is presented in this paper. His result was used in [5] but has remained unmentioned by modern textbooks, and has never been applied to the probability of error for channel codes.

This letter is organized as follows. In Section II the tighter bounds for discrete channels are introduced and proofs for them are provided. For simplicity the statements and proofs shown in Section II are for discrete channels. Section III generalizes the results for continuous channels. Section IV delivers the conclusions.

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II. Improved Bhattacharyya Bounds for Discrete Channels

We begin with a Lemma that suggests the looseness of (1) by including an additional term on the left hand-side. Denote the set of possible codewords of a certain code C as $\mathcal{X}$.

Lemma 2.1: Given any two codewords $X \in \mathcal{X}$ and $\tilde{X} \in \mathcal{X}$, then:

$$P(X \to \tilde{X}) + P(\tilde{X} \to X) \le \sum_{Y \in \mathcal{Y}} \sqrt{p(Y|\tilde{X})p(Y|X)}. \tag{2}$$

Proof of Lemma 2.1:

Given two possible codewords X and $\tilde{X}$, assume without loss of generality that when $p(Y|X) = p(Y|\tilde{X})$, then X is chosen by the decoder between the two. Using the union bound over both pairwise error probabilities, we get the following inequality:

$$\begin{aligned} &P(X \to \tilde{X}) + P(\tilde{X} \to X) \le \\ &\sum\nolimits_{Y \in \mathcal{Y}} \Big[I\Big(p(Y|\tilde{X}) > p(Y|X)\Big) p(Y|X) + \\ &\qquad I\Big(p(Y|X) \ge p(Y|\tilde{X})\Big) p(Y|\tilde{X}) \Big]. \end{aligned} \tag{3}$$

where $I(\cdot)$ is the indicator function. Notice that when one of the indicator functions in (3) is 1, the other indicator function is 0. Therefore,

$$P(X \to \tilde{X}) + P(\tilde{X} \to X) \le \sum_{Y \in \mathcal{Y}} \min\Big(p(Y|\tilde{X}), p(Y|\tilde{X})\Big) \tag{4}$$

Now,

$$\min\Big(p(Y|X), p(Y|\tilde{X})\Big) \le \sqrt{p(Y|\tilde{X})p(Y|X)}. \tag{5}$$

Applying the inequality in (5) to (4), we get (2), which proves the Lemma.

We will use *Lemma 2.1* to find the bound on the frame error probability, defined as:

$$P_e = \sum_{X \in \mathcal{X}} P(\text{error}|X\text{transmitted})P(X\text{transmitted}). \tag{6}$$

Theorem 2.2: Given any discrete channel, and any code $\mathcal{X}$ with equal a-priori probabilities, *i.e.* $P(X) = P(\tilde{X}) = 1/|\mathcal{X}|$, $\forall X, \tilde{X} \in \mathcal{X}$, the frame error probability can be upper bounded by:

$$P_e \le \frac{1}{2|\mathcal{X}|} \left(\sum_{X \in \mathcal{X}} \sum_{\tilde{X} \ne X} \sum_{Y \in \mathcal{Y}} \sqrt{p(Y|\tilde{X})p(Y|X)} \right). \tag{7}$$

Proof of Theorem 2.2

Using the union bound of the pairwise error probabilities, the frame error probability is bounded by:

$$P_e \leq \sum_{X \in \mathcal{X}} \sum_{\tilde{X} \neq X} P(X \to \tilde{X}) P(X). \tag{8}$$

Since $P(X) = P(\tilde{X}), \forall X, \tilde{X} \in \mathcal{X}$, (8) can be rewritten as:

$$P_e \leq \frac{1}{|\mathcal{X}|} \sum_{X \in \mathcal{X}} \sum_{\tilde{X} \neq X} P(X \to \tilde{X}) \tag{9}$$

$$= \frac{1}{2|\mathcal{X}|} \sum_{X \in \mathcal{X}} \sum_{\tilde{X} \neq X} \Big(P(X \to \tilde{X}) + P(\tilde{X} \to X) \Big), \tag{10}$$

where (10) includes every term in the sum twice and divides by two. Applying Lemma 2.1 to (10) we obtain (7), which proves the theorem.

Theorem 2.3: For symmetric channels, the pairwise error probability $P(X \to \tilde{X})$ can be upper bounded by:

$$P(X \to \tilde{X}) \leq \frac{1}{2} \sum_{Y \in \mathcal{Y}} \sqrt{p(Y|\tilde{X}) p(Y|X)}. \tag{11}$$

Proof of Theorem 2.3

For symmetric channels,

$$P(X \to \tilde{X}) = P(\tilde{X} \to X) = \frac{1}{2} \Big(P(X \to \tilde{X}) + P(\tilde{X} \to X) \Big). \tag{12}$$

The theorem is proved by applying Lemma 2.1 to the right-hand side of Eq. (12).

III. Improved Bhattacharyya Bound for Continuous Channels

For simplicity, we have stated and shown the bounds in the discrete domain. All the results shown in the previous section, and their proofs, can be easily generalized to codebooks transmitted over continuous channels. The lemma and theorems for the continuous case are stated in this section. Their proofs are identical to the discrete case, except that $\sum_{Y \in \mathcal{Y}}$ changes to $\int_{Y \in \mathcal{Y}}$ and PMFs change to the corresponding PDFs.

Lemma 3.1: Given any two codewords $X \in \mathcal{X}$ and $\tilde{X} \in \mathcal{X}$, then:

$$P(X \to \tilde{X}) + P(\tilde{X} \to X) \leq \int_{y \in \mathcal{Y}} \sqrt{p(y|\tilde{X}) p(y|X)} dy. \tag{13}$$

where $p(y|X)$ is the probability density function of receiving y given that X was transmitted.

Theorem 3.2: Given any channel, and any code $\mathcal{X}$ with equal a-priori probabilities, *i.e.* $P(X) = P(\tilde{X}) = 1/|\mathcal{X}|$, $\forall X, \tilde{X} \in \mathcal{X}$, the frame error probability can be upper bounded by:

$$P_e \leq \frac{1}{2|\mathcal{X}|} \left(\sum_{X \in \mathcal{X}} \sum_{\tilde{X} \neq X} \int_{y \in \mathcal{Y}} \sqrt{p(y|\tilde{X}) p(y|X)} dy \right). \tag{14}$$

Theorem 3.3: For symmetric channels, the pairwise error probability $P(X \to \tilde{X})$ can be upper bounded by:

$$P(X \to \tilde{X}) \leq \frac{1}{2} \int_{y \in \mathcal{Y}} \sqrt{p(y|\tilde{X}) p(y|X)}. \tag{15}$$

IV. Conclusions

We have shown that the Bhattacharyya bound on the frame error probability as shown in textbooks on channel coding, can be improved by a factor of 1/2. We have also shown that this factor of 1/2 can be applied in the pairwise error probability for symmetric channels. For simplicity, proofs have been provided for discrete channels but they easily generalize to continuous channels.

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