A Mutual Information Invariance Approach to Symmetry in DMCs

Bike Xie and Richard Wesel
**Input-Invariance Symmetry**

\[ X \xrightarrow{P(y|x)} Y \]

- Let \( I_i(X;Y) \) be \( I(X;Y) \) when input distribution is \( p_i \).
- Suppose that for any \( p_1, \exists \{ p_2, \ldots p_k \} \) such that:

\[
I_i(X;Y) = I(X;Y) \quad \frac{1}{k} \sum_{i=1}^{k} p_i = u
\]

- Then the uniform distribution \( u \) is capacity-achieving.
Proof that uniform is optimal

\[ I_1(X;Y) = \frac{1}{k} \sum_{i=1}^{k} I_i(X;Y) \]
\[ \leq I_u(X;Y) \]

by Jensen's since \( u = \frac{1}{k} \sum_{i=1}^{k} p_i \).
Cyclic-shifts

\[ p(x) = \begin{cases} 
  a & \text{if } x = 1 \\
  b & \text{if } x = 2 \\
  c & \text{if } x = 3 
\end{cases} \]

\[ p(x) = [a \ b \ c] \]

\[ p^{(1)}(x) = [c \ a \ b] \]

\[ p^{(2)}(x) = [b \ c \ a] \]

\[ p^{(3)}(x) = [a \ b \ c] \]
Cyclic-shift symmetry

• If for every $p$, $p$ and $p^{(1)}$ have the same mutual information, then the uniform distribution is optimal

• Proof: $p^{(2)}$... $p^{(k)}$ also have the same mutual information and the average of all cyclic shifts is the uniform.

$$\frac{1}{k} \sum_{i=1}^{k} p^{(i)} = u$$
Cyclic-shift symmetry is common

Binary Symmetric Channel

Binary Erasure Channel

Noisy Typewriter
Wang-Kulkarni-Poor (WKP) cyclic symmetry

- [WKP-2007] An $n$-input, $m$-output memoryless channel matrix $T$ is WKP cyclic symmetric if there exists a permutation matrix $Q$ such that:

$$Q^n = I$$

$$T(1, j) = \left[ TQ^{i-1} \right](i, j)$$

- **WKP cyclic symmetry is identical to cyclic-shift symmetry.**
**Permutation-symmetry**

- Now we generalize the permutation beyond cyclic shifts.

- If there are $k$ permutations $\Pi_i \ i = 1, ... k$ such that each permutation of the input distribution preserves the mutual information and the average of these permutations is the uniform

- Then the uniform distribution is capacity-achieving.
A channel without cyclic-shift symmetry

\[ T = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \]

And the four permutations…

\[ \Pi_1 = I \]

\[ \Pi_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ \Pi_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ \Pi_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]
Cover & Thomas (CT) Symmetry

- [CT-1991] Rows of transition matrix are permutations of each other.
- Columns are permutations of each other.
- Does not include the erasure channel.
  \[
  \begin{bmatrix}
  1 - \alpha & \alpha & 0 \\
  0 & \alpha & 0 
  \end{bmatrix}
  \]
- Includes channels that do not have input-invariance permutation symmetry.
CT Weak Symmetry

- [CT-1991] Rows of transition matrix are permutations of each other.
- Columns all have the same sum.
- Does not include the erasure channel (except $\alpha = 1/3$).
- Includes even more channels that do not have input-invariance permutation symmetry.

\[
\begin{bmatrix}
1 - \alpha & \alpha & 0 \\
0 & \alpha & 1 - \alpha
\end{bmatrix}
\]
A. Witsenhausen-Wyner symmetry

Let $\Phi_n$ denote the representation of the symmetric group of permutations of $n$ objects by the $n \times n$ permutation matrices. For an $n \times m$ stochastic matrix $T$ (an $n$ input, $m$ output channel), let $\mathcal{G}_1$ be the set $\{ G \in \Phi_n : \exists \pi \in \Phi_m, \text{s.t.} \ GT = T\Pi \}$ and $\mathcal{G}_o$ be the set $\{ \Pi \in \Phi_m : \exists G \in \Phi_n, \text{s.t.} \ GT = T\Pi \}$. If $G_1T = T\Pi_1, G_2T = T\Pi_2$, then $G_1G_2T = T\Pi_1\Pi_2$, which shows that $\mathcal{G}_1$ and $\mathcal{G}_o$ are subgroups of the finite groups $\Phi_n$ and $\Phi_m$ respectively [1].

Definition 4 Witsenhausen-Wyner (WW) Input Symmetry [1]: A discrete memoryless channel $T$ is WW input symmetric if the set $\mathcal{G}_1$ is transitive, i.e., each element of $\{1, \cdots, n\}$ can be mapped to every other element of $\{1, \cdots, n\}$ by some member of $\mathcal{G}_1$. 

All channels with WW input symmetry also have input-invariance permutation symmetry.
Matrix interpretation of WW input Symmetry

- The channel transition matrix can be decomposed into sub-matrices each of which has CT symmetry.

- There exists a set of column-preserving row permutations \( \prod_{i=1}^{k} \Pi_i \) (including the identity) such that for any input distribution \( p \), the average of the permuted distributions is the uniform.

- Any WW input symmetric matrix has these properties, and they imply input-invariance permutation symmetry.
Gallager Symmetry [G-1968]

• The channel transition matrix can be decomposed into sub-matrices each of which has CT symmetry.

• A larger class than cover symmetry, certainly larger than input-invariance permutation symmetry.
Chen-Yang Symmetry

• The channel transition matrix can be decomposed into sub-matrices each of which has CT weak symmetry.

• An even larger class. The most general form of symmetry.
Input-Invariance Symmetry

• Of course, every channel that has the uniform as a capacity achieving distribution obeys the most general formulation of input-invariance symmetry.

• However, this most general formulation is not operationally helpful (yet)…