Optimal Natural Encoding Scheme for Discrete Multiplicative Degraded Broadcast Channels

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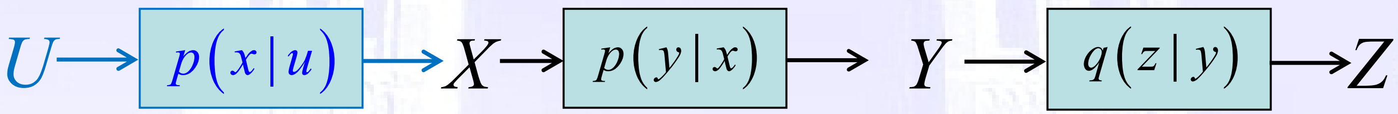
Outline

- The Big Question
  - When is natural encoding scheme optimal?
- A Useful Tool
  - F^* function and the region $C^*_{\boldsymbol{q}}$
  - $\psi(\boldsymbol{q}, \lambda)$ function and its dual
- Discrete Multiplicative (DM) DBC
  - Binary case
  - General case
- Conclusion
  - Natural encoding is optimal for DM-DBC.

Degraded Broadcast Channels

- Capacity Region [Cover72][Bergmans73][Gallager74]



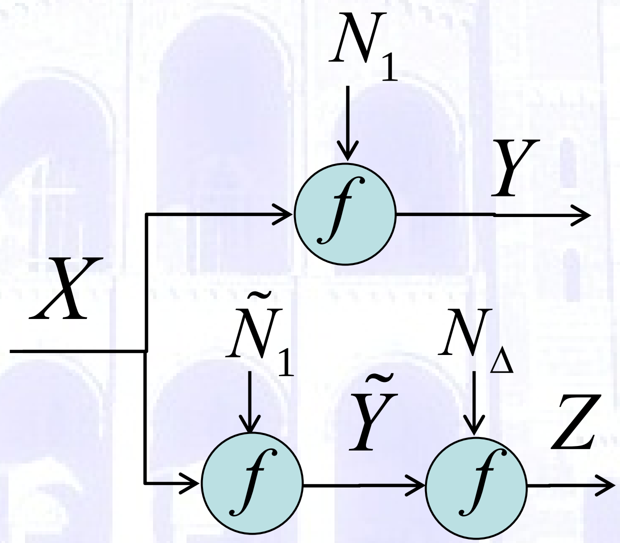
  - The capacity region is the convex hull of the closure of all rate pairs (R_1, R_2) satisfying

$$R_1 \le I(X;Y|U),$$
$$R_2 \le I(U;Z),$$

for some joint distribution $p(u)p(x|u)p(y,z|x)$, where the auxiliary random variable U has cardinality bounded by $|\mathcal{U}| \le \min\{|\mathcal{X}|,|\mathcal{Y}|,|\mathcal{Z}|\}$.

Natural Encoding Scheme

- The natural encoding (NE) scheme combines independent codebooks (one for each receiver) using the same single-letter function that adds distortion to the channel.



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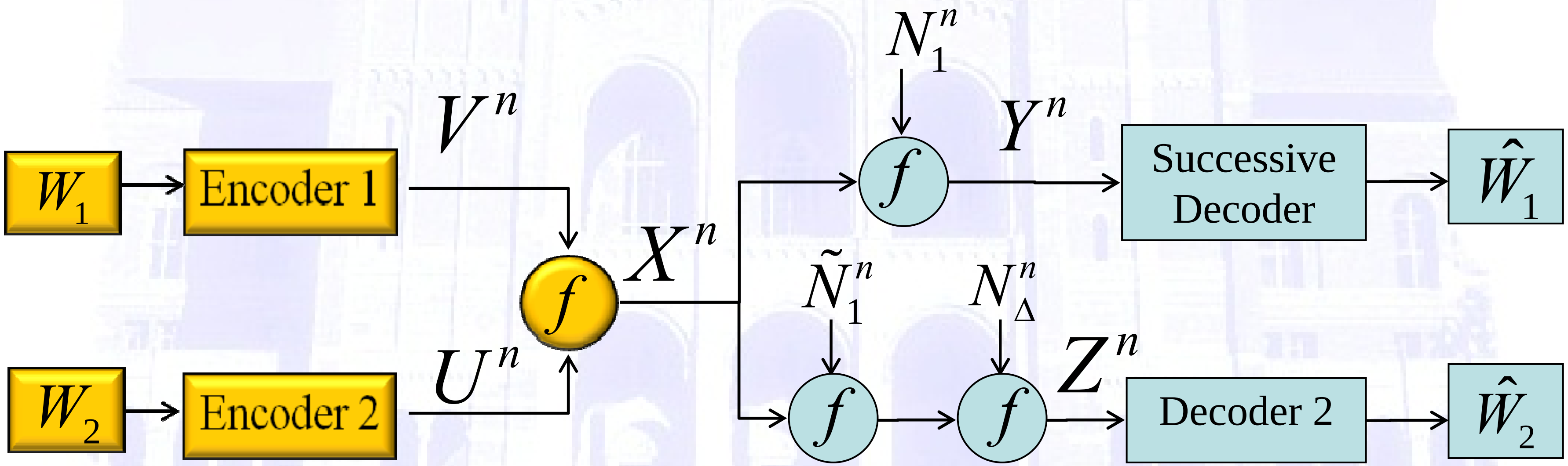
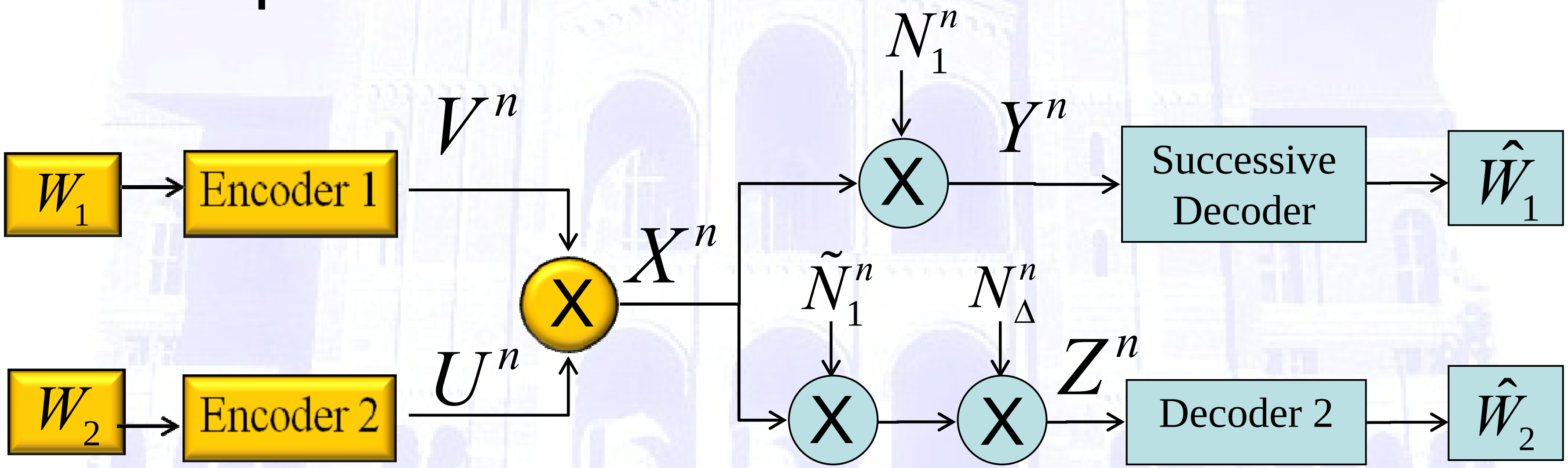


Known Cases of Optimal Natural Encoding

- Broadcast Gaussian channel [Bergmans74]
- Broadcast binary-symmetric channel [Wyner73] [Witsenhausen74]
- Broadcast Z channel [Xie08]
- Discrete additive degraded broadcast channels [Benzel79]
- Our approach is inspired by [Witsenhausen74] and [Witsenhausen & Wyner 75], which are also seminal to [Benzel79] and [Liu&Ulukus07].

Main Result

- Natural encoding achieves the boundary of the capacity region for discrete multiplicative DBC.



Introduce s and $F^*(\mathbf{q},s)$ to optimize (R_1, R_2)

- Given an input distribution $X \sim \mathbf{q}$,

$$
\begin{aligned}
R_1 &\leq I(X;Y|U) \\
&= H(Y|U) - H(Y|X,U) \\
&= H(Y|U) - H(Y|X) \\
&= s - H(Y|X),
\end{aligned}
$$

$$
\begin{aligned}
R_2 &\leq I(U;Z) \\
&= H(Z) - H(Z|U) \\
&= H(Z) - F^*(\mathbf{q},s).
\end{aligned}
$$

Defining F^* and $C^*_{\boldsymbol{q}}$

- Definition of F^*

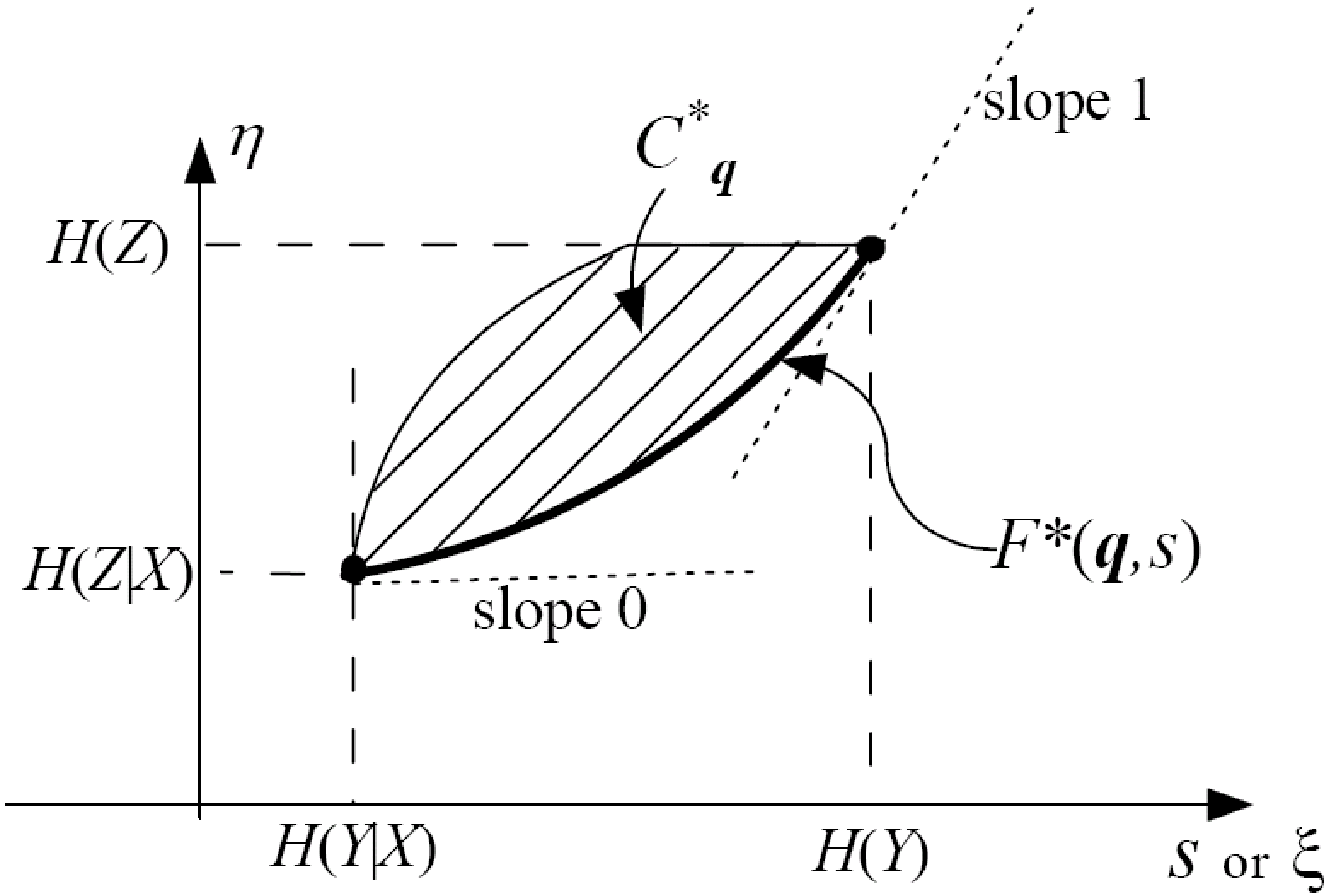
$$F^*(\boldsymbol{q}, s) = \min \; H(Z \mid U), \quad \text{s.t.} \; H(Y \mid U) = s, \quad X \sim \boldsymbol{q}.$$

- F^* is increasing in s for any fixed $\boldsymbol{q}$.
- F^* is jointly convex in $(\boldsymbol{q}, s)$.
- Definition of $C^*_{\boldsymbol{q}}$

$$C^*_{\boldsymbol{q}} = \{(s,\eta) \mid s = H(Y \mid U), \eta = H(Z \mid U), X \sim \boldsymbol{q}\}.$$

- $C^*_{\boldsymbol{q}}$ is a convex set.

F^* is the lower (optimal) boundary of C^*_q.



Introduce $\psi(\mathbf{q}, \lambda)$ to maximize $R_2+\lambda R_1$

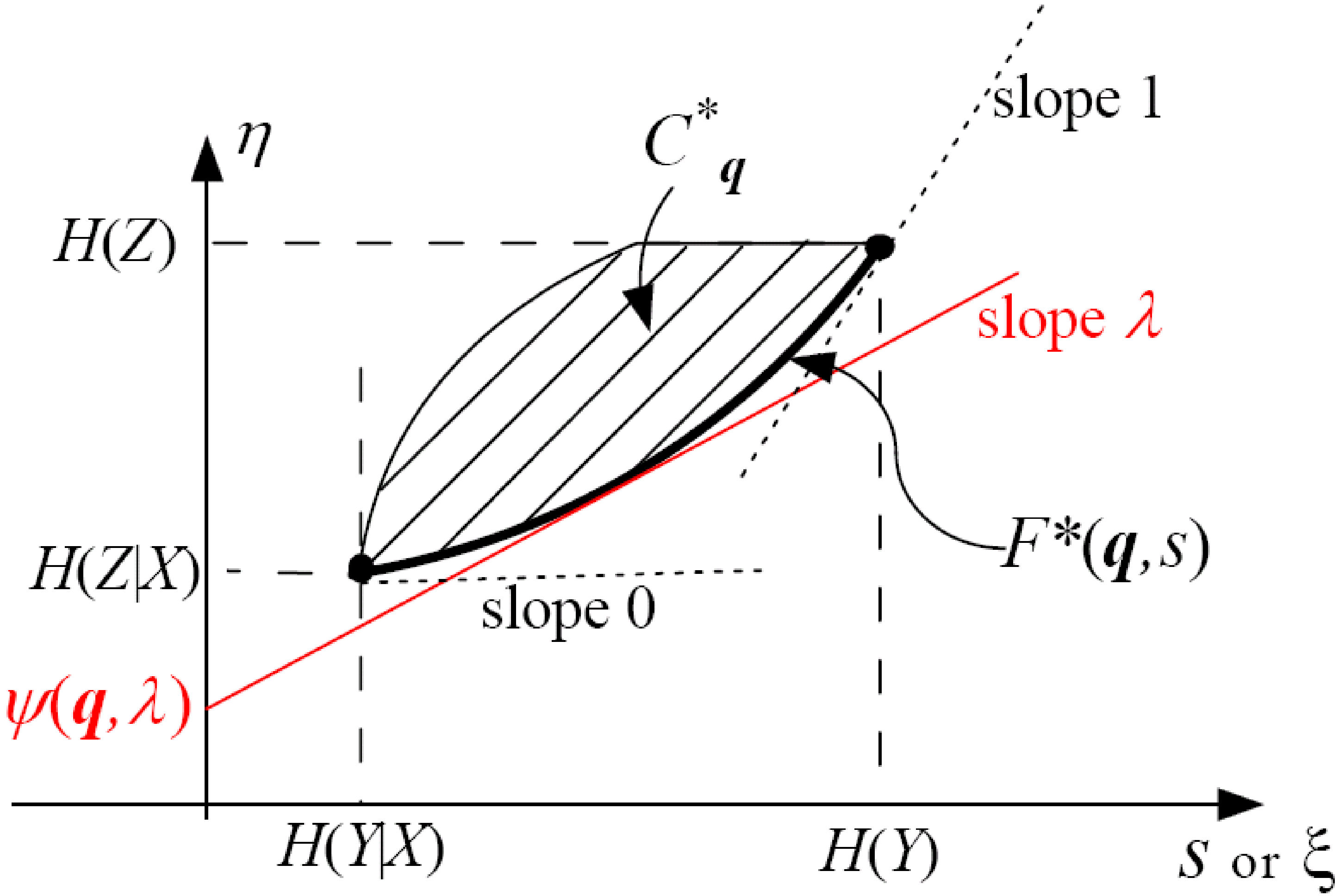
- Given an input distribution $X \sim \mathbf{q}$,
- Given a non-negative number λ ,

$$\max\left\{R_2 + \lambda R_1\right\}$$
$$= \max\left\{H(Z) - F^*(\mathbf{q}, s) + \lambda\left(s - H(Y|X)\right)\right\}$$
$$= H(Z) - \lambda H(Y|X) - \min_s\{F^*(\mathbf{q}, s) - \lambda s\}.$$

- Definition of $\psi(\mathbf{q}, \lambda)$

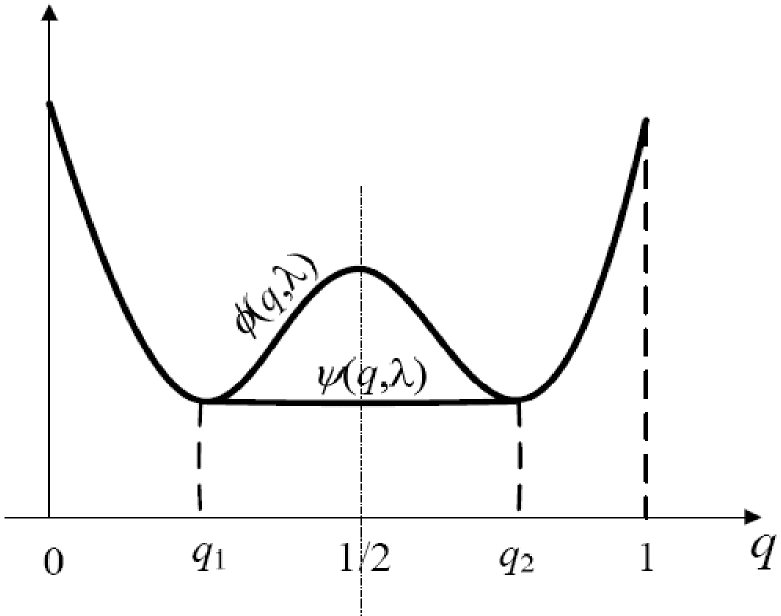
$$\psi(\mathbf{q}, \lambda) = \min_s\{F^*(\mathbf{q}, s) - \lambda s\},$$
$$= \min\{H(Z|U) - \lambda H(Y|U) | X \sim \mathbf{q}\}.$$

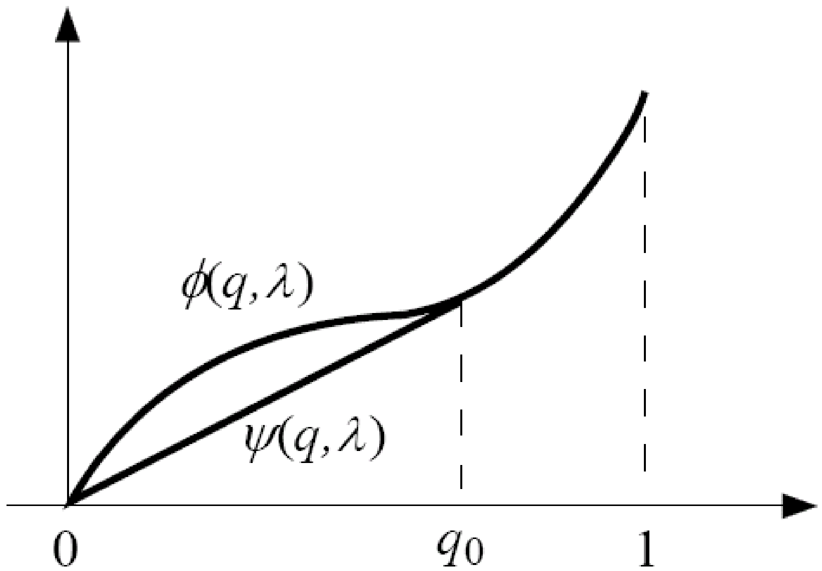
$\psi(\boldsymbol{q}, \lambda)$ is the η-intercept of the tangent line with slope λ for F^*.



Evaluate $\psi(\boldsymbol{q}, \lambda)$

- $\psi(\boldsymbol{q},\lambda) = \min\{H(Z|U) - \lambda H(Y|U) \mid X \sim \boldsymbol{q}\}$.
- Define $\phi(\boldsymbol{q},\lambda) = \{H(Z) - \lambda H(Y) \mid X \sim \boldsymbol{q}\}$.
- $\psi(\boldsymbol{q},\lambda)$ is the lower convex envelope of $\phi(\boldsymbol{q},\lambda)$ in $\boldsymbol{q}$ for each λ.





Key Theorem

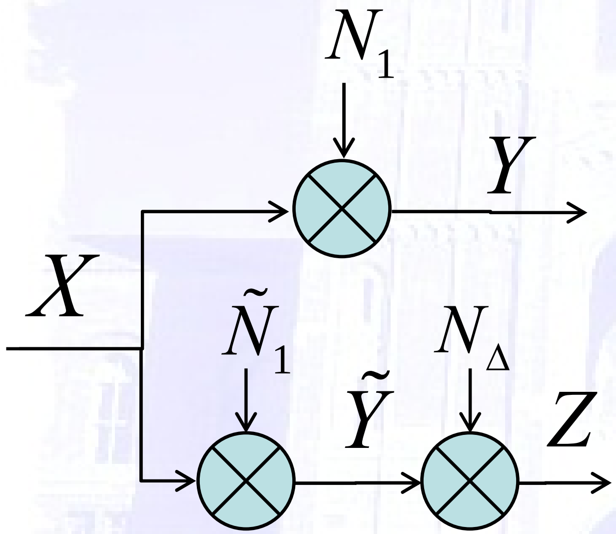
- An encoding scheme ($U \rightarrow X$) is determined by

$$\boldsymbol{p}_U = \begin{bmatrix} w_1 & \cdots & w_l \end{bmatrix},$$
$$P_{U \to X} = \begin{bmatrix} \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_l \end{bmatrix}.$$

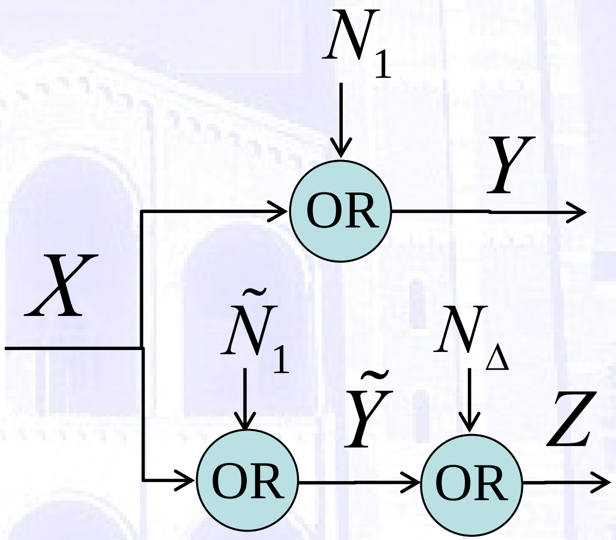
- An encoding scheme ($U \rightarrow X$) maximizes $R_2 + \lambda R_1$ and achieves $X \sim \boldsymbol{q}$ if and only if the point on the graph of $\psi(\boldsymbol{q}, \lambda)$ is the linear combination of the l points $\{\phi(\boldsymbol{q}_1, \lambda), \cdots, \phi(\boldsymbol{q}_l, \lambda)\}$ with weights $\{w_1, \cdots, w_l\}$.

Discrete Multiplicative DBC

- General Case

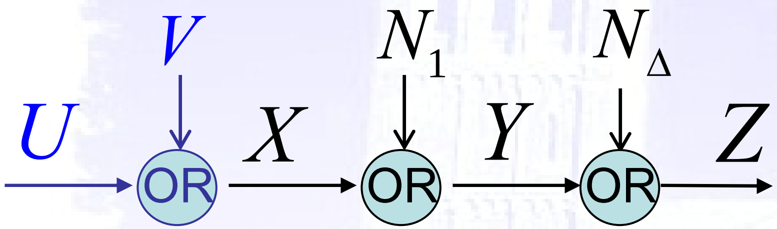


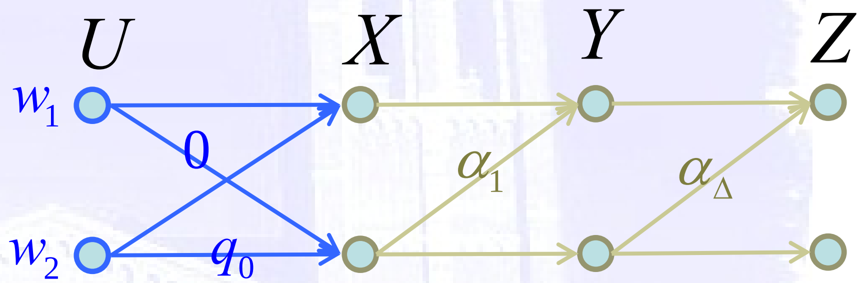
- Binary Case



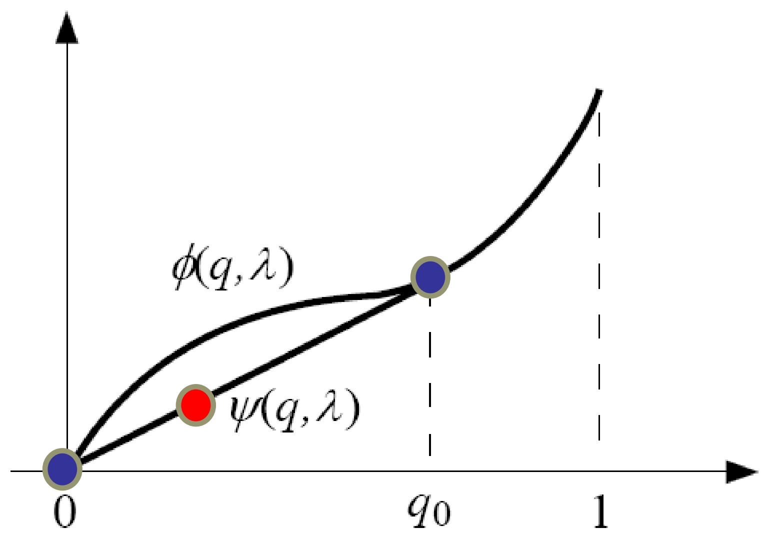
Binary DM-DBC

- Channel Model



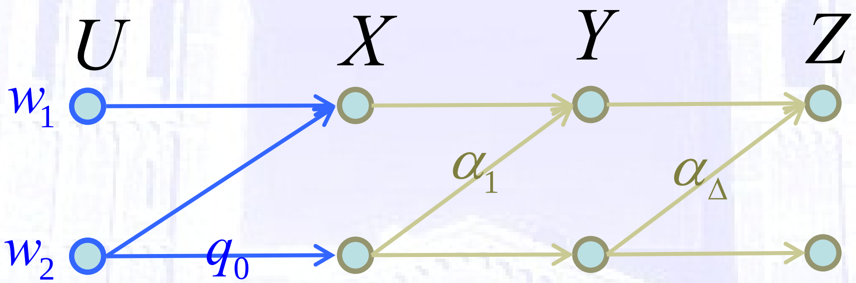


- The graph of $\psi(\mathbf{q}, \lambda)$

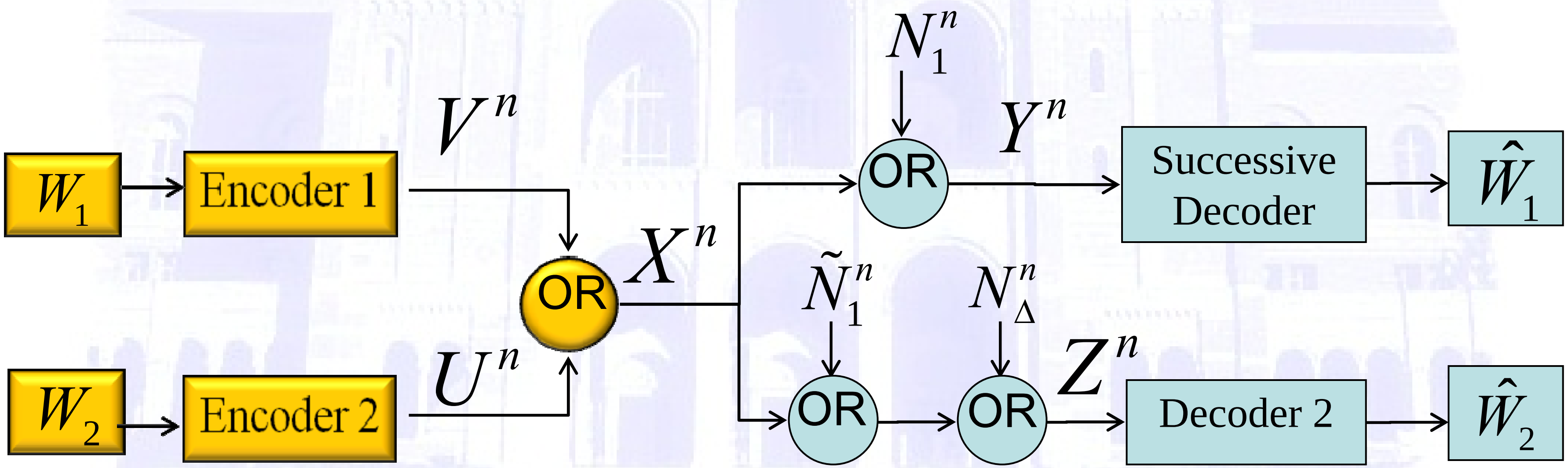


Binary DM-DBC

- Optimal Encoding Scheme

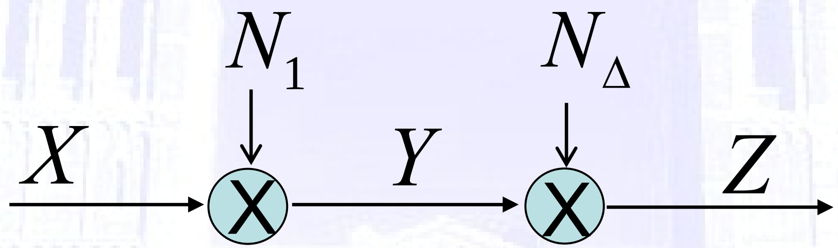


- Natural encoding is optimal.

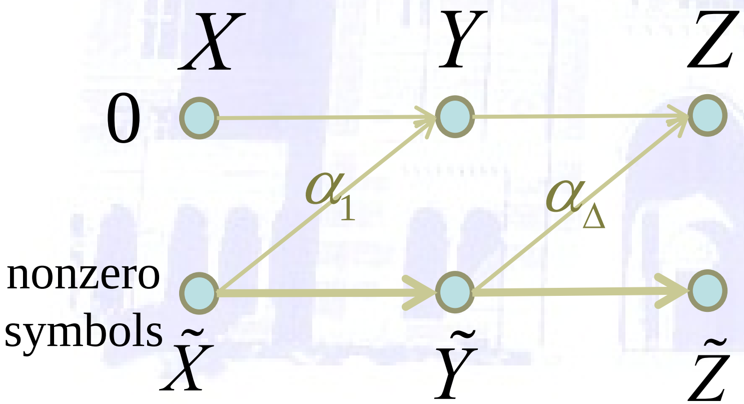


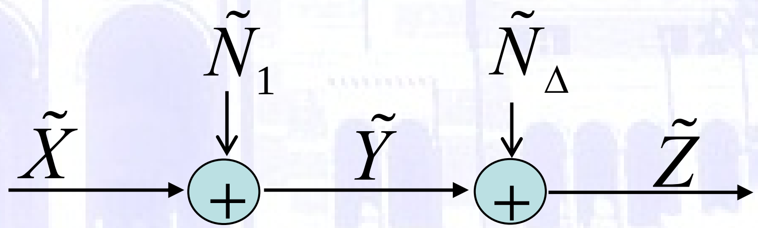
General DM-DBC

- Channel Model

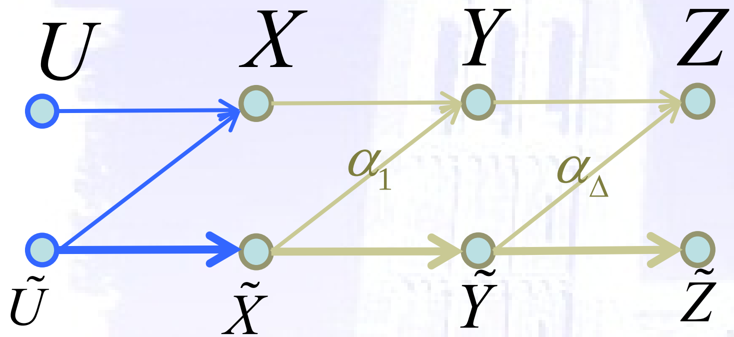


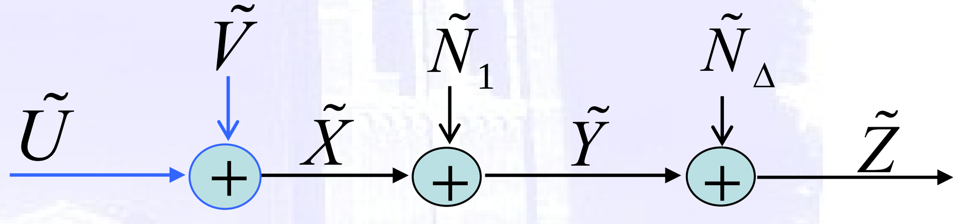
- Decompose it into a binary DM-DBC with a group-additive degraded broadcast sub-channel.

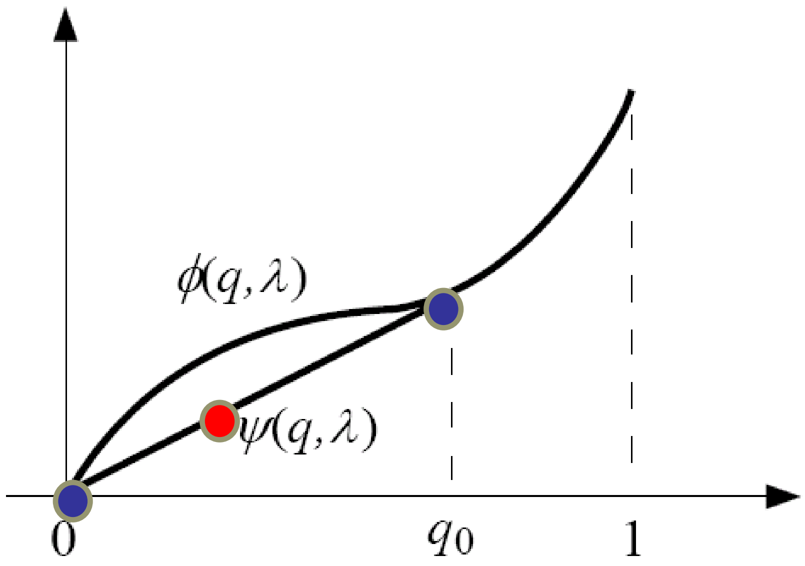


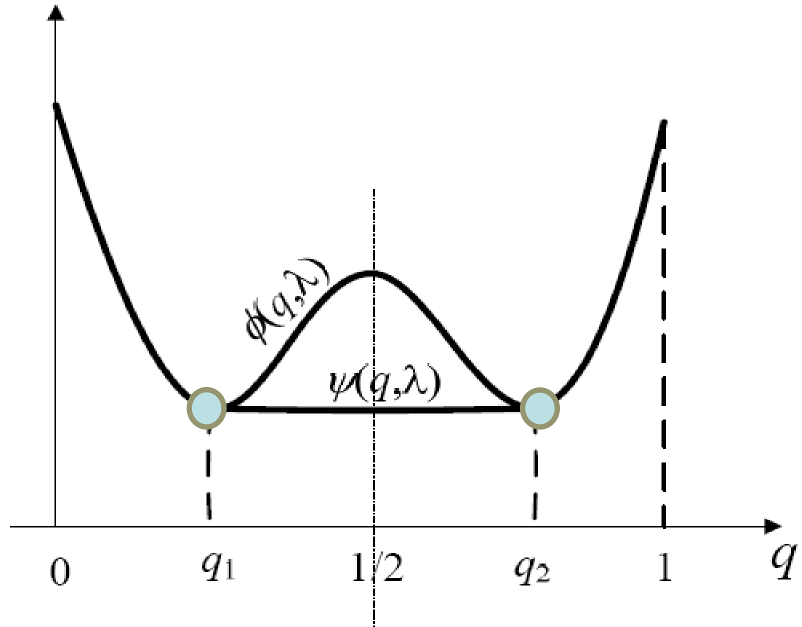


General DM-DBC



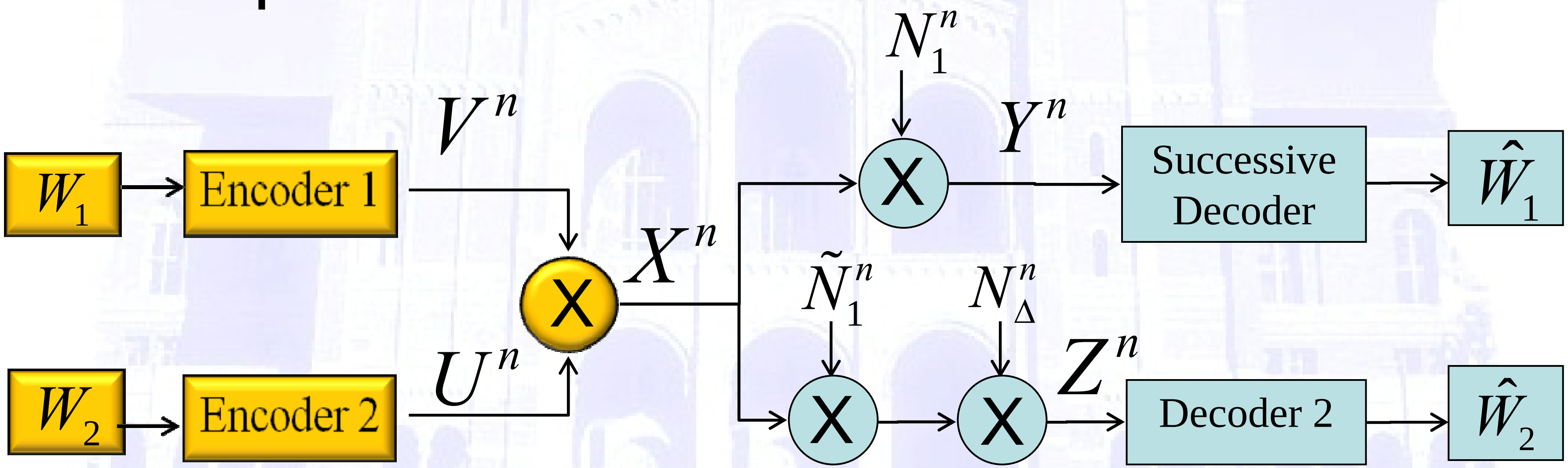






Conclusion

- Natural encoding achieves the boundary of the capacity region for discrete multiplicative DBC.



The rest of the story…

- Natural encoding is optimal for several classes of DBCs. [arXiv:0811.4162v4]

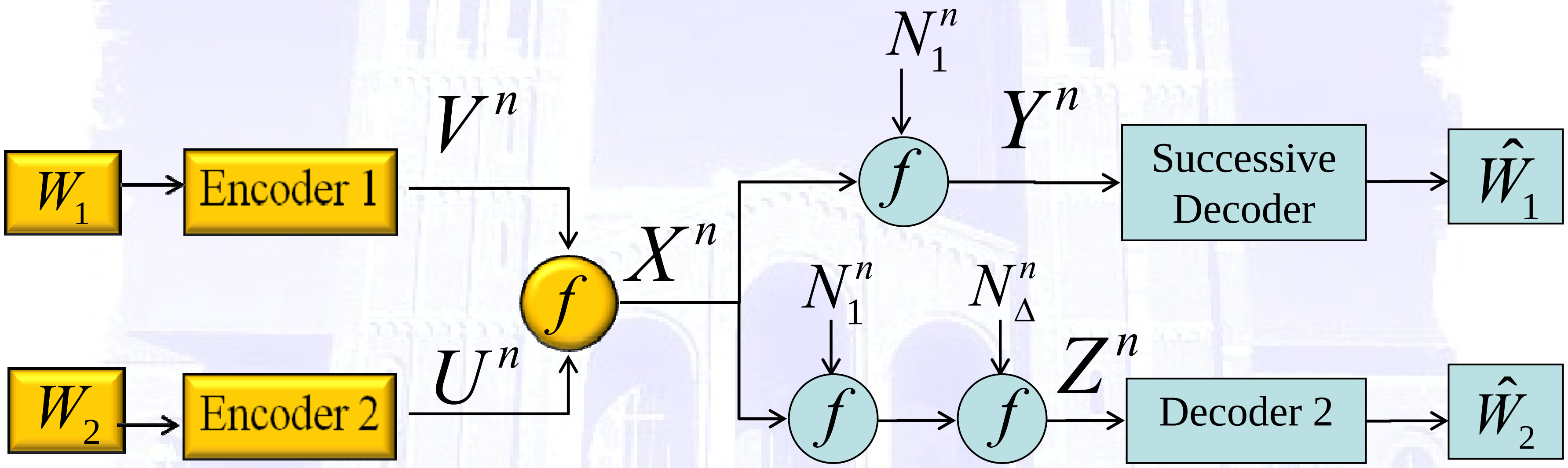
- A more general approach, permutation encoding, is optimal for all input-symmetric DBCs. [arXiv:0811.4162v4]

Remaining Problems

- Can we find some general result for discrete DBC?
  - Natural encoding is optimal if the channel function f has properties …?
- How about continuous DBC?

Thank you.

Natural Encoding Scheme



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