RCA Analysis of the Polar Codes and the use of Feedback to aid Polarization at Short Blocklengths

Kasra Vakilinia, Dariush Divsalar*, and Richard D. Wesel
Department of Electrical Engineering, University of California, Los Angeles, Los Angeles, California 90095
*Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California 91109

***Abstract*—This paper uses an extension of Reciprocal Channel Approximation (RCA) to accurately and efficiently predict the frame error rate (FER) performance of polar codes by analyzing the probability density function (p.d.f) of log likelihood ratios (LLR) associated with information bits. A feedback scheme uses the RCA to predict the p.d.f of LLRs in conjunction with a repetition coding system to decrease the blocklength required for a target FER by a factor of 16. Using a rate-0.5 128-bit polar code as the initially transmitted code, the FER of the system with feedback is obtained by theoretical analysis and verified by simulation. Including the additional incremental transmissions the average blocklength for the system with feedback is 137.55 bits and the rate is 0.4653. Without feedback, a polar code with blocklength 2048 is required to achieve a comparable FER at a comparable rate. Intuitively, feedback allows the polar code to use fewer frozen bits in the initial transmission and then uses repetition codes to provide the needed reliability to resolve unreliable unfrozen bits identified by feedback.**

I. Introduction

Polar codes are a class of modern error correcting codes introduced by Arikan [1]. These codes can achieve the capacity of binary-input symmetric discrete memoryless channels universally. Polar codes have an explicit construction using a kernel matrix and have low encoding and decoding complexity. Originally, Arikan [1] used the 2 x 2 kernel matrix F in (1) to show the effect of polarization.

$$F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{1}$$

When a large number of these polarizing matrices are concatenated, the channel polarizes in the sense that the fraction of the transmitted bits that experience a noiseless channel to the total number of channel uses is equal to the capacity of the channel. The rest of the fraction of the bits are deemed useless as they experience very noisy synthetic channels. The information (unfrozen) bits are allocated to the noiseless synthetic channels and pre-determined values (usually zeros) are allocated to the noisy channels.

Researchers [2] have studied the polarization phenomenon with larger kernel matrices than (1). The general requirement for a square matrix to be a valid polarizing kernel matrix is that none of the column permutations of the matrix should result in an upper triangular matrix. Korada et al. [3] show that larger polarizing matrices can increase the speed of polarization. Asymptotically, the rate of polarization shows how fast the probability of error decays to zero as the blocklength B of the polar code approaches infinity. Arikan and Telatar [4] derived an upper bound suggesting that this rate scales exponentially in $-B^{\beta}$ where β is the rate of polarization. For kernel matrices smaller than 16×16, $\beta < 1/2$.

This paper uses the original polarization kernel matrix (1) to analyze how feedback can help in achieving a target FER at a much smaller blocklength with feedback than without feedback. The figure of merit used in this work is the expected total number of forward channel uses required to achieve a particular FER. This total number of forward channel transmissions includes the blocklength of the polar code initially transmitted as well as the subsequent forward transmissions based on the feedback. The FER performance gain is compared with the FER of the no-feedback system with an average rate similar to the scheme where feedback is used. In other words, we take into account the overhead due to the additional transmissions and the consequent rate reduction in our comparison.

Reciprocal channel approximation (RCA) was used in [5], [6] to design LDPC codes. This paper uses RCA to accurately estimate the FER of short blocklength polar codes (without feedback). Moreover, RCA gives an approximation on the distribution of the LLR values of the information bits (the unfrozen bits).

The approximate distribution of the LLR values obtained by RCA informs the use of feedback during decoding. Feedback requests additional bits only when the reliability of the successive-cancellation-decoded information bits falls in a specified low-reliability region of the RCA-obtained distribution. A repetition code of a specified length is used to increase the reliability of those particular bits. Significant coding gain or equivalently polarization-rate improvement is obtained by this scheme. Furthermore, an RCA-based analysis accurately predicts the simulation performance of this feedback scheme.

The rest of the paper is organized as follows: Sec. II gives an introduction to successive cancelation (SC) decoding of polar codes and the LLR calculation of the information bits. Sec. III provides the system model and discusses the application of RCA to polar codes. This section also compares the FER performance predicted by RCA with simulation results. Sec. IV

This material is based upon work supported by the National Science Foundation under Grant Numbers 1162501 and 1161822. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation. This research was carried out in part at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with NASA JPL Task Plan 82-17473.

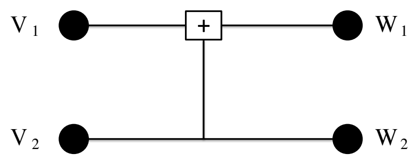

Fig. 1: The graph of the kernel matrix.

presents the feedback scheme, demonstrates its significant improvement of FER, and compares the RCA analysis of that scheme with simulation results. Sec. V draws conclusions and the Appendix in Sec. VI shows the mathematical derivations for the results obtained in the previous sections.

II. POLAR CODES

Fig. 1 shows the graphical representation of the polarization matrix used in this paper. The bits V_1 and V_2 are combined according to the kernel matrix (1) to give W_1 and W_2

$$W_1 = V_1 \oplus V_2 \tag{2}$$
$$W_2 = V_2 \tag{3}$$

From (2) and (3), V_1 can be calculated by $V_1 = W_1 \oplus W_2$. The probability of error for V_1 is high because if either W_1 or W_2 is decoded incorrectly, V_1 will be decoded incorrectly.

The LLR value for V_1 can be calculated as if V_1 is connected to a check node in an LDPC code that also connects to the variable nodes W_1 and W_2. The LLR value of V_1 can be calculated as:

$$LLR_{V_1} = 2\tanh^{-1}(\tanh(\frac{LLR_{W_1}}{2})\tanh(\frac{LLR_{W_2}}{2})). \tag{4}$$

If V_1 is in fact decoded correctly (or it has a value already determined) the probability of error for V_2 is significantly lower since the decision about V_2 is from two independent channel observations:

$$V_2 = W_2 \tag{5}$$
$$V_2 = W_1 \oplus V_1\,. \tag{6}$$

Assuming V_1 is previously determined, the LLR value of LLR_{V_2} is calculated as follows:

$$LLR_{V_2} = \begin{cases} LLR_{W_1} + LLR_{W_2} & \text{if } V_1 = 0 \\ -LLR_{W_1} + LLR_{W_2} & \text{if } V_1 = 1 \end{cases} \tag{7}$$

Computation of the LLRs of all the bits in a polar code follow from repeated applications of (4) and (7).

In this paper we consider a channel with binary input and additive white Gaussian noise (BI-AWGN) having signal-to-noise ratio (SNR) s_{ch} where equiprobable information bits are coded according to the polar code designed with the polarization matrix (1).

III. RCA FOR POLAR CODES

The RCA for BI-AWGN channel in the study and design of LDPC codes uses a single real-valued parameter s, the SNR, to approximate the distribution of LLR messages exchanged between variable and check nodes. RCA for LDPC codes is a low complexity alternative to the density evolution algorithm. The RCA approach can be applied to polar codes with SC decoders.

Assume a BI-AWGN channel with input x and output y,

$$y = x + z, \tag{8}$$

where $z \sim N(0, \sigma_{ch}^2)$ is a white Gaussian noise sample and $x \in \{1, -1\}$ are binary-phase-shift-keying (BPSK) modulated channel inputs corresponding to 0 and 1 respectively. Note that for this scenario $s_{ch} = 1/\sigma_{ch}^2$. The bit LLR of a message received at the receiver is given by

$$L = \mathrm{Ln}(\frac{P(y|x=+1)}{P(y|x=-1)}) = \frac{2y}{\sigma_{ch}^2} = 2ys_{ch}\,. \tag{9}$$

Since polar codes are linear block codes, without loss of generality assume that the all-zero codeword is transmitted where all information and frozen bits are set to zero. Under the assumption that only $x = +1$ is transmitted over the channel, the channel output y will have a Gaussian distribution, $y \sim N(1, \sigma_{ch}^2)$. Consequently, the LLR message L is normally distributed with mean $E(L)$ and variance var(L) given by

$$\mathrm{E}(L) = \frac{2\mathrm{E}(y)}{\sigma_{ch}^2} = \frac{2}{\sigma_{ch}^2} \tag{10}$$
$$\mathrm{Var}(L) = \frac{4}{\sigma_{ch}^4}\mathrm{Var}(y) = \frac{4}{\sigma_{ch}^2}\,. \tag{11}$$

Define the reciprocal SNR as $r \in R$ such that

$$C(s) + C(r) = 1\,, \tag{12}$$

where $C(s)$ is the capacity of the BI-AWGN channel with SNR s. For computational simplicity and numerical precision we prefer to express $C(s)$ as

$$C(s) = 1 - \int_{-\infty}^{\infty} \log_2\left(1 + e^{-(2\sqrt{2s}u+2s)}\right)\frac{e^{-u^2}}{\sqrt{\pi}}du, \tag{13}$$

which is obtained from [7, (15)] through a change of variables. The self-inverting reciprocal energy function

$$R(s) = C^{-1}\left(1 - C(s)\right) \tag{14}$$

in [5] transforms between s and r: $r = R(s)$ and $s = R(r)$.

Let s_{ch} be the channel SNR and s_{W_i} for $i \in 1, 2$ be the SNR corresponding to the nodes W_i in Fig. 1. Using $C(s)$ and $C^{-1}(s)$ we can calculate the SNR observed at V_1. The function $R(s)$ makes the corresponding values D_i in Fig. 2 additive at the check node (+) in Fig. 2. Thus the SNR value at V_1 is calculated by $C^{-1}(1 - C(D_1 + D_2))$ as in Fig. 2. The distribution of V_1 is approximately Gaussian with a mean of $2s_{V_1}$ and variance of $4s_{V_1}$. Under the assumption that V_1 is correctly decoded ($V_1 = 0$), W_1 and W_2 provide two

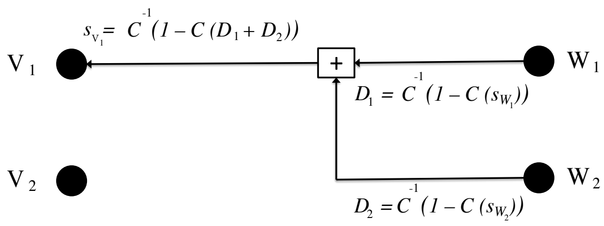

Fig. 2: RCA to calculate the LLR p.d.f. of V_1

independent looks at V_2 and the SNR at V_2 is the summation of the SNR values observed at W_1 and W_2:

$$s_{V_2} = s_{W_1} + s_{W_2} \,. \tag{15}$$

The LLR distribution of each information bit (assumed by RCA to be Gaussian) is calculated at each stage of SC decoding. If any of the information bits is decoded in error the entire block will be decoded in error as the errors propagate in an SC decoder. The frame error probability is the probability that none of the bits is decoded incorrectly:

$$P_{FE} = 1 - \prod_{1}^{k} P(\hat{u}_i = 0 | u_1 = 0, \ldots, u_{i-1} = 0) \,. \tag{16}$$

Note that $\hat{u}_i$ is the decision about the information bit u_i for $i \in \{1, \ldots, k\}$, made by the SC decoder. Assuming all information bits from 1 to $i-1 < k$ are decoded correctly, the SNR for each information (unfrozen) bit is s_i. The distribution is approximately $N(\mu_{i,\mathrm{sc}}, 2\mu_{i,\mathrm{sc}})$ where $\mu_{i,\mathrm{sc}} = 2s_{u_i}$. The probability that the i^{th} successively decoded information bit is in error under the assumption that all the previous information bits are decoded correctly is approximated by

$$P(\hat{u}_i = 0 | u_1 = 0, \ldots, u_{i-1} = 0) \approx Q\left(-\frac{\mu_{i,\mathrm{sc}}}{\sqrt{2\mu_{i,\mathrm{sc}}}}\right). \tag{17}$$

Thus the FER (P_{FE}) is approximately

$$P_{FE} \approx 1 - \prod_{i=1}^{k} Q\left(-\frac{\mu_{i,\mathrm{sc}}}{\sqrt{\mu_{i,\mathrm{sc}}}}\right) \,. \tag{18}$$

Fig. 3 shows the FER predicted by RCA and illustrates how closely it follows the FER obtained from simulations and also density evolution (DE) for different SNR values.

Fig. 4 shows the probability of error for each information bit and its comparison to the RCA-predicted probability of error of (17). RCA can be effectively used to select which bits are more unreliable. Those bits can be selected as frozen bits to lower the rate and increase reliability.

IV. Using Feedback to Reduce the Blocklength Required to Achieve a Target FER

In this section we show how feedback can be used to reduce the blocklength required to achieve a specified FER. Essentially, feedback allows the polar code to use fewer frozen bits in the initial transmission and then uses repetition codes to provide the needed reliability to resolve unreliable unfrozen bits identified by feedback.

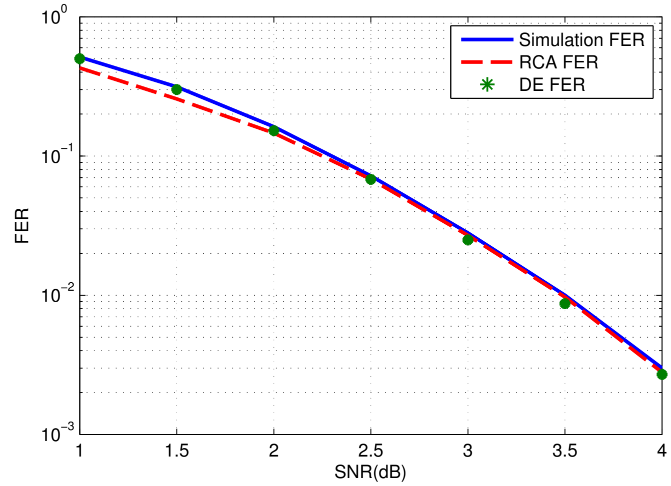

Fig. 3: Simulation FER and FER predicted by RCA and DE for a rate-0.5 polar code of length 128 bits constructed using the original kernel matrix of (1).

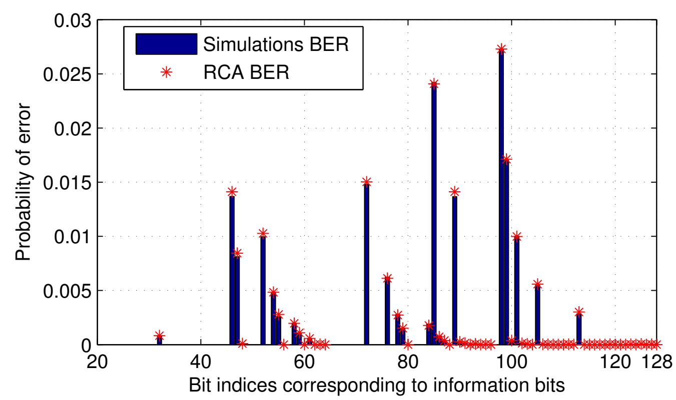

Fig. 4: Information bit probability of error from simulation (bar plot) and RCA predicted probability of error as in (17).

In order to use feedback to reduce the error probability of the SC decoded information bit u_i, a symmetric LLR threshold range around zero ($[-t_i, t_i]$) is determined. If the LLR from SC decoding is in this range, feedback instructs the transmitter to increase reliability by sending a repetition code of size n_i indicate the value of bit currently being decoded.

The probability that the LLR value of the SC-decoded bit u_i falls in the less reliable range of $[-t_i, t_i]$, where t_i is the threshold, is given by

$$P(-t_i < L_i < t_i) = Q\left(\frac{-t_i - \mu_{i,\mathrm{sc}}}{\sigma_{i,\mathrm{sc}}}\right) - Q\left(\frac{t_i - \mu_{i,\mathrm{sc}}}{\sigma_{i,\mathrm{sc}}}\right). \tag{19}$$

If the LLR L_i of u_i is in the range of $[-t_i, t_i]$, feedback initiates the use of a repetition code to increase the number of independent observations of the channel and consequently increase the reliability of u_i. The probability of having the increased LLR value (L_i') become positive and hence be decoded correctly is

$$P(L_i' > 0| - t_i < L_i < t_i), \tag{20}$$

where L_i' includes the additional reliability $L_{i,\mathrm{rc}}$ from the repetition code, i.e. $L_i' = L_i + L_{i,\mathrm{rc}}$.

The total probability of correctly decoding the information bit u_i under the assumption that all the previous information bits are decoded correctly is

$$P(\hat{u}_i = u_i) = P(L_i > t_i) + P(L_i' > 0, |L_i| < t_i)\,. \tag{21}$$

The second term (joint probability) in (21) can be decomposed into the product of two terms $P(L_i' > 0| - t_i < L_i < t_i)P(-t_i < L_i < t_i)$.

Note that $P(\hat{u}_i = u_i)$ depends on the length of the repetition code as well as the threshold t_i. The selection of these parameters will be discussed later in this in section.

Under the assumption that $u_i = 0$, for a repetition-coded signal with a blocklength of n_i, $L_{i,\mathrm{rc}}$ is normally distributed with a mean of $\mu_{i,\mathrm{rc}} = n_i(\frac{2}{\sigma_{ch}^2}) = 2n_i s_{ch}$ and a variance of $\sigma_{i,\mathrm{rc}}^2 = n_i(\frac{4}{\sigma_{ch}^2})$. For simplicity of notation, the index i is omitted in the following. For the general case where $t_i, \mu_{i,\mathrm{sc}}$ and $\sigma_{i,\mathrm{sc}}$ of (19) are represented by t, μ_{sc}, and σ_{sc} (for successive cancellation) respectively, the probability $P(L' > 0| - t < L < t)$ has the following expression:

$$\int_0^{\infty} c_1 e^{c_2(v)} \left[Q\left(\frac{v - G_1}{G_2}\right) - Q\left(\frac{v + G_1}{G_2}\right)\right] dv\,, \tag{22}$$

where

$$c_1 = \frac{\sqrt{\frac{1}{\sigma_{sc}^2 + \sigma_{rc}^2}}}{\sqrt{2\pi}\left(Q\left(\frac{-t - \mu_{\mathrm{sc}}}{\sigma_{\mathrm{sc}}}\right) - Q\left(\frac{t - \mu_{\mathrm{sc}}}{\sigma_{\mathrm{sc}}}\right)\right)} \tag{23}$$

$$c_2(v) = \frac{-(v - \mu_{\mathrm{sc}} - \mu_{\mathrm{rc}})^2}{2(\sigma_{\mathrm{sc}}^2 + \sigma_{\mathrm{rc}}^2)} \tag{24}$$

$$G_1 = \frac{t(\sigma_{\mathrm{sc}}^2 + \sigma_{\mathrm{rc}}^2)}{\sigma_{\mathrm{sc}}^2} \tag{25}$$

$$G_2 = \sqrt{(\sigma_{\mathrm{sc}}^2 + \sigma_{\mathrm{rc}}^2)}\frac{\sigma_{\mathrm{rc}}}{\sigma_{\mathrm{sc}}}\,. \tag{26}$$

Equation (22), which is derived in the Appendix, shows the probability that a particular bit u is decoded correctly if the original LLR value from the initial SC decoding is within the thresholds $[-t, t]$ and a repetition code of length n is used. c_1 and G_1 are variables that depend on the choice of the threshold t and the channel SNR. G_2 only depends on the SNR of the channel and the LLR distribution of the SC decoded bit u.

The probability that a particular information bit u is correctly and reliably decoded by the initial SC decoding or with the increased reliability from the repetition code when its LLR is within the threshold $[-t_i, t_i]$ is

$$P_{\mathrm{fb}(t,n)}(\hat{u} = u) \tag{27}$$

$$= P(L > t) + P(L' > 0| - t < L < t)P(|L| < t) \tag{28}$$

$$= Q\left(\frac{t - \mu_{\mathrm{sc}}}{\sigma_{\mathrm{sc}}}\right) + \frac{\int_0^{\infty} e^{c_2(v)}\left[Q\left(\frac{v - G_1}{G_2}\right) - Q\left(\frac{v + G_1}{G_2}\right)\right] dv}{2\pi\sigma_{\mathrm{sc}}\sigma_{\mathrm{rc}}}\,.$$

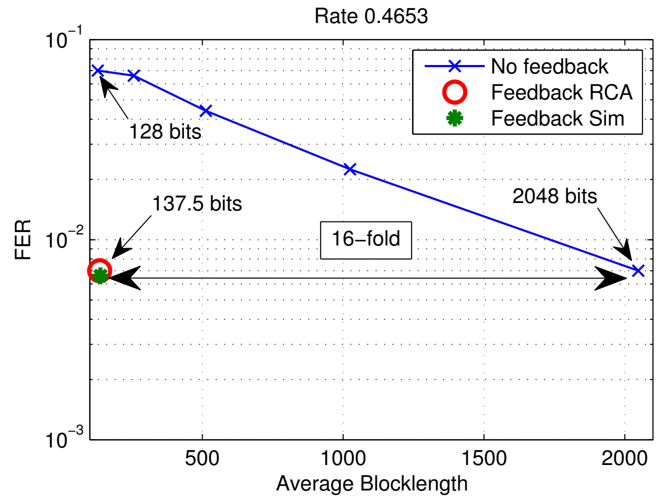

Fig. 5: FER of the feedback system obtained by simulations and its comparison to the analysis of Sec. IV. The feedback system shows about 16-fold reduction in blocklength required to achieve a target FER of 7×10^{-3} compared to the case without feedback.

The expected number of additional bits transmitted in the forward direction to increase the reliability of the single information bit u in response to feedback is

$$\Delta_{\mathrm{fb}}(u, t, n) = n\left(Q\left(\frac{-t - \mu_{\mathrm{sc},u}}{\sigma_{\mathrm{sc}}}\right) - Q\left(\frac{t - \mu_{\mathrm{sc},u}}{\sigma_{\mathrm{sc}}}\right)\right). \tag{29}$$

For a particular target FER ϵ, the optimization problem is to find the t_i and n_i values such that the total expected number of additional bits is minimized. Therefore, the optimization problem reduces to

$$\underset{\{t_1,\ldots,t_k,n_1,\ldots,n_k\}}{\text{Minimize}} \quad \sum_{i=1}^{k} \Delta_{\mathrm{fb}}(u_i, t_i, n_i) \tag{30}$$

$$\text{s.t.} \quad 1 - \prod_{i=1}^{k} P_{\mathrm{fb}(t_i,n_i)}(\hat{u}_i = u_i) < \epsilon\,. \tag{31}$$

The size of the above optimization problem is very large where n_i can be any positive integer number and each t_i is any positive real number. To simplify the optimization space, in this paper, we assume $t_i = t$ and $n_i = n$ for all i, even though we will continue to reflect in our calculations that $\mu_{\mathrm{sc},i}$ and $\sigma_{\mathrm{sc},i}$ are distinct for each i. We have found that constraining the optimization to a single t and a single n in fact does not substantially diminish performance.

Fig. 5 shows simulation results on a 2 dB binary-input AWGN channel including a polar code with feedback using the values $t = 5.1$ and $n = 5$, which optimize $\sum_{i=1}^{k} \Delta_{\mathrm{fb}}(u_i, t, n)$ for the target FER of $\epsilon = 7 \times 10^{-3}$ at 2 dB for the original rate 0.5 polar code shown in Fig. 3. This target was chosen because it is an order of magnitude below the FER of a rate-0.4653 blocklength-128 polar code at 2 dB. Based on RCA analysis, the effective rate is 0.4658 after accounting for the expected additional bits, which total $\sum_{i=1}^{k} \Delta_{\text{fb}}(u_i, t, n) = 9.40$ due to the repetition codes. The simulation results used an average of 9.55 additional bits and the effective rate of the simulation is 0.4653.

Fig 5 also shows the FER simulation results for polar codes without feedback for various blocklengths ranging from 128 to 2048 with rates set as closely as possible to 0.4653. The actual rates were 0.4688 for the blocklength 128, 0.4648 for blocklengths 256 and 512, and 0.4658 for blocklengths 1024 and 2048.

For rates near 0.465, the polar code with feedback achieved an FER of 7×10^{-3} with an average blocklength of $128 + 9.55 = 137.55$. In contrast, a polar code without feedback requires a blocklength of 2048 for a comparable FER. Thus the reduction in blocklength achieved by feedback is a about factor of 16.

V. Conclusions

In this paper we have shown how to use an extension of RCA to closely and efficiently predict the FER performance of polar codes by analyzing the p.d.f of LLR values associated with information bits. Using this distribution and a feedback scheme incorporating a repetition code, the blocklength required to achieve a target FER is decreased by a factor of 16 as compared to a polar code without feedback. The use of feedback allows a smaller faction of the initial bits to be frozen because feedback provides the ability to effectively freeze a few more bits after the initial transmission through the use of a repetition codes to provide additional needed reliability.

VI. Appendix

In this section we derive the p.d.f. corresponding to (22). Equations (32)-(33) show the p.d.f.s of L_{rc} and L_{sc}.

$$f_{L_{rc}}(x) = \frac{1}{\sqrt{2\pi\sigma_{rc}^2}} e^{-\frac{(x-\mu_{rc})^2}{2\sigma_{rc}^2}} \tag{32}$$

$$f_{L_{sc}}(y) = \frac{c_1}{\sqrt{2\pi\sigma_{sc}^2}} e^{-\frac{(y-\mu_{sc})^2}{2\sigma_{sc}^2}}, -t < y < t\,, \tag{33}$$

where

$$c_1 = \frac{1}{Q\left(\frac{-t-\mu_{sc}}{\sigma_{sc}}\right) - Q\left(\frac{t-\mu_{sc}}{\sigma_{sc}}\right)}\,. \tag{34}$$

Let $L' = L_{rc} + L_{sc}$. The p.d.f of L' is the convolution of the p.d.fs of L_{rc} and L_{sc} and is given by

$$f_{L'}(v) = \int_{-\infty}^{\infty} f_{L_{rc}}(u) f_{L_{sc}}(v-u) du\,. \tag{35}$$

The integrand of (35) can be expressed as a normal term multiplied by the term c_6 as follows:

$$f_{L_{rc}}(u) f_{L_{sc}}(v-u) = c_2 \frac{1}{\sqrt{2\pi c_3^2}} e^{-\frac{(u-c_4)^2}{2c_3^2}}, \tag{36}$$

where

$$c_2 = c_5 e^{c_6} \sqrt{2\pi c_3^2} \tag{37}$$

$$c_5 = \frac{c_1}{2\pi\sigma_{sc}\sigma_{rc}} \tag{38}$$

$$c_6 = \frac{-(v-\mu_{sc}-\mu_{rc})^2}{2(\sigma_{sc}^2+\sigma_{rc}^2)} \tag{39}$$

$$c_3^2 = \frac{\sigma_{rc}^2\sigma_{sc}^2}{\sigma_{sc}^2+\sigma_{rc}^2} \tag{40}$$

$$c_4 = \frac{\sigma_{sc}^2\mu_{rc} + (v-\mu_{sc})\sigma_{rc}^2}{\sigma_{sc}^2+\sigma_{rc}^2}\,. \tag{41}$$

The variable u is only defined in the range of $[v-t, v+t]$ consistent with (33). Integrating (36) over $u \in [v-t, v+t]$, the p.d.f of $f_{L'}(v)$ is derived as

$$f_{L'}(v) = c_2 \left(Q\left(\frac{v-t-c_4}{c_3}\right) - Q\left(\frac{v+t-c_4}{c_3}\right)\right). \tag{42}$$

By simplifying terms, (42) reduces to

$$f_{L'}(v) = c_2 \left(Q\left(\frac{v+G_1}{G_3}\right) - Q\left(\frac{v+G_2}{G_3}\right)\right), \tag{43}$$

where (44)

$$G_1 = \frac{-\sigma_{sc}^2\mu_{rc} + \mu_{sc}\sigma_{rc}^2 - t(\sigma_{sc}^2+\sigma_{rc}^2)}{\sigma_{sc}^2} \tag{45}$$

$$G_2 = \frac{-\sigma_{sc}^2\mu_{rc} + \mu_{sc}\sigma_{rc}^2 + t(\sigma_{sc}^2+\sigma_{rc}^2)}{\sigma_{sc}^2} \tag{46}$$

$$G_3 = \sqrt{(\sigma_{sc}^2+\sigma_{rc}^2)}\frac{\sigma_{rc}}{\sigma_{sc}}\,. \tag{47}$$

Under the assumptions that $\sigma_{sc}^2 = 2\mu_{sc}$ and $\sigma_{rc}^2 = 2\mu_{rc}$,

$$G_2 = -G_1 = \frac{t(\sigma_{sc}^2+\sigma_{rc}^2)}{\sigma_{sc}^2}\,. \tag{48}$$

Finally, the probability of (22) is given by

$$\int_0^{\infty} c_6 \left(Q\left(\frac{v-G_1}{G_3}\right) - Q\left(\frac{v+G_1}{G_3}\right)\right) dv. \tag{49}$$

References

[1] E. Arikan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *CoRR*, vol. abs/0807.3917, 2008. [Online]. Available: http://arxiv.org/abs/0807.3917

[2] B. Serbetci and A. Pusane, "On the selection of generator matrices for polar coded communication systems," in *Signal Processing and Communications Applications Conference (SIU), 2012 20th*, April 2012, pp. 1–4.

[3] S. Korada, E. Sasoglu, and R. Urbanke, "Polar codes: Characterization of exponent, bounds, and constructions," *Information Theory, IEEE Transactions on*, vol. 56, no. 12, pp. 6253–6264, Dec 2010.

[4] E. Arikan and I. Telatar, "On the rate of channel polarization," in *Information Theory, 2009. ISIT 2009. IEEE International Symposium on*, June 2009, pp. 1493–1495.

[5] S. Y. Chung, "On the construction of some capacity-approaching coding schemes," Ph.D. dissertation, MIT, Cambridge, MA, 2000.

[6] T. Chen, K. Vakilinia, D. Divsalar, and R. D. Wesel, "Protograph-based raptor-like LDPC codes," *CoRR*, vol. abs/1403.2111, 2014. [Online]. Available: http://arxiv.org/abs/1403.2111

[7] A. Ashikhmin, G. Kramer, and S. ten Brink, "Extrinsic information transfer functions: model and erasure channel properties," *IEEE Trans. Inf. Theory*, vol. 50, no. 11, pp. 2657–2673, Nov 2004.