Nonlinear Turbo Codes for the broadcast Z Channel

Richard Wesel
Miguel Griot
Bike Xie
Andres Vila Casado
Outline

- The stochastically degraded Broadcast channel.
- The broadcast Z channel (B-Z channel)
  - Optimal transmission strategy.
  - Capacity region.
  - Channel coding design, nonlinear turbo codes:
    - Controlled ones density.
    - Designed for the Z channel and the Z channel with erasures.
  - Simulation results.
- Conjecture: optimal transmission strategy for a particular set of broadcast channels.
- Conclusions
The stochastically degraded broadcast channel

Stochastically degraded if:

\[ \exists p'(y_2 | y_1) \text{ such that } p(y_2 | x) = \sum_{y_1} p(y_1 | x) p'(y_2 | y_1) \]
The stochastically degraded channel

Capacity region for sending independent information over the degraded channel $X \rightarrow Y_1 \rightarrow Y_2$ is the convex hull of the closure of the rate pairs

$$R_1 \leq I(X; Y_1 | X_2)$$
$$R_2 \leq I(X_2; Y_2)$$
The broadcast Z channel

Theorem: Optimal surface can be achieved by:

\[ (W_1, W_2) \xrightarrow{Encoder} X \xrightarrow{OR} Y_1 \xrightarrow{OR} Y_2 \xrightarrow{Decoder} \hat{W}_1 \xrightarrow{Decoder} \hat{W}_2 \]

\[ N_1 \sim \text{Bernoulli}(\alpha) \quad N_2 \sim \text{Bernoulli}(\beta) \quad N_3 \sim \text{Bernoulli}\left(\frac{\beta - \alpha}{1 - \alpha}\right) \]

\[ \alpha < \beta \]

\[ X_1 \sim \text{Bernoulli}(p_1) \quad X_2 \sim \text{Bernoulli}(p_2) \]
Optimal transmission strategy

Sketch of proof:

General case

We need to prove that $q_1 = 0$ (or $p_1 = 1$)

Without loss of generality $q_1 \leq 1 - p_1$

Consider any $(R_1, R_2)$ point achieved with $p_2 \neq 0, p_2 \neq 1, q_1 \neq 0, p_1 \neq 1, q_1 \neq 1 - p_1$

\[ R_2 \leq I(X_2; Y_2) \]
\[ R_1 \leq I(X; Y_1 | X_2) \]
Proof for the B-Z channel

\[ A : (R_1, R_2) \mid p_2, q_1, p_1 \]

\[ R_1 \leq I(X_2; Y_2) \]

\[ R_2 \leq I(X; Y_1 \mid X_2) \]

\[ p_2 \neq 0, p_2 \neq 1, q_1 \neq 0, p_1 \neq 1, q_1 \neq 1 - p_1 \]

\[ \Delta_1 : \begin{cases} p_2 \rightarrow p_2 \\ q_1 \rightarrow q_1 - (1 - p_2)\bar{\epsilon} \\ p_1 \rightarrow p_1 - p_2\bar{\epsilon} \end{cases}, \bar{\epsilon} > 0 \]

\[ \Delta_2 : \begin{cases} p_2 \rightarrow p_2 + (1 - p_2)\hat{\epsilon} \\ q_1 \rightarrow q_1 \\ p_1 \rightarrow p_1 + (1 - p_1 - q_1)\hat{\epsilon} \end{cases}, \hat{\epsilon} > 0 \]
Perceived channels

- Receiver 2: Z channel

\[ R_2 \leq I(X_2; Y_2) \]
\[ = H[(1 - p_2)(1 - p_1)(1 - \beta)] - (1 - p_2)H[(1 - p_1)(1 - \beta)] \]

- Receiver 1: Z channel + erasure channel.

\[ R_1 \leq I(X; Y_1 | X_2) \]
\[ = (1 - p_2)\{H[(1 - p_1)(1 - \alpha)] - (1 - p_1)H[1 - \alpha]\} \]
Implementation

- Encoding: OR of two parallel concatenated nonlinear trellis codes [GlobeCom’06].

- Decoding receiver 1 (hard):

  \[
  \hat{x}_1 = \begin{cases} 
  y_1 & \text{if } \hat{x}_2 = 0, \\
  e & \text{if } \hat{x}_2 = 1.
  \end{cases}
  \]
Parallel Concatenated Nonlinear Trellis Codes

- Presented in GlobeCom’06 (for Z channel).
- The NLTC consists of:
  - A $2^\nu$-state trellis structure (block $S$).
  - A look-up table (LUT) stores an output per branch.
  - The outputs satisfy the required ones density $\rho$ (non-systematic)
- PC-NLTC: Two constituent $(n_0, k_0)$ non-linear trellis codes (NLTC) linked by an interleaver ($\Pi$) of length $K$.  

\[
\text{rate} = \frac{k_0}{2n_0}
\]
Example

Capacity region and simulated rates, $\alpha = 0.15, \beta = 0.6$

- **Capacity surface**
- ★ Simulated rates
- ○ Optimal rates

- (1/12, 1/5)
- (1/6, 1/6)
- (1/3, 1/9)
- (1/2, 1/22)
## Results

- **8-state nonlinear turbo codes.**
- $k_0 = 2$

<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>BER$_1$</th>
<th>BER$_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/12</td>
<td>1/5</td>
<td>0.106</td>
<td>0.56</td>
<td>4800</td>
<td>1700</td>
<td>$2.54 \times 10^{-5}$</td>
<td>$1.24 \times 10^{-5}$</td>
</tr>
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<td>1/6</td>
<td>1/6</td>
<td>0.196</td>
<td>0.5</td>
<td>2048</td>
<td>2048</td>
<td>$7.01 \times 10^{-6}$</td>
<td>$5.33 \times 10^{-6}$</td>
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<tr>
<td>1/3</td>
<td>1/9</td>
<td>0.336</td>
<td>0.3739</td>
<td>1536</td>
<td>1536</td>
<td>$7.13 \times 10^{-6}$</td>
<td>$6.70 \times 10^{-6}$</td>
</tr>
<tr>
<td>1/2</td>
<td>1/22</td>
<td>0.463</td>
<td>0.1979</td>
<td>5632</td>
<td>1024</td>
<td>$9.27 \times 10^{-7}$</td>
<td>$3.27 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
The broadcast Z channel

Optimal surface can be achieved by:

Also true for (AWGN, + operator), (BSC, XOR).
Conjecture

Points on the optimal surface can be achieved by:

\[ f(x, n_1) \]

\[ f(x, n_2) \]

\[ f(x, n_3) \]

\[ X \]

\[ Y_1 \]

\[ Y_2 \]

\[ N_1 \sim p_1(n_1) \]

\[ N_2 \sim p_2(n_2) \]

\[ N_3 \sim p_3(n_3) \]

\[ |X| = |Y_1| = |Y_2| = |N_1| = |N_2| \]
Conclusions

- We have presented an optimal transmission strategy for the Broadcast Z Channel.
- Simple encoding and decoding.
- A practical implementation that works close to capacity has been presented.
- Nonlinear turbo codes, specifically designed for the Z channel and the Z channel + erasures, have been designed.
- Conjecture: simple transmission strategy could be used to a set of stochastically degraded broadcast channels.