A Mutual Information Invariance Approach to Symmetry in DMCs

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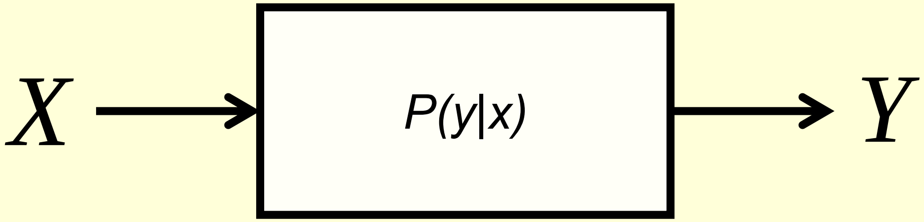
Input-Invariance Symmetry



- Let $I_i(X;Y)$ be $I(X;Y)$ when input distribution is p_i.
- Suppose that for any p_1, $\exists\ \{p_2, \ldots p_k\}$ such that:

$$I_i(X;Y) = I_1(X;Y) \qquad\qquad \frac{1}{k}\sum_{i=1}^{k} p_i = u$$

- Then the uniform distribution u is capacity-achieving.

Proof that uniform is optimal

$$I_1(X;Y) = \frac{1}{k}\sum_{i=1}^{k} I_i(X;Y)$$

$$\le I_u(X;Y)$$

by Jensen's since $u = \frac{1}{k}\sum_{i=1}^{k} p_i$.

Cyclic-shifts

$$p(x) = \begin{cases} a \text{ if } x = 1 \\ b \text{ if } x = 2 \\ c \text{ if } x = 3 \end{cases}$$

$$p(x) = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$p^{(1)}(x) = \begin{bmatrix} c & a & b \end{bmatrix}$$

$$p^{(2)}(x) = \begin{bmatrix} b & c & a \end{bmatrix}$$

$$p^{(3)}(x) = \begin{bmatrix} a & b & c \end{bmatrix}$$

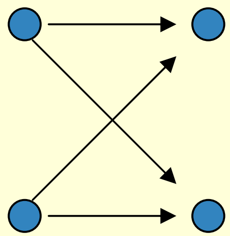
Cyclic-shift symmetry

- If for every *p, p* and $p^{(1)}$ have the same mutual information, then the uniform distribution is optimal

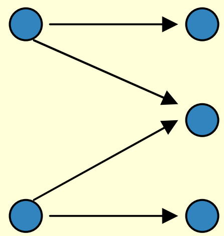
- Proof: $p^{(2)} \ldots p^{(k)}$ also have the same mutual information and the average of all cyclic shifts is the uniform.

$$\frac{1}{k}\sum_{i=1}^{k} p^{(i)} = u$$

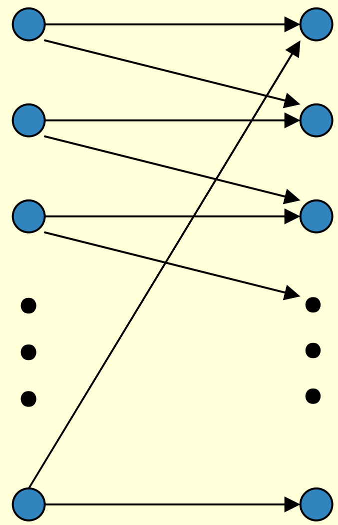
Cyclic-shift symmetry is common



Binary Symmetric Channel



Binary Erasure Channel



Noisy Typewriter

Wang-Kulkarni-Poor (WKP) cyclic symmetry

- *[WKP-2007] An n-input, m-output memory-less channel matrix T is WKP cyclic symmetric if there exists a permutation matrix Q such that:*

$$Q^n = I$$

$$T(1, j) = \left[TQ^{i-1} \right](i, j)$$

- *WKP cyclic symmetry is identical to cyclic-shift symmetry.*

Permutation-symmetry

- Now we generalize the permutation beyond cyclic shifts.
- If there are *k* permutations $\Pi_i \; i=1,\ldots k$ such that each permutation of the input distribution preserves the mutual information and the average of these permutations is the uniform
- Then the uniform distribution is capacity-achieving.

A channel without cyclic-shift symmetry

$$T = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}$$

And the four permutations…

$$\Pi_1 = I \qquad \Pi_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \Pi_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Cover & Thomas (CT) Symmetry

- [CT-1991]Rows of transition matrix are permutations of each other.
- Columns are permutations of each other.
- Does not include the erasure channel.

$$\begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & \alpha & 0 \end{bmatrix}$$

- Includes channels that do not have input-invariance permutation symmetry.

$$\begin{bmatrix} a & b & c & d & e \\ b & a & d & e & c \\ c & e & a & b & d \\ d & c & e & a & b \\ e & d & b & c & a \end{bmatrix}$$

CT Weak Symmetry

- [CT-1991]Rows of transition matrix are permutations of each other.
- Columns all have the same sum.
- Does not include the erasure channel (except α=1/3).
- Includes even more channels that do not have input-invariance permutation symmetry.

$$\begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & \alpha & 1-\alpha \end{bmatrix}$$

Witsenhausen-Wyner Symmetry [WW-1975]

A. Witsenhausen-Wyner symmetry

Let Φ_n denote the representation of the symmetric group of permutations of n objects by the $n \times n$ permutation matrices. For an $n \times m$ stochastic matrix T (an n input, m output channel), let $\mathcal{G}_{\rm i}$ be the set $\{G \in \Phi_n : \exists \pi \in \Phi_m, \text{s.t. } GT = T\Pi\}$ and $\mathcal{G}_{\rm o}$ be the set $\{\Pi \in \Phi_m : \exists G \in \Phi_n, \text{s.t. } GT = T\Pi\}$. If $G_1 T = T\Pi_1$, $G_2 T = T\Pi_2$, then $G_1 G_2 T = T\Pi_1 \Pi_2$, which shows that $\mathcal{G}_{\rm i}$ and $\mathcal{G}_{\rm o}$ are subgroups of the finite groups Φ_n and Φ_m respectively [1].

Definition 4 *Witsenhausen-Wyner (WW) Input Symmetry [1]: A discrete memoryless channel T is WW input symmetric if the set $\mathcal{G}_{\rm i}$ is transitive, i.e., each element of $\{1, \cdots, n\}$ can be mapped to every other element of $\{1, \cdots, n\}$ by some member of $\mathcal{G}_{\rm i}$.*

All channels with WW input symmetry also have input-invariance permutation symmetry.

Matrix interpretation of WW input Symmetry

- The channel transition matrix can be decomposed into sub-matrices each of which has CT symmetry.
- There exists a set of column-preserving row permutations Π_i $i=1,\ldots k$ (including the identity) such that for any input distribution *p*, the average of the permuted distributions is the uniform.
- Any WW input symmetric matrix has these properties, and they imply input-invariance permutation symmetry.

Gallager Symmetry [G-1968]

- The channel transition matrix can be decomposed into sub-matrices each of which has CT symmetry.
- A larger class than cover symmetry, certainly larger than input-invariance permutation symmetry.

Chen-Yang Symmetry

- The channel transition matrix can be decomposed into sub-matrices each of which has CT weak symmetry.
- An even larger class. The most general form of symmetry.

Input-Invariance Symmetry

- Of course, every channel that has the uniform as a capacity achieving distribution obeys the most general formulation of input-invariance symmetry.
- However, this most general formulation is not operationally helpful (yet)…