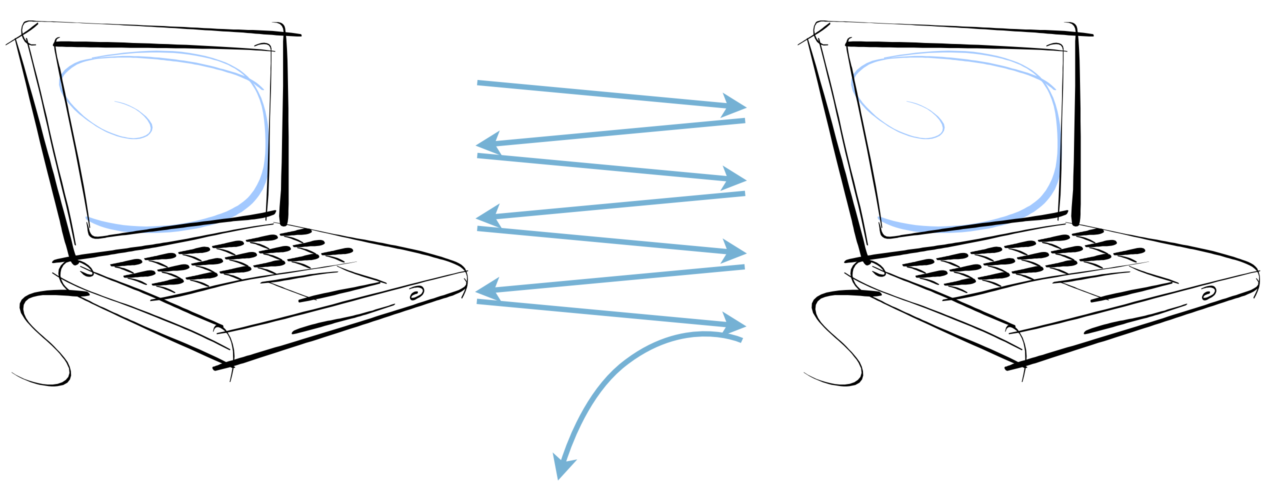
Communication Complexity Theory: An Invitation

Alexander Sherstov
UCLA

Yao's theory (1979)

$f : X \times Y \to \{0, 1\}$

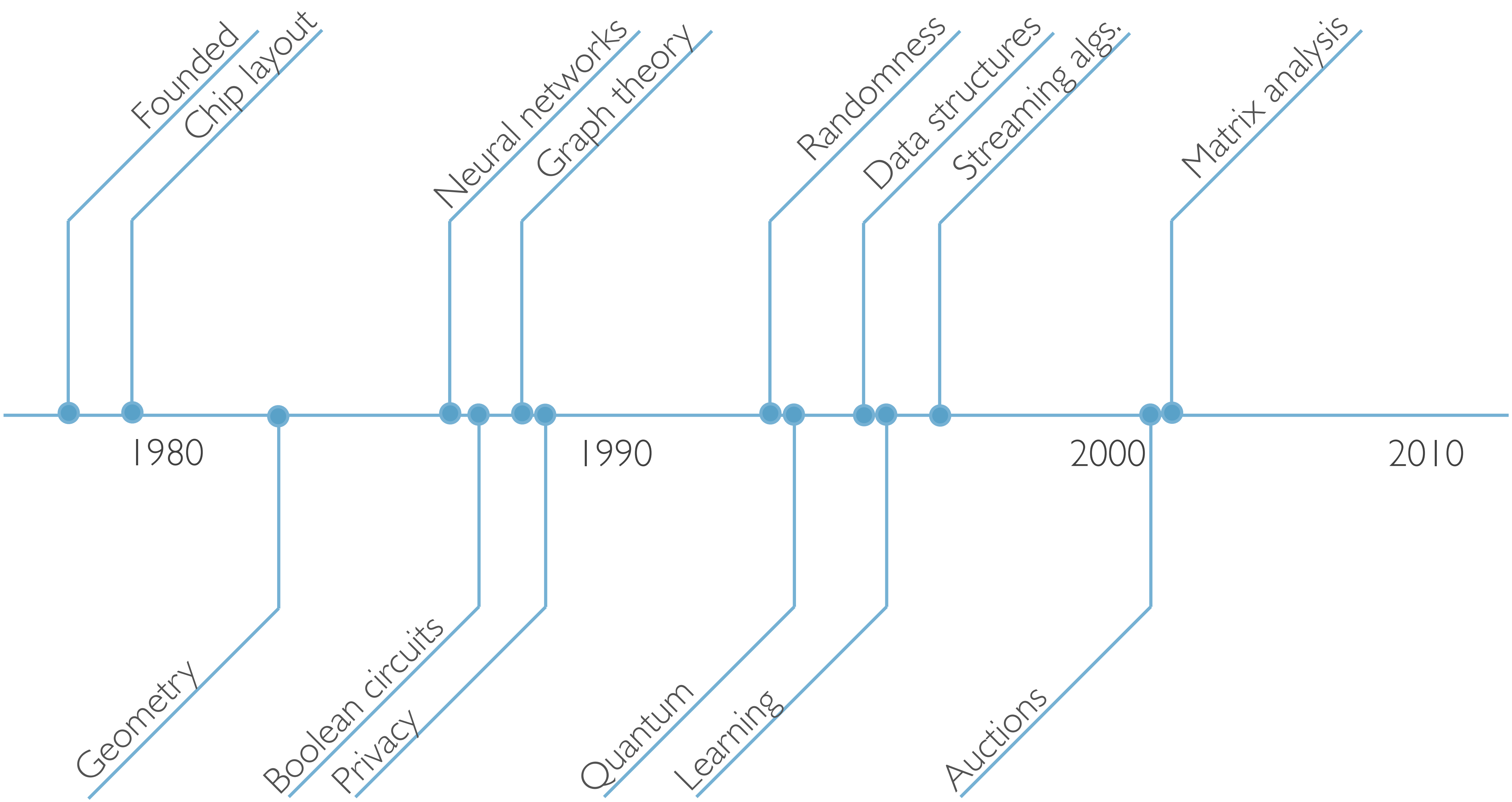
$x \in X$ $\qquad$ $y \in Y$



$f(x, y)$

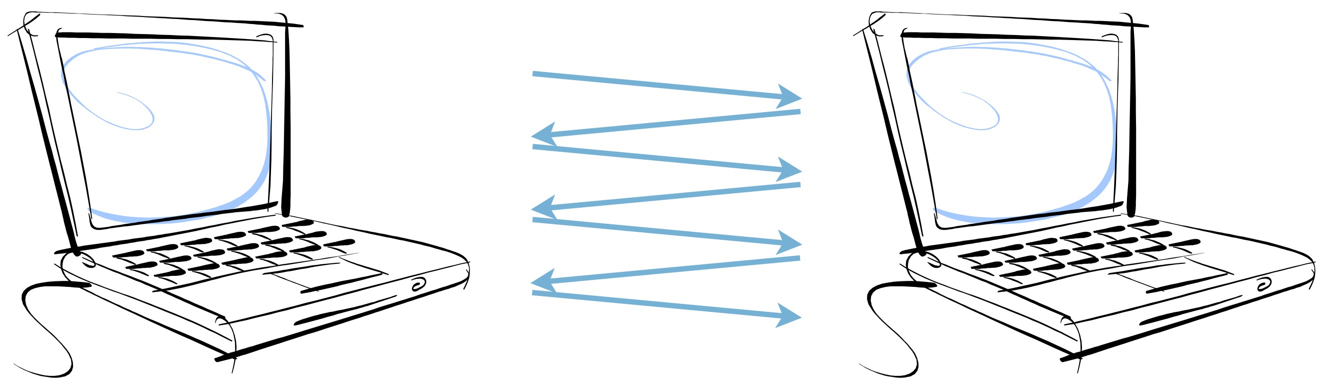
Communication is the only resource

43 years of communication



Disjointness problem

$A \subseteq \{1, 2, \ldots, n\}$ $B \subseteq \{1, 2, \ldots, n\}$



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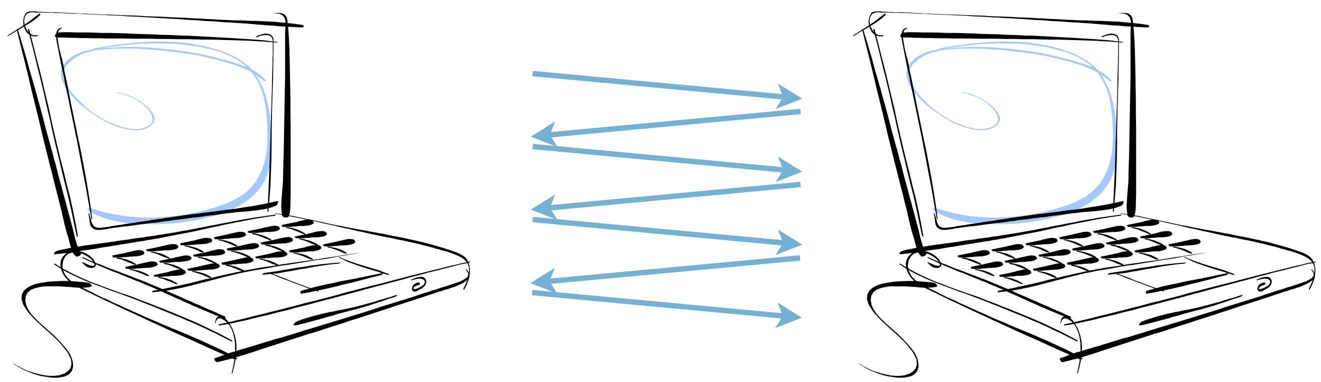

Goal:
determine if
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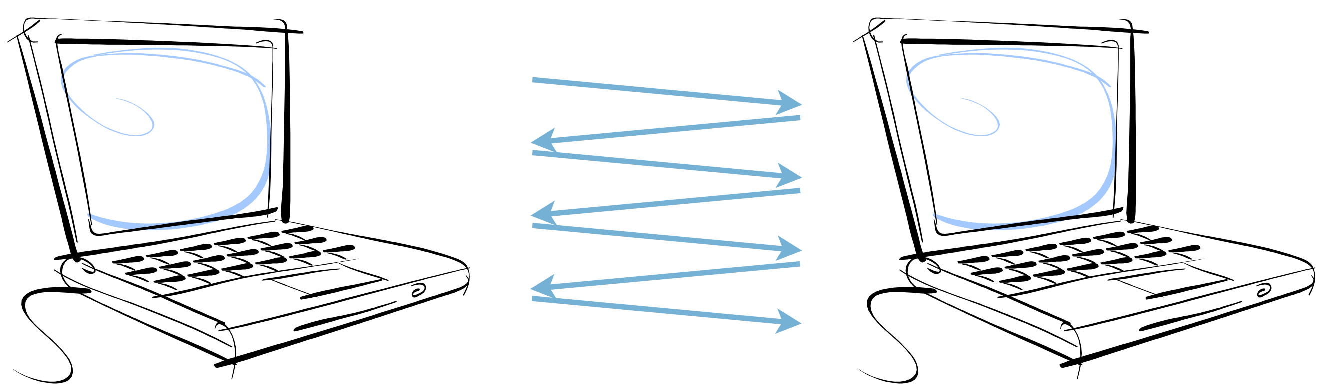

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Requires $\Omega(n)$ bits, even for correctness 51%

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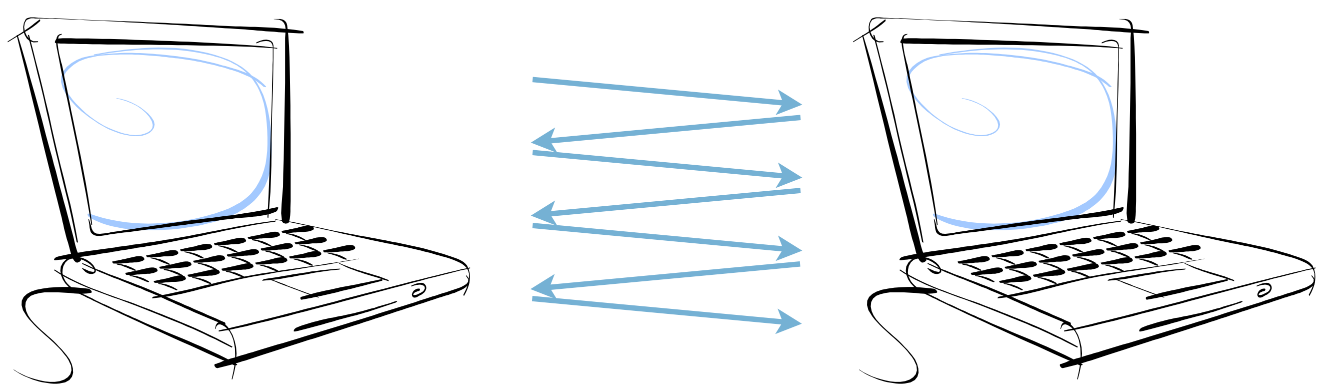

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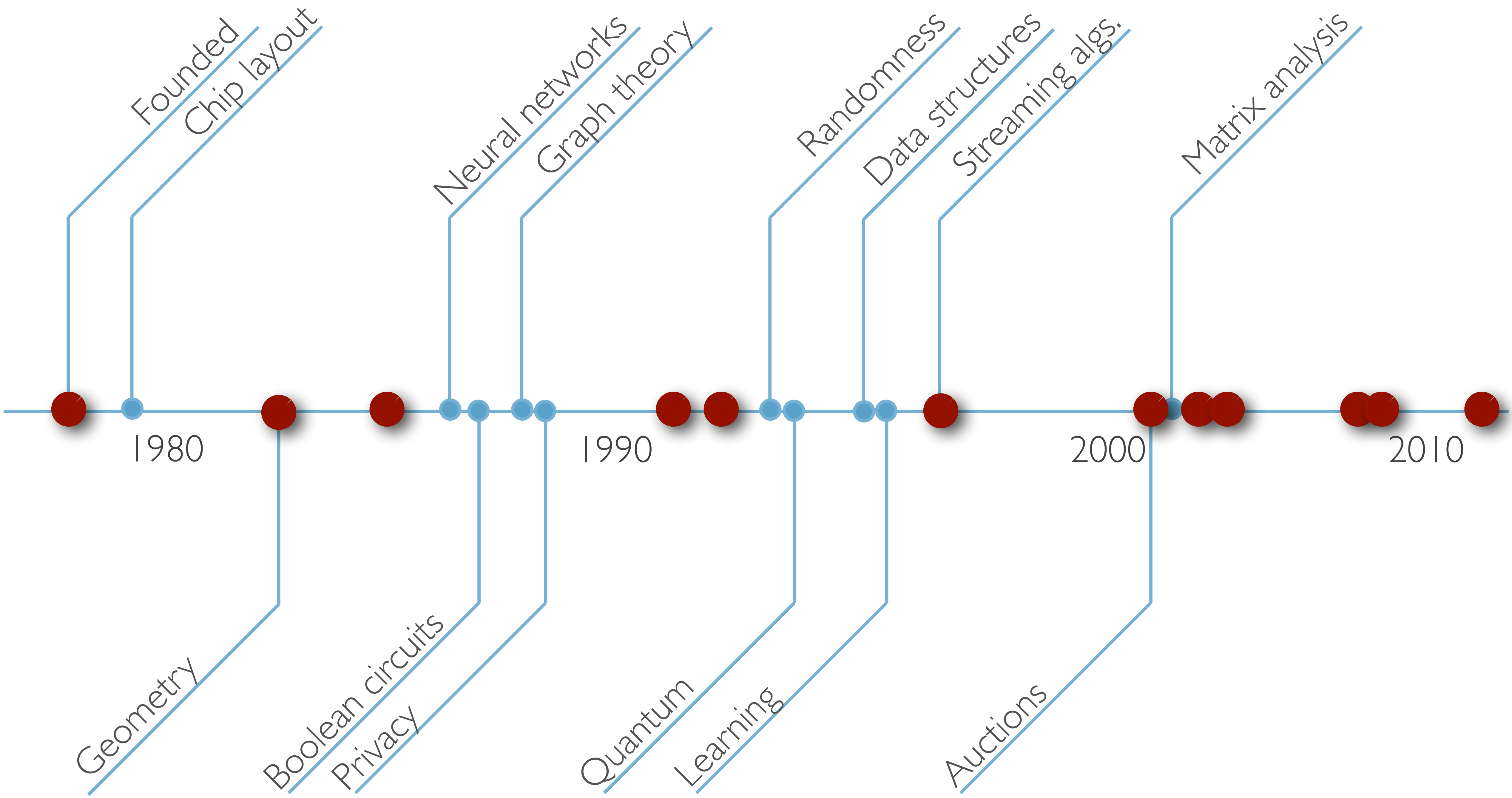
Think: scheduling a meeting

43 years of ~~communication~~ **set disjointness!**



I. Deterministic communication

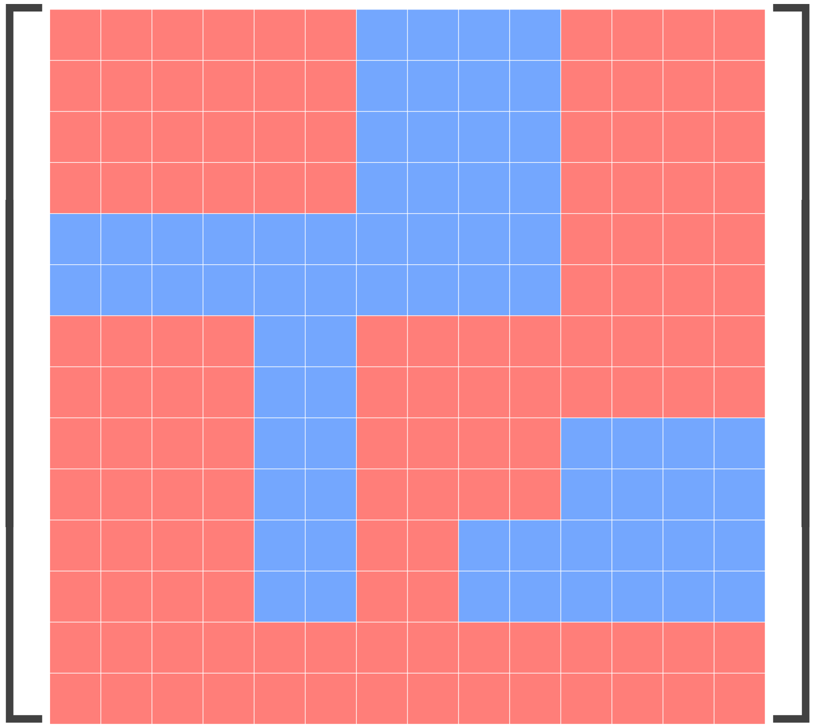
Matrix view of communication

$f : X \times Y \to \{0, 1\}$

Characteristic matrix:

$$\begin{array}{cc} \begin{array}{c} x \in X \end{array} & \\ M_f = \begin{bmatrix} & \vdots & \\ \cdots\cdots & f(x,y) & \cdots\cdots \\ & \vdots & \end{bmatrix} & y \in Y \end{array}$$

Matrix view of communication



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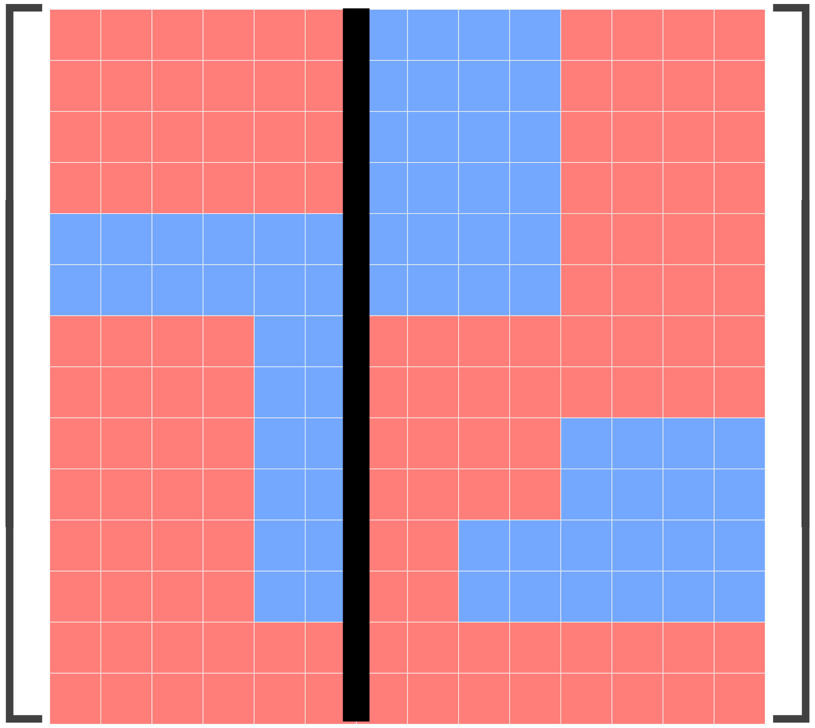

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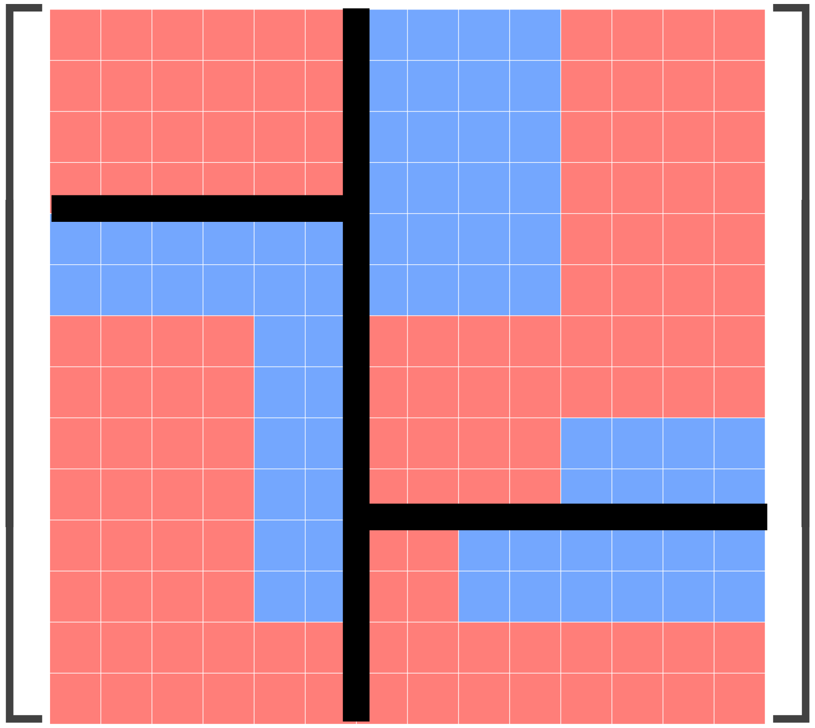

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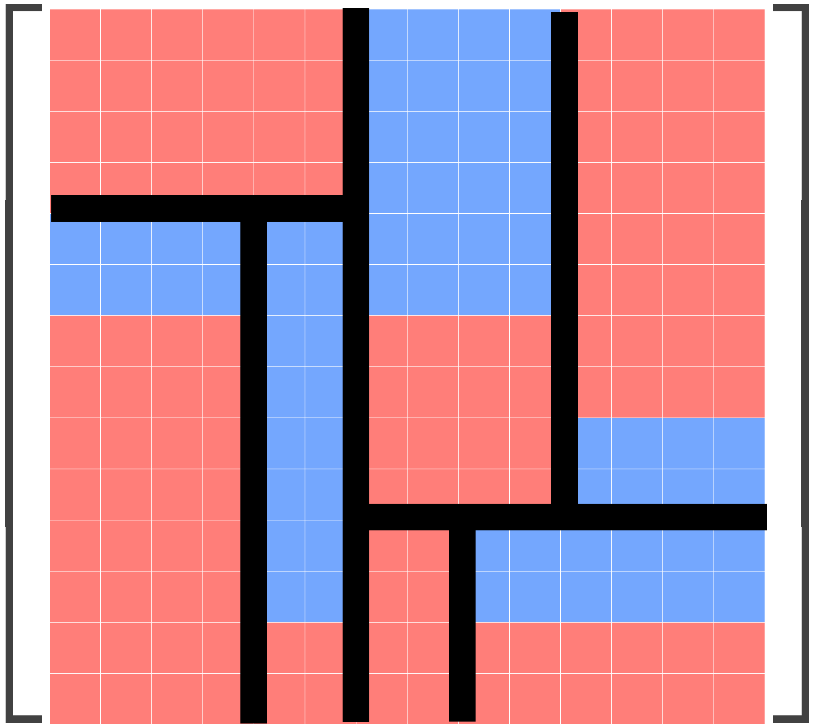

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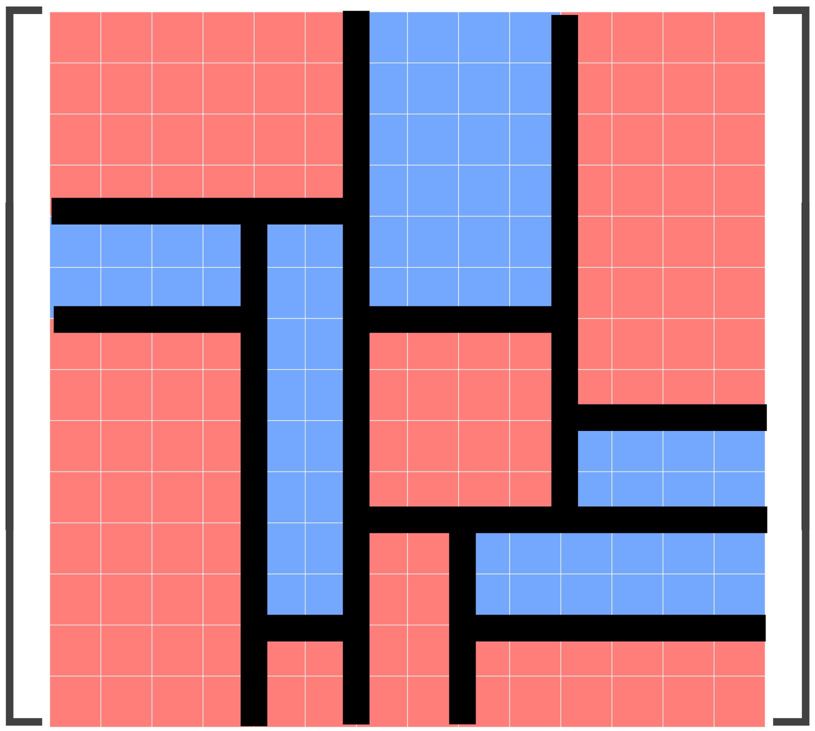

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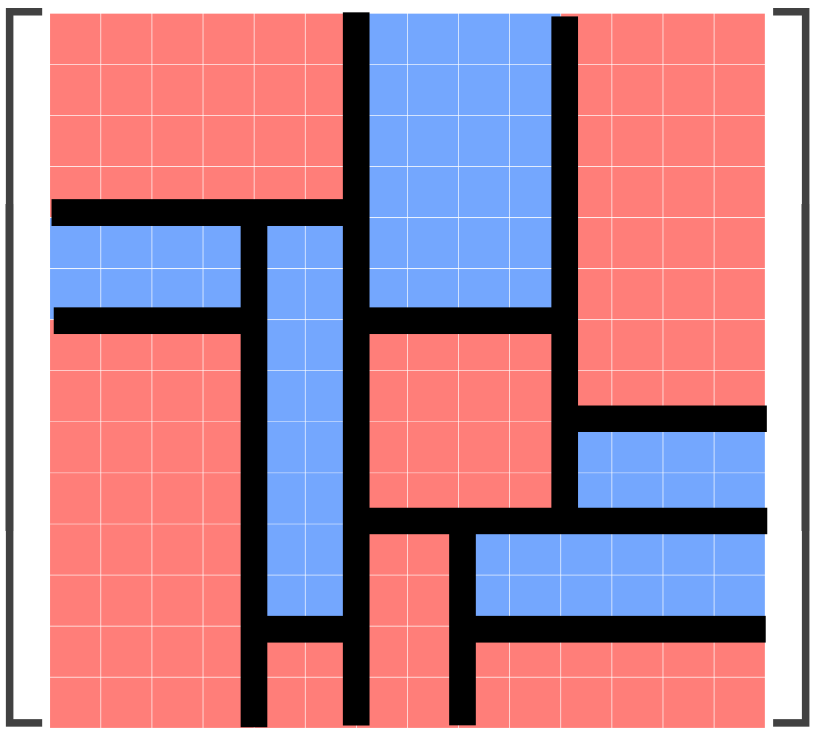

Fact. Any cost-c deterministic protocol for f partitions M_f into 2^c monochromatic submatrices.

Set disjointness

Theorem. M_{DISJ_n} cannot be partitioned into $< 2^n$ monochromatic submatrices.

Set disjointness

$\therefore\ D(\mathrm{DISJ}_n) \geq n$

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Proof (Yao '79).

$$M_{\mathrm{DISJ}_n} = \begin{array}{cc}
\begin{array}{ccccccc} & & & S & & & \end{array} & \\
\begin{bmatrix}
1 & & & \vdots & & & \\
 & 1 & & \vdots & & & \\
 & & \ddots & \vdots & & & \\
\cdots & \cdots & \cdots & 1 & \cdots & \cdots & \cdots \\
 & & & \vdots & \ddots & & \\
 & & & \vdots & & 1 & \\
 & & & \vdots & & & 1
\end{bmatrix} & \begin{array}{c} \\ \\ \\ \overline{S} \\ \\ \\ \\ \end{array}
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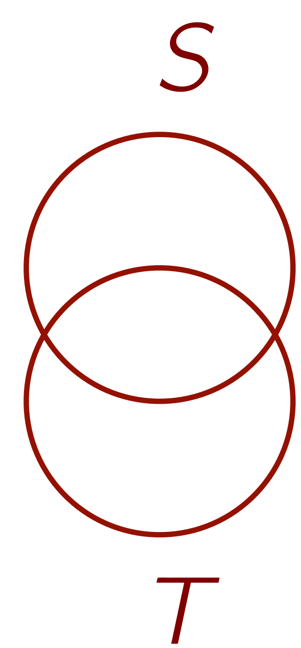

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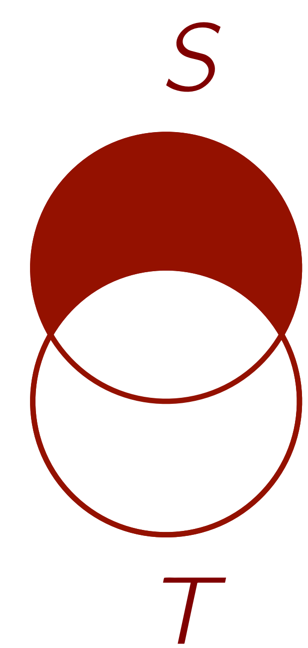

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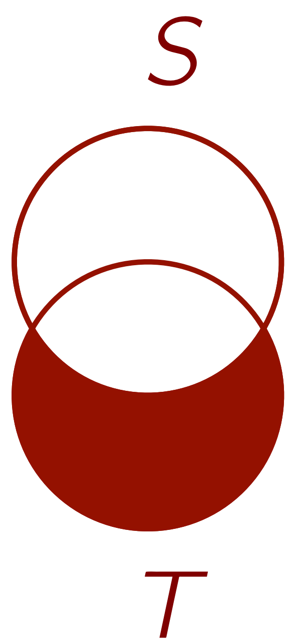

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∴ Every monochromatic submatrix contains ≤ 1 diagonal entry. ∎

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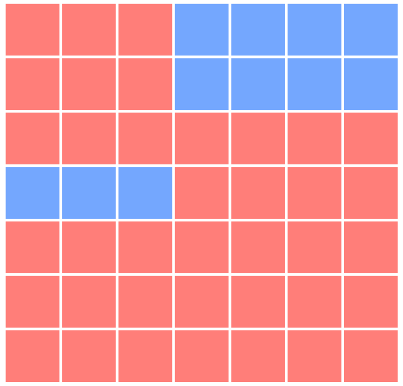

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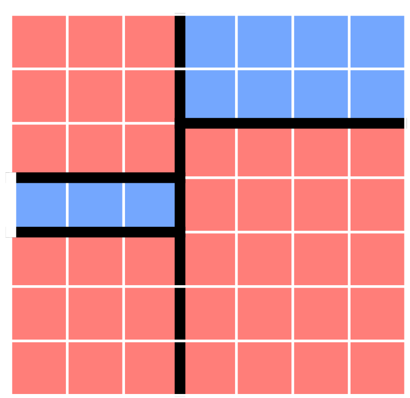

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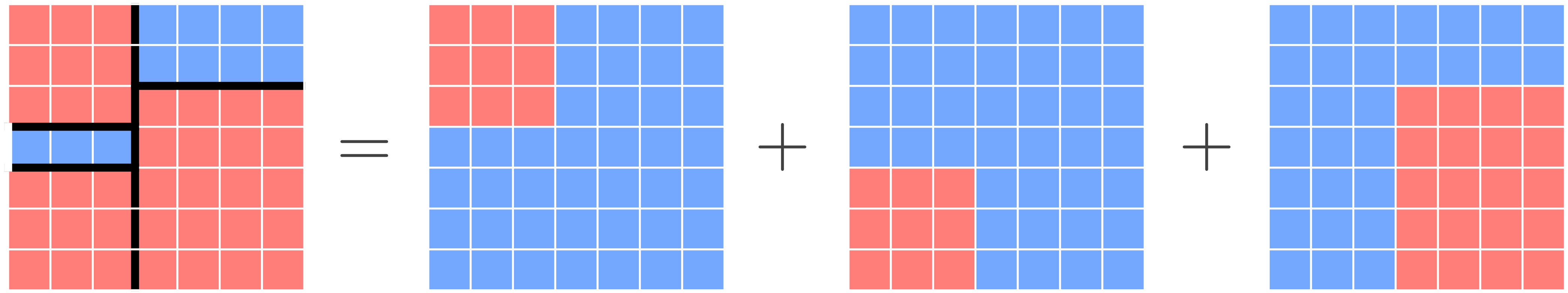

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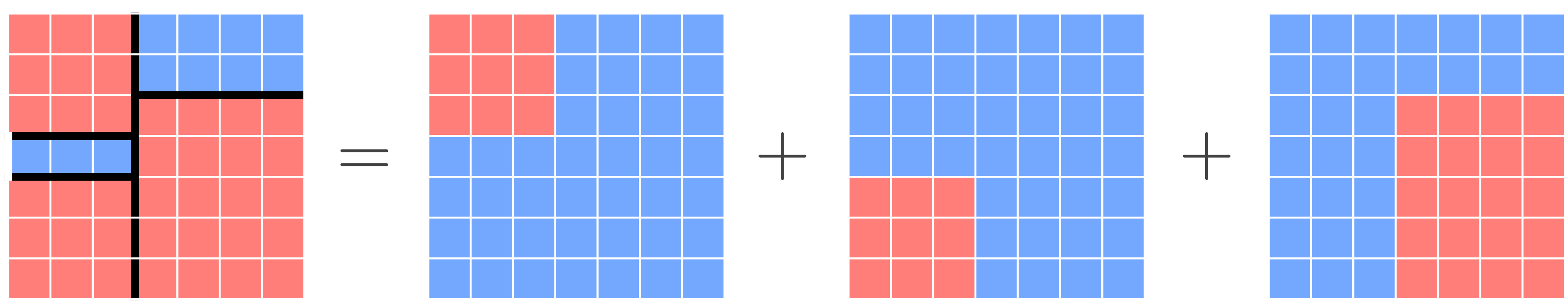

$\therefore$ Need $\mathrm{rk}\left(M_{\mathrm{DISJ}_n}\right)$ submatrices.

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$$\therefore \quad \mathrm{rk}\left(M_{\mathrm{DISJ}_n}\right) = 2^n \qquad \blacksquare$$

Log-rank conjecture

Conjecture (Lovász-Saks 1988). For all f,

$$D(f) \leq (\log \text{ rk } M_f)^{O(1)}$$

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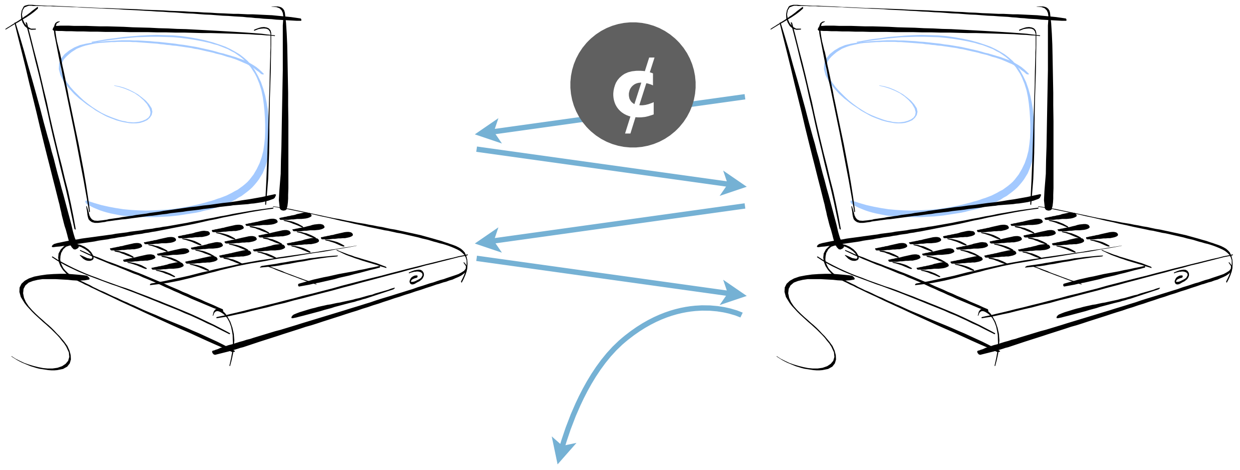
- Construction based on DISJ_n

II. Nondeterministic communication

Nondeterminism

$f : X \times Y \to \{0, 1\}$

$x \in X$ $\qquad$ $y \in Y$



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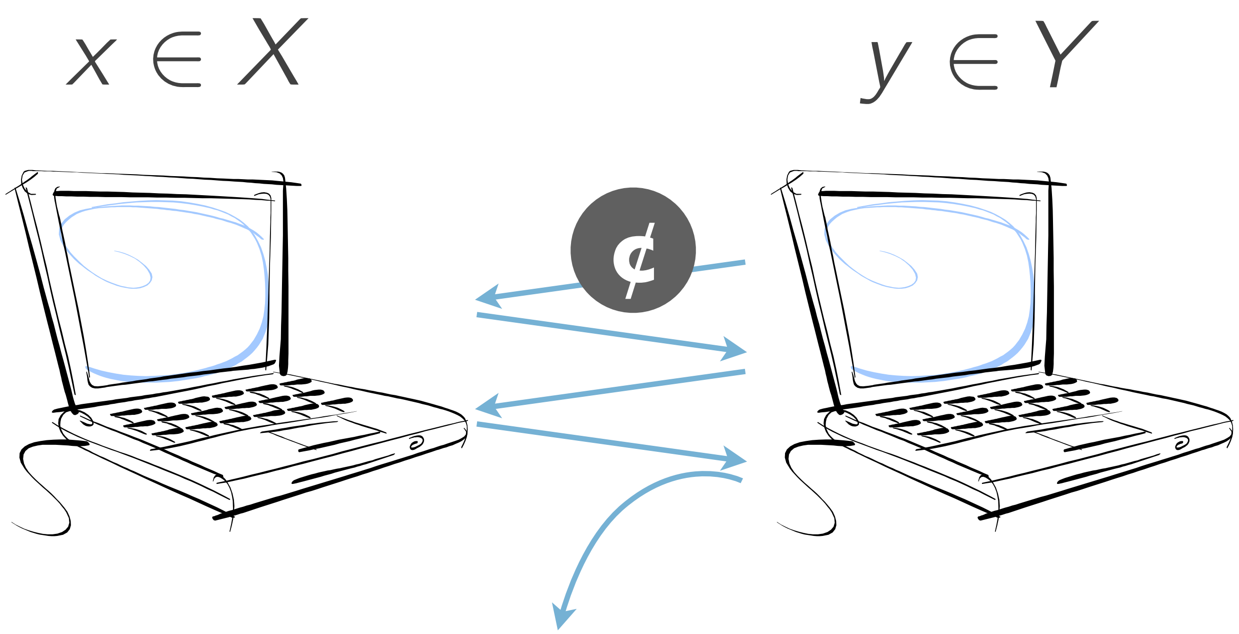

- Can send coin flips

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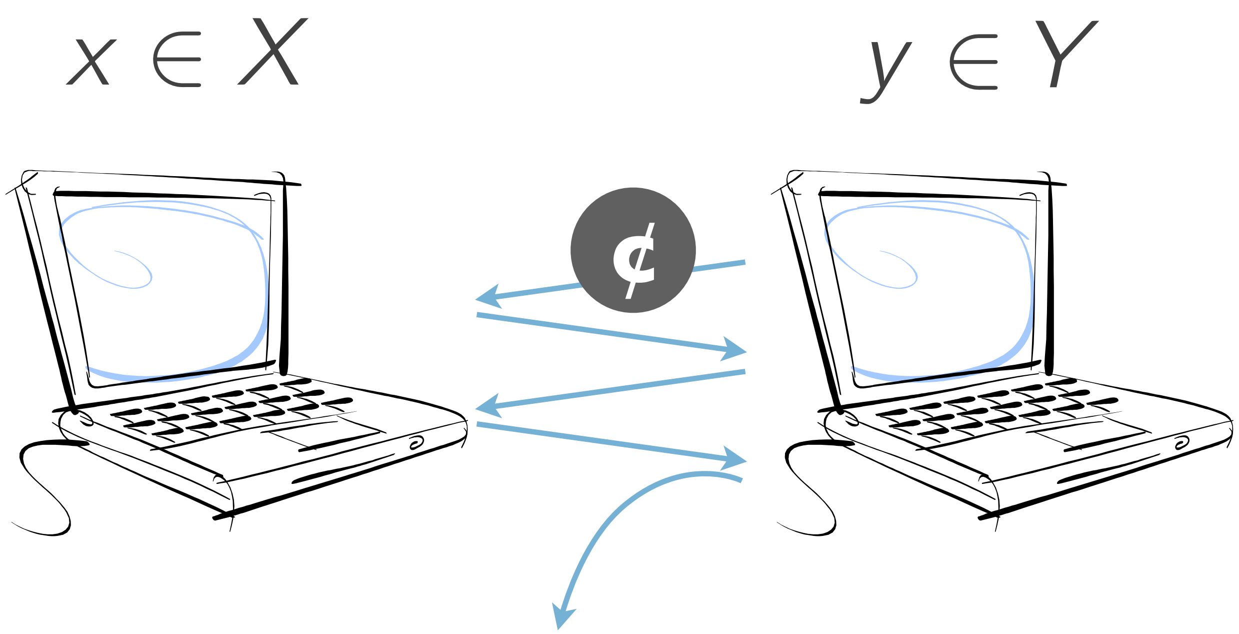

- Can send coin flips
- Correct on all coin flips when $f(x, y) = 0$
- Correct on some coin flips when $f(x, y) = 1$

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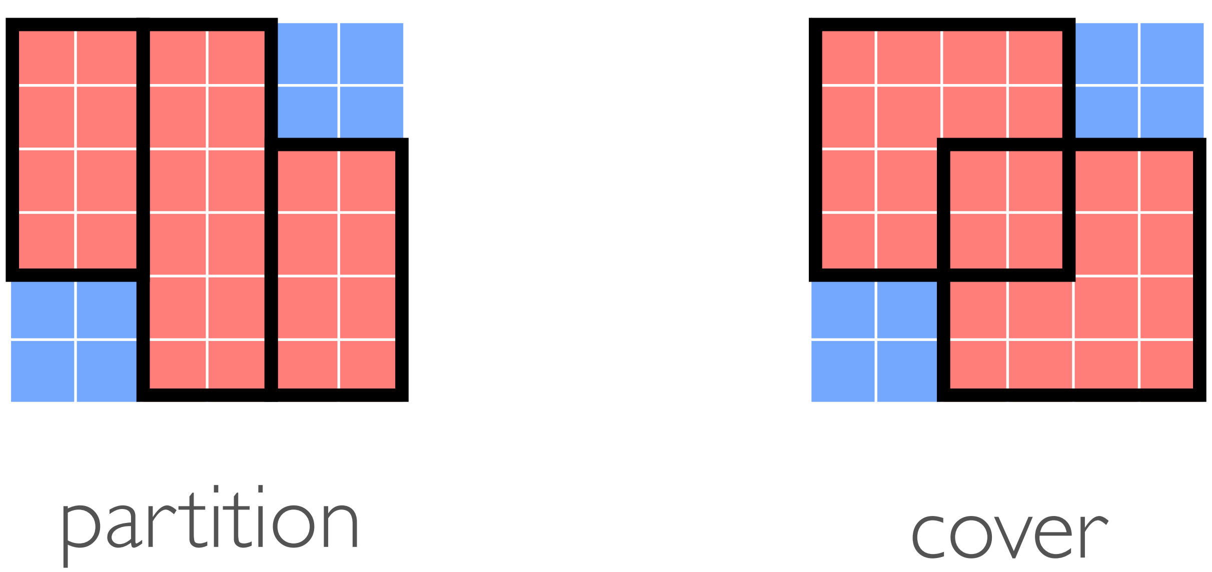

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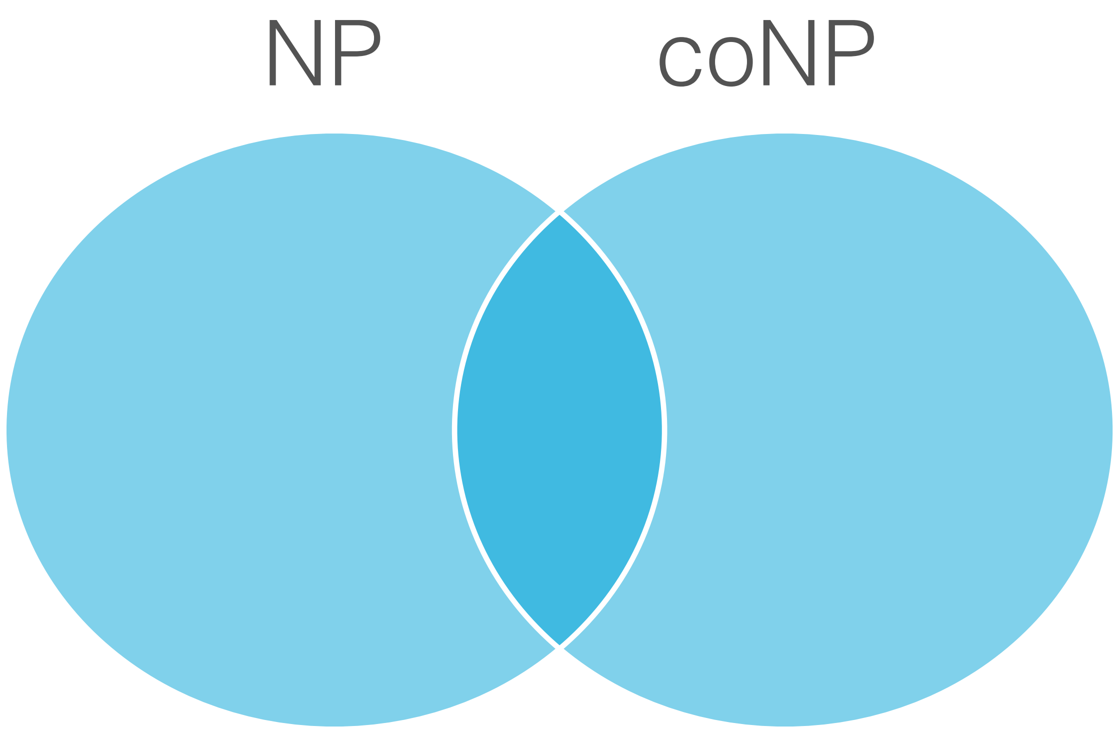
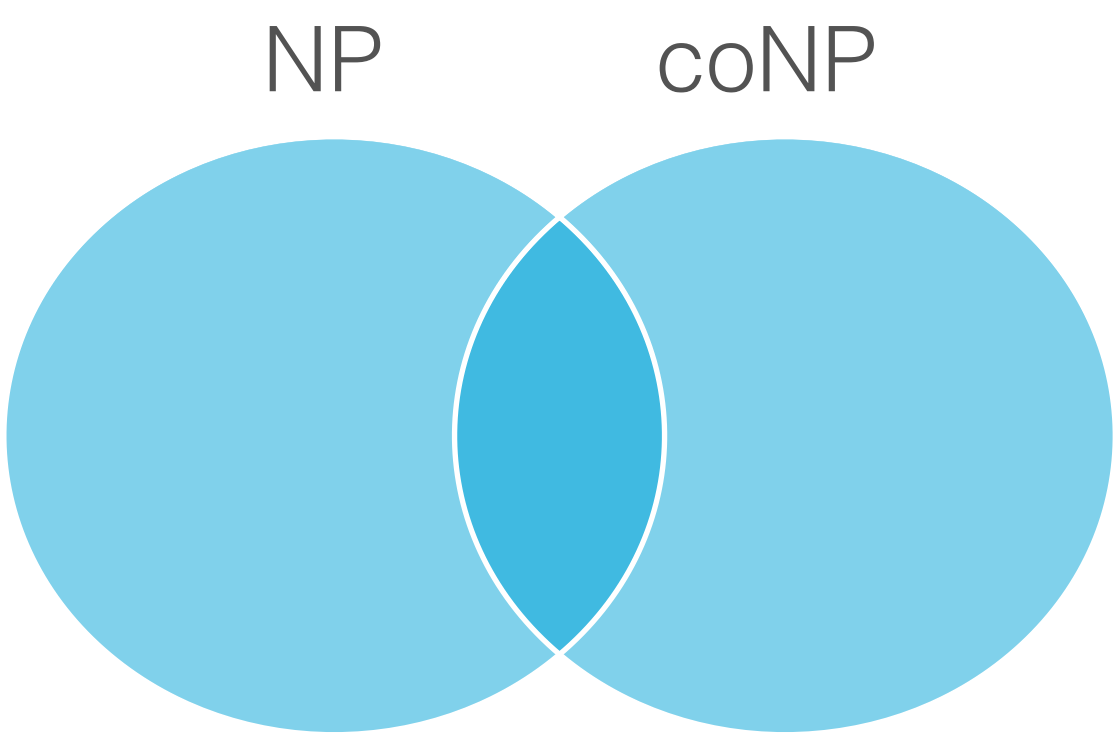
Communication classes



Classes due to
Babai et al. '86

Communication classes



Fact (Aho '83).

$$D(f) \leq 4N(f)N(\neg f)$$

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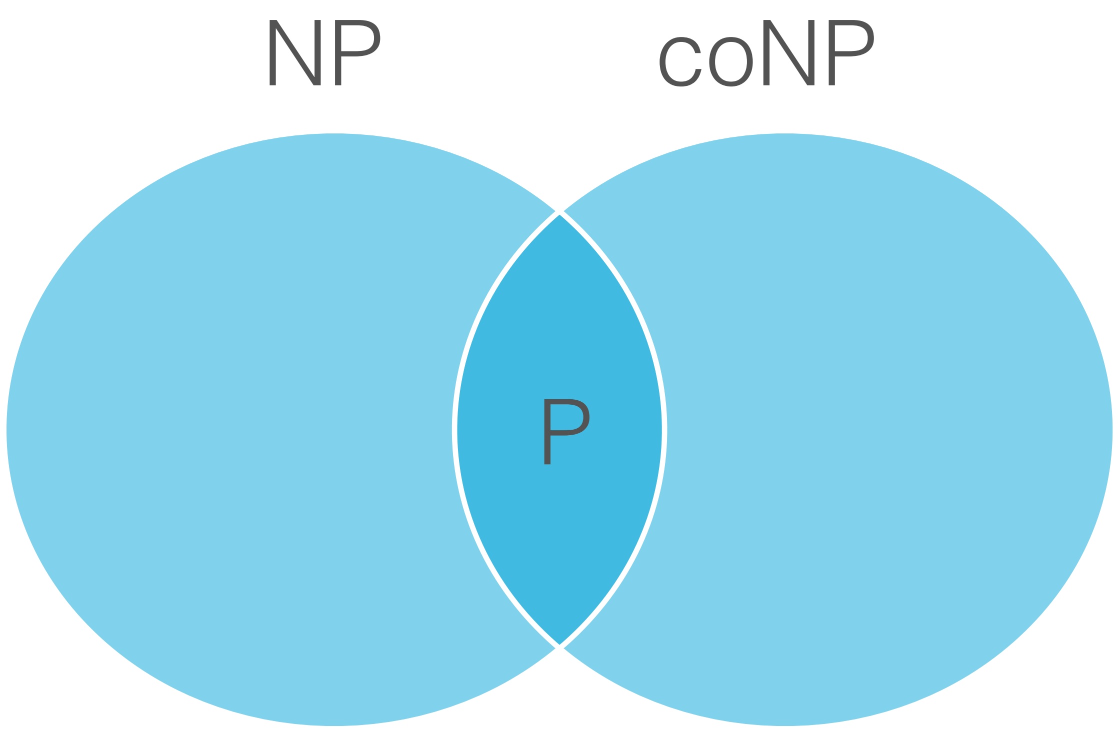

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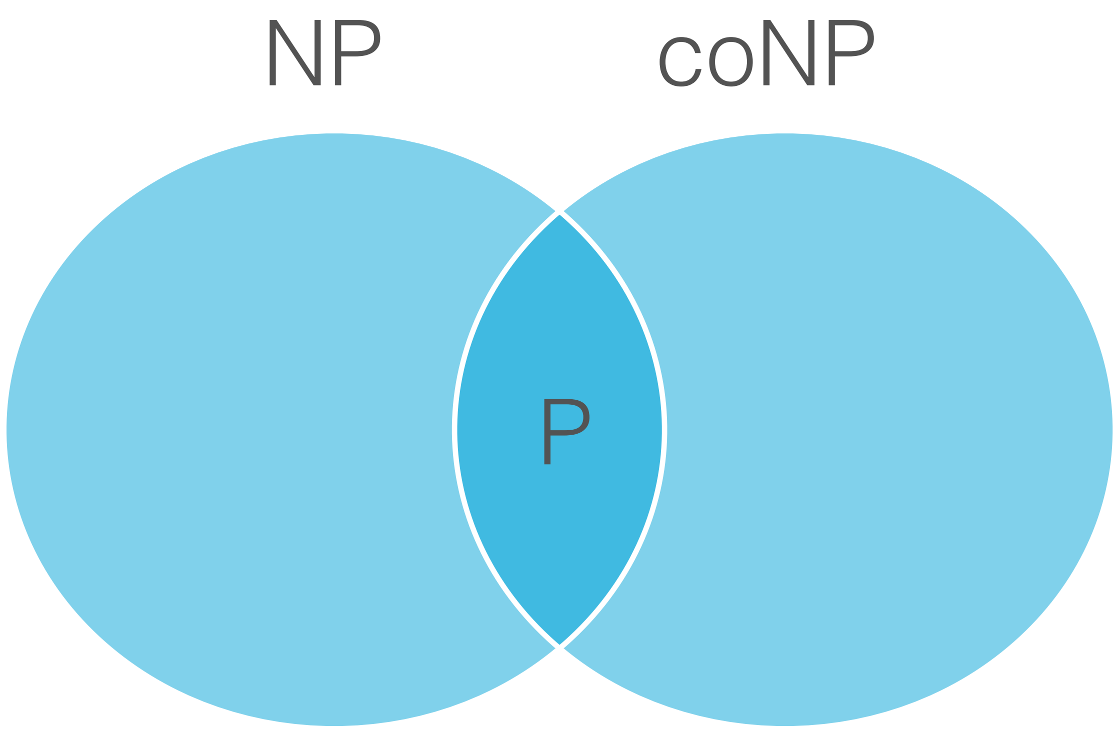

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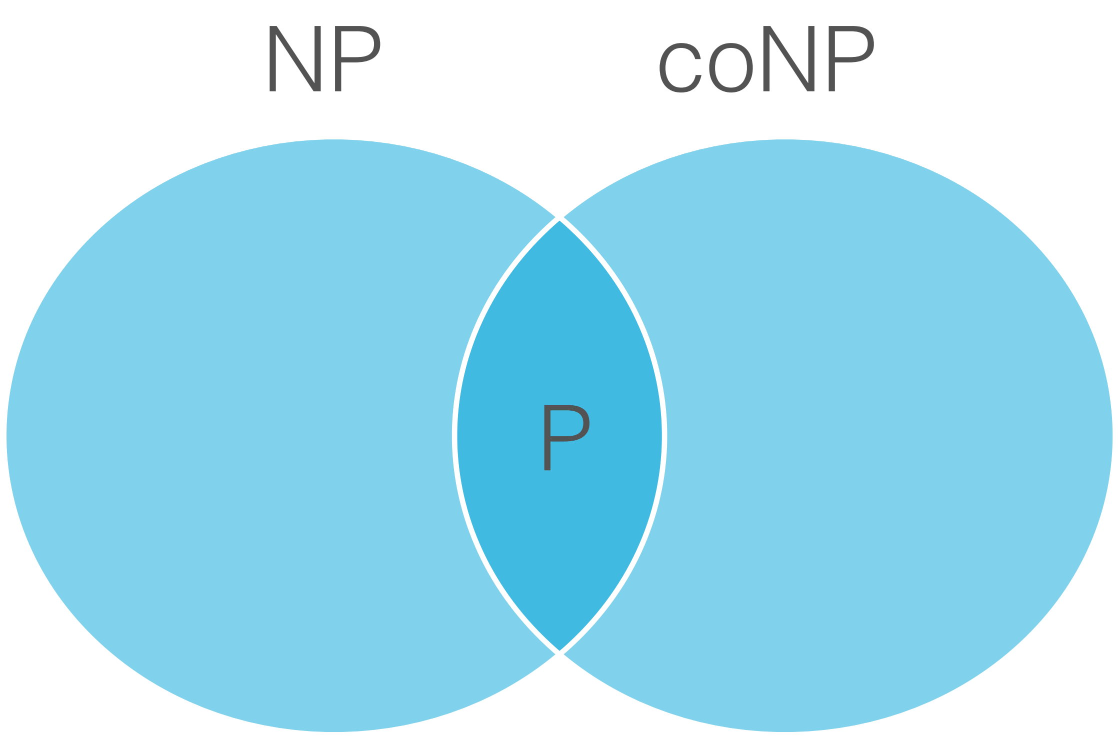

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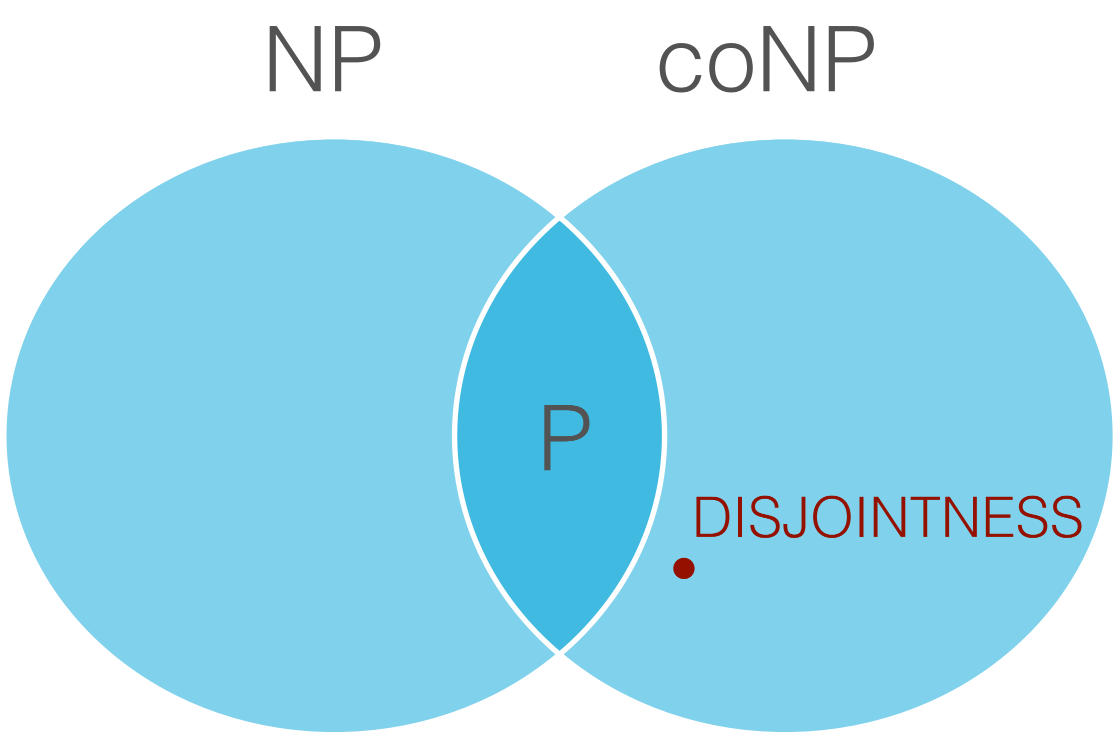
Fact.

$$D(\mathrm{DISJ}_n) = n + 1$$
$$N(\mathrm{DISJ}_n) = n$$
$$N(\neg\mathrm{DISJ}_n) = \log_2 n$$

Communication classes



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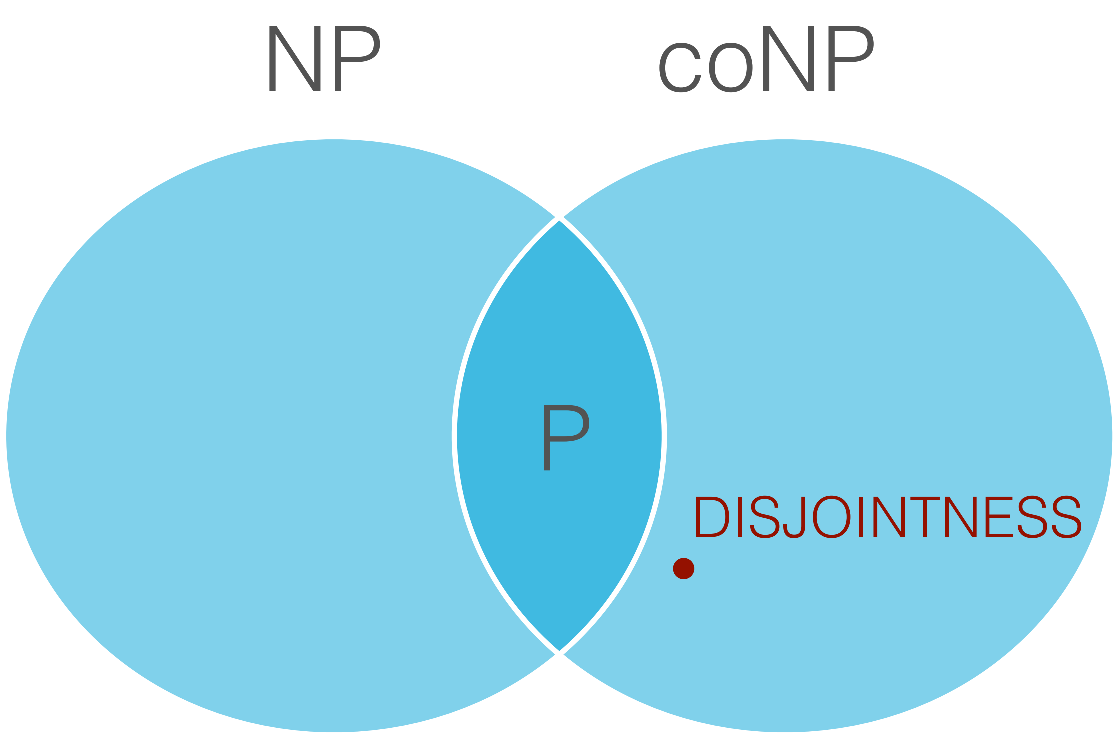
Communication classes



Classes due to Babai et al. '86

tight for *k*-disjointness!

Fact (Aho '83).

$$D(f) \leq 4N(f)N(\neg f)$$

largest gaps possible

Fact.

$$D(\mathrm{DISJ}_n) = n + 1$$

$$N(\mathrm{DISJ}_n) = n$$

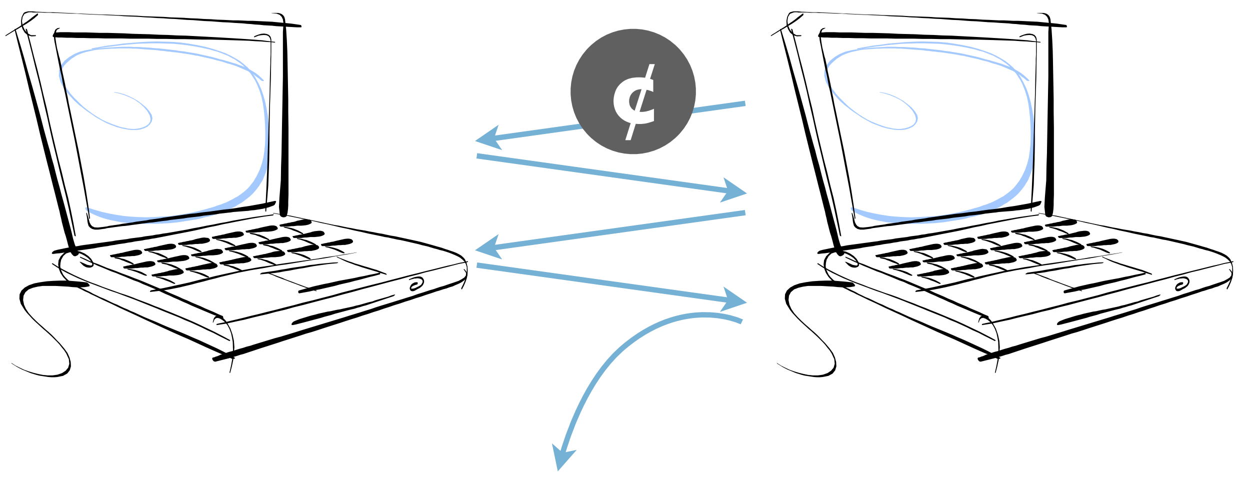
$$N(\neg\mathrm{DISJ}_n) = \log_2 n$$

III. Randomized communication

Randomness

$f : X \times Y \to \{0, 1\}$

$x \in X$ $y \in Y$



- Can send coin flips
- Error probability ≤1/3 on every input

Alternate view

A **randomized** protocol of cost c is a distribution on **deterministic** protocols of cost c.

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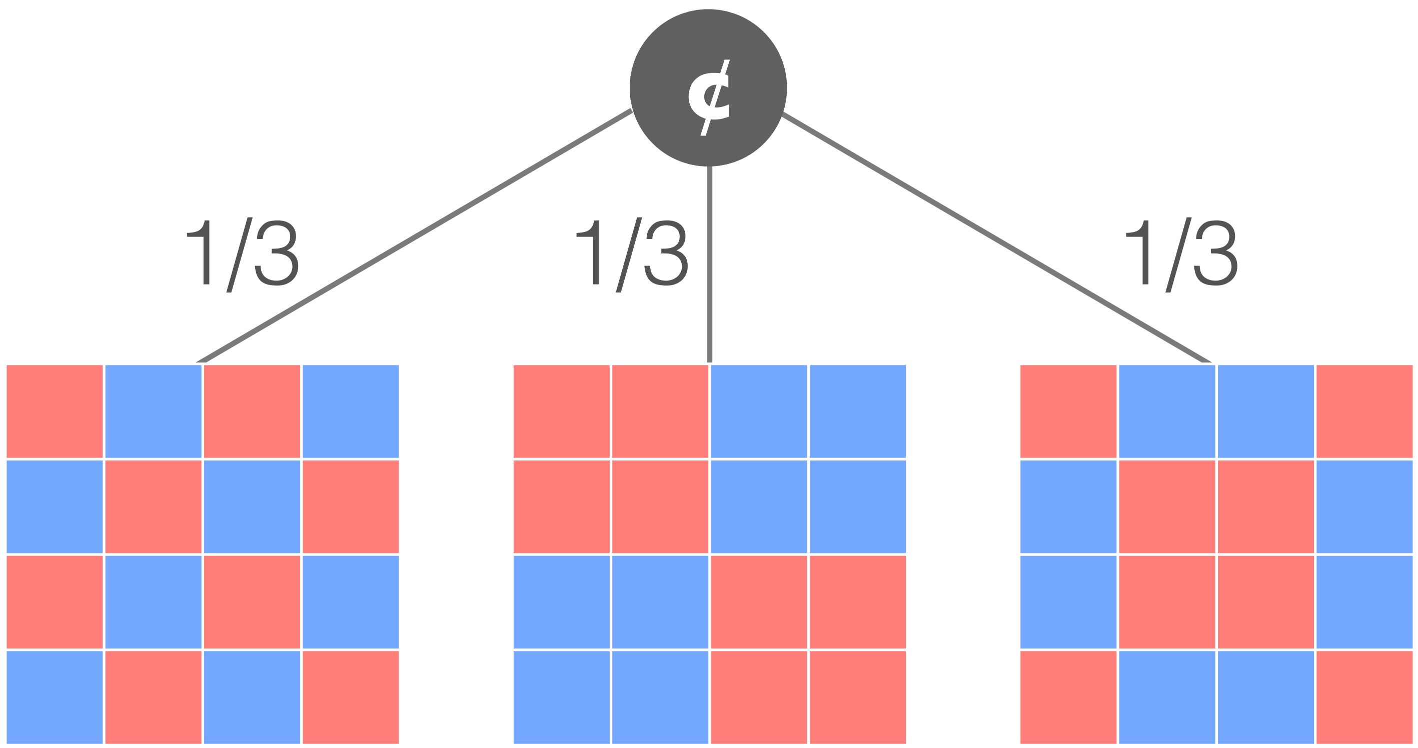

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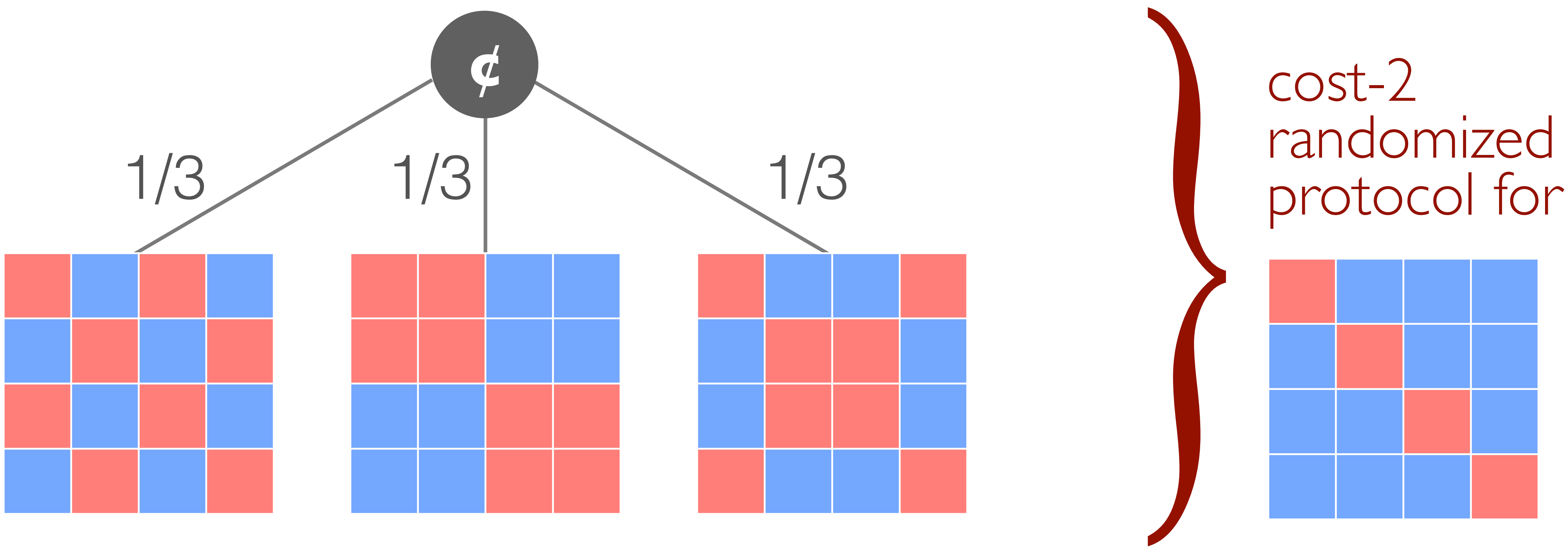

Yao’s minimax principle

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To prove that f is hard for randomized protocols, find a distribution μ on $X \times Y$ w.r.t. which f is hard for deterministic protocols.

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Theorem (Yao 1983).

Let $\mu(f^{-1}(1)) \geq 0.01,$

$$\mu(M \cap f^{-1}(0)) \geq 0.01\mu(M) - \delta \qquad \forall \text{ submatrix } M.$$

Then $R(f) \geq \Omega\left(\log \frac{1}{\delta}\right).$

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many 1's;
every large submatrix
corrupted by 0's.

Theorem (Yao 1983).

Let $\mu(f^{-1}(1)) \geq 0.01,$

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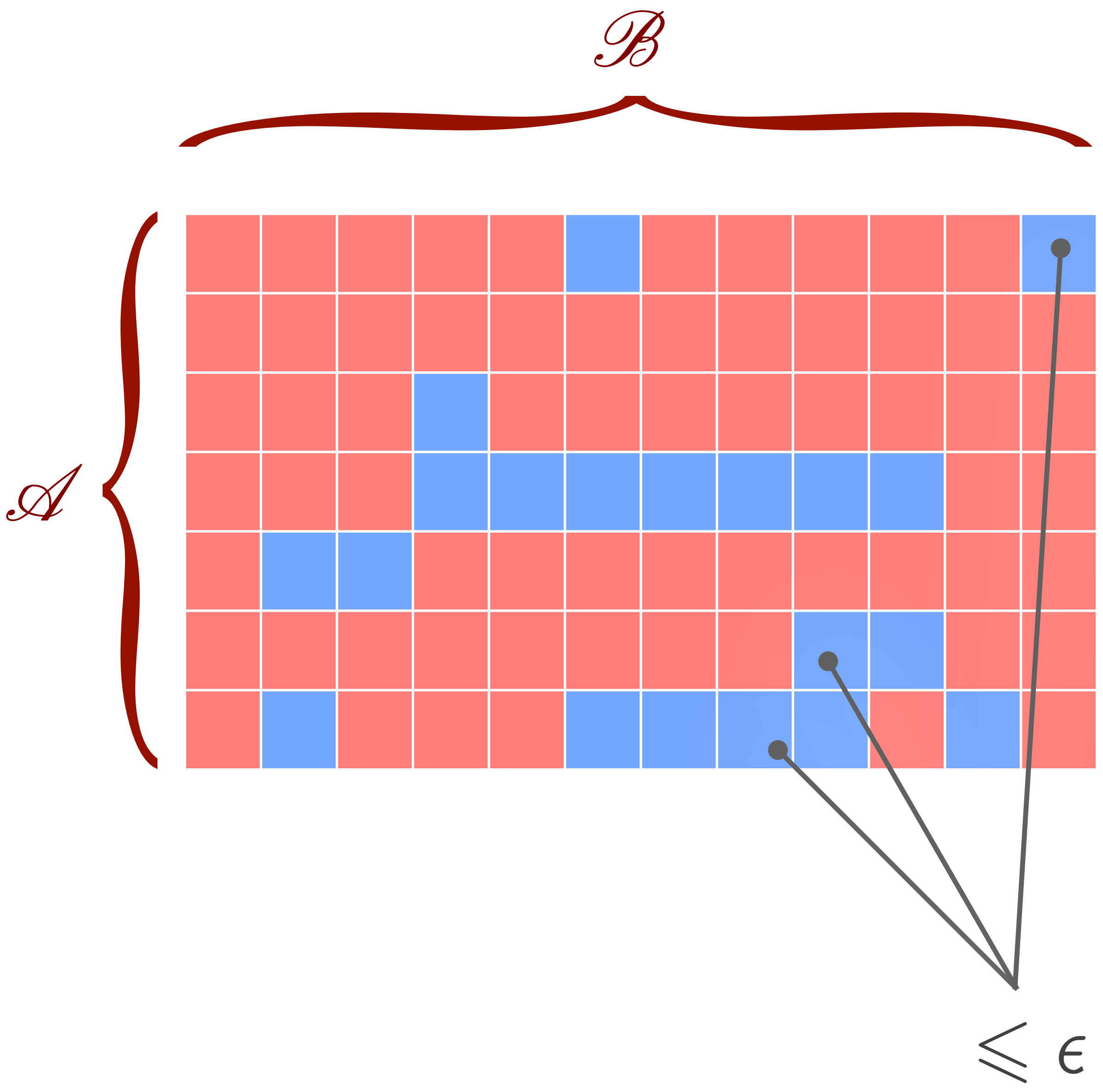
Will prove:

Theorem (Babai, Frankl, Simon 1986).

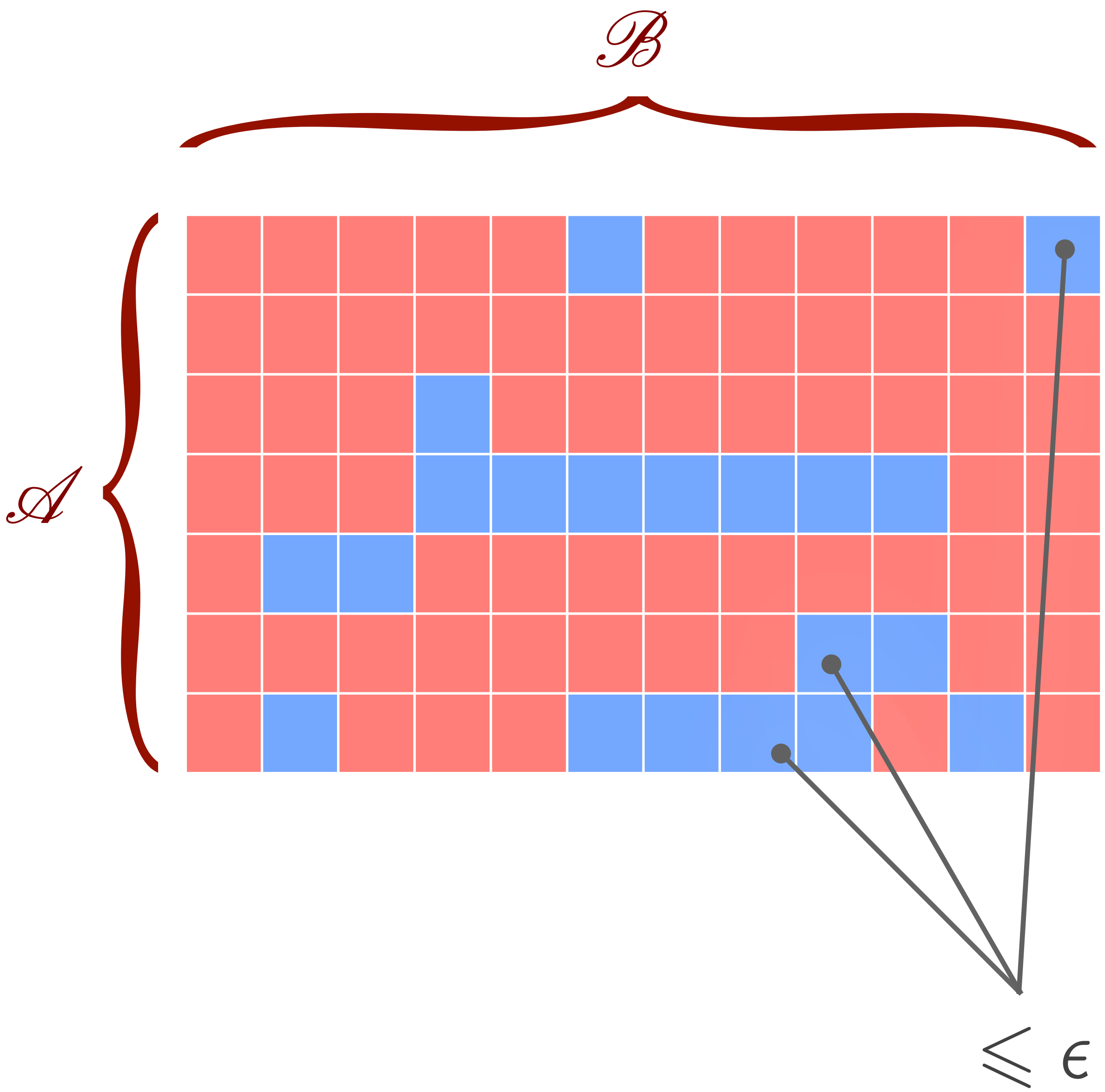
$$R(\mathrm{DISJ}_n) = \Omega(\sqrt{n})$$

Corruption lemma



$$\mathscr{A}, \mathscr{B} \subseteq \binom{\{1, 2, \ldots, n\}}{\sqrt{n}}$$

Corruption lemma



$$\mathscr{A}, \mathscr{B} \subseteq \binom{\{1, 2, \ldots, n\}}{\sqrt{n}}$$

Lemma (Babai et al).

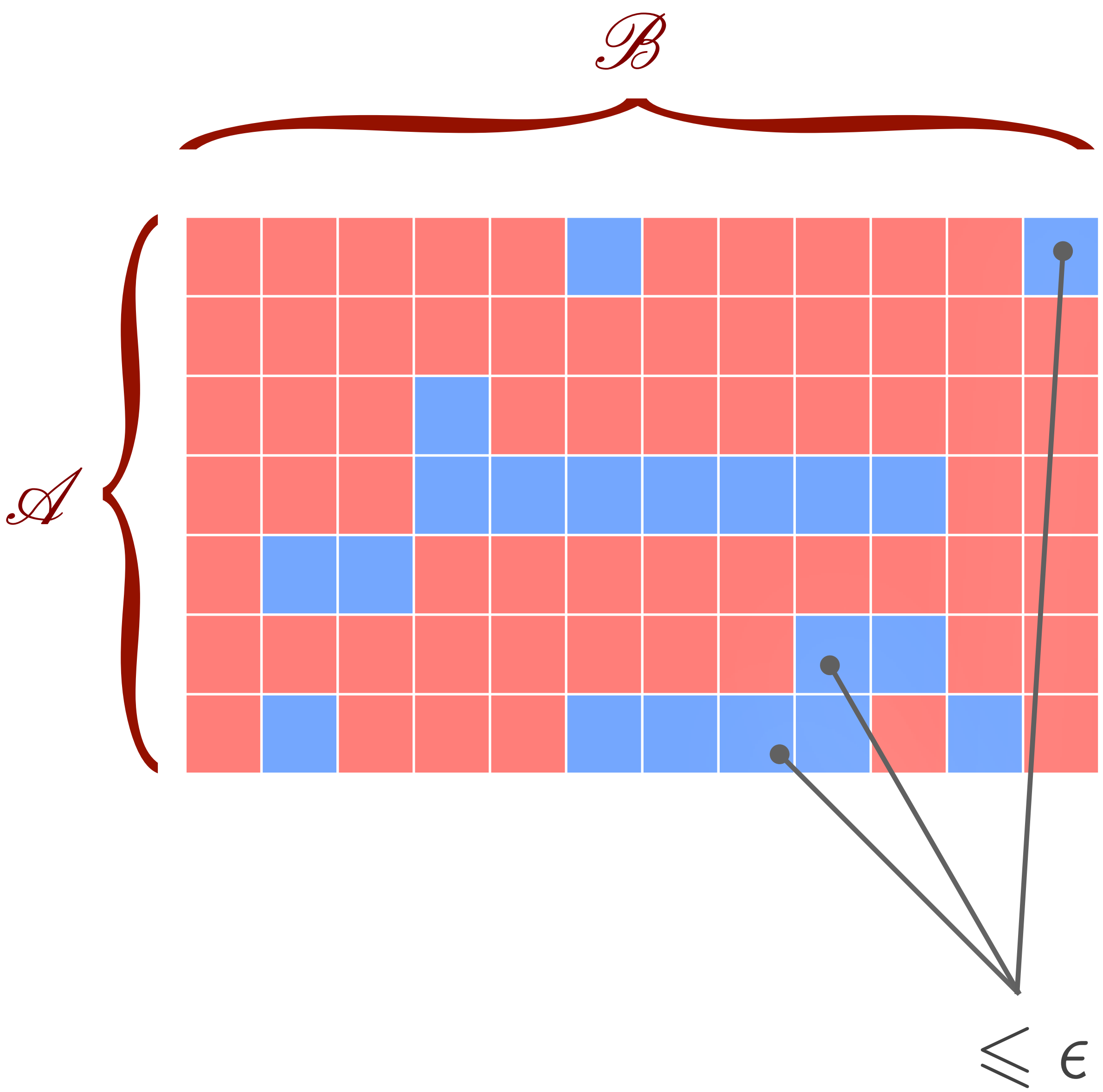
Let

$$\underset{\mathscr{A}\times\mathscr{B}}{\mathbf{P}}[\mathrm{DISJ}_n = 1] \geq 1 - \epsilon.$$

Then

$$\frac{|\mathscr{A} \times \mathscr{B}|}{\binom{n}{\sqrt{n}} \times \binom{n}{\sqrt{n}}} \leq 2^{-\epsilon\sqrt{n}}.$$

Corruption lemma



$$\mathscr{A}, \mathscr{B} \subseteq \binom{\{1, 2, \ldots, n\}}{\sqrt{n}}$$

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Prove: $|\mathscr{B}| \leq 2^{-\epsilon\sqrt{n}} \binom{n}{\sqrt{n}}$

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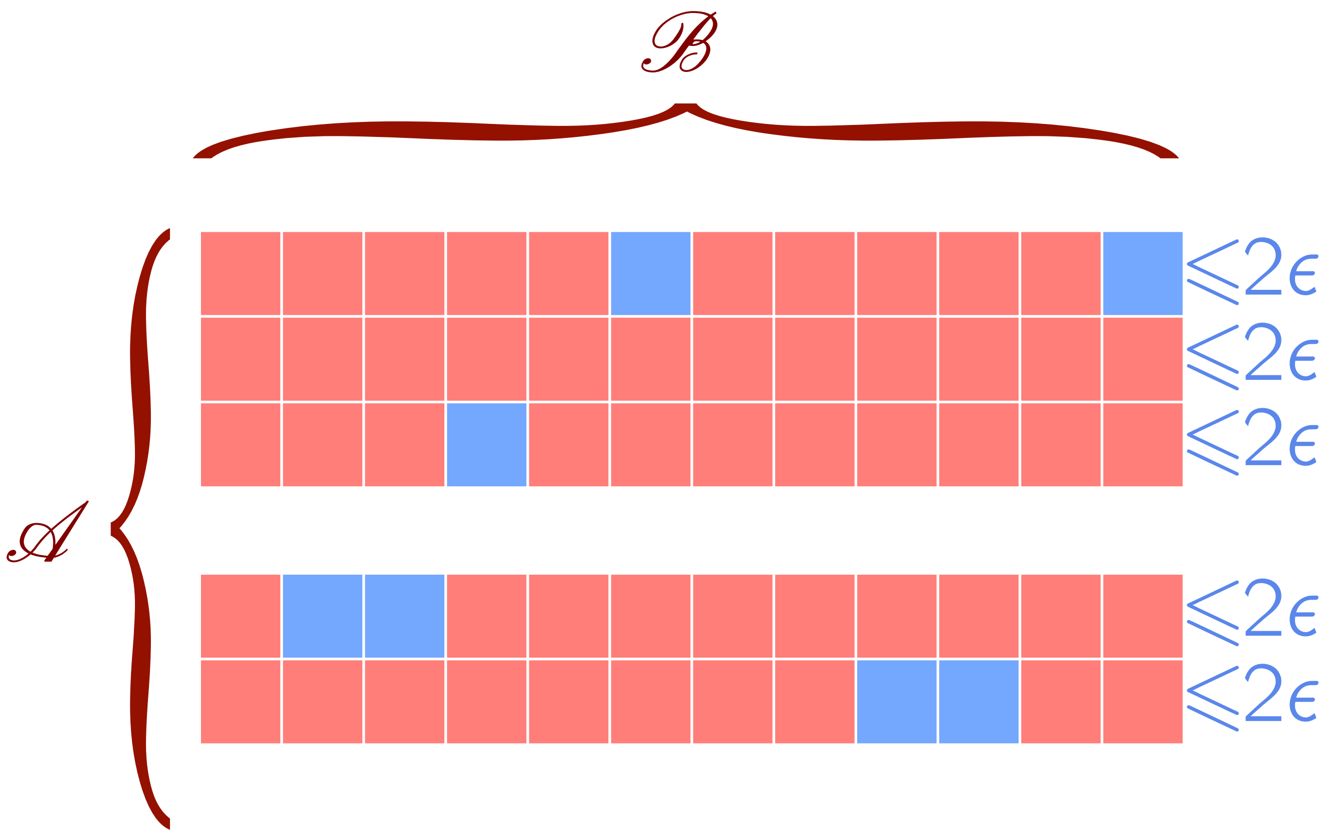

$$\mathscr{A}, \mathscr{B} \subseteq \binom{\{1, 2, \ldots, n\}}{\sqrt{n}}$$

$$|\mathscr{A}| \geq 2^{-\epsilon\sqrt{n}} \binom{n}{\sqrt{n}} \times 1/2$$

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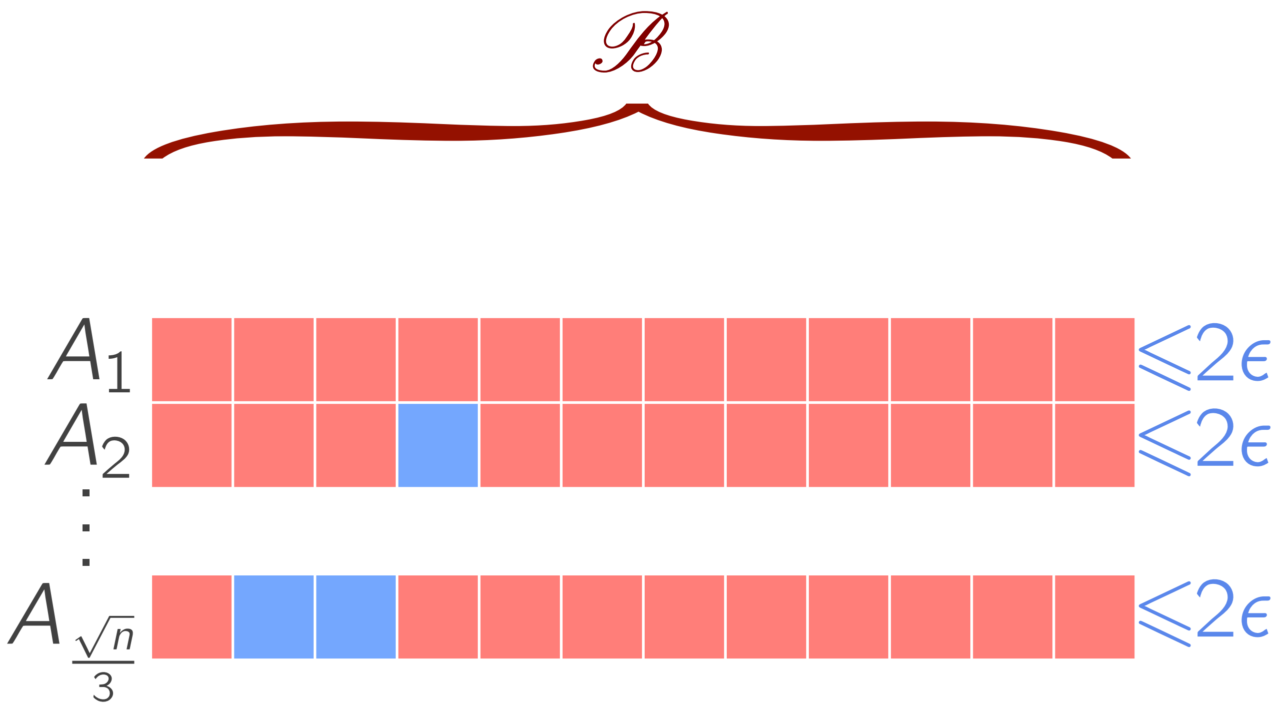

$$|A_i \setminus (A_1 \cup \cdots \cup A_{i-1})| \geq \frac{\sqrt{n}}{2}$$

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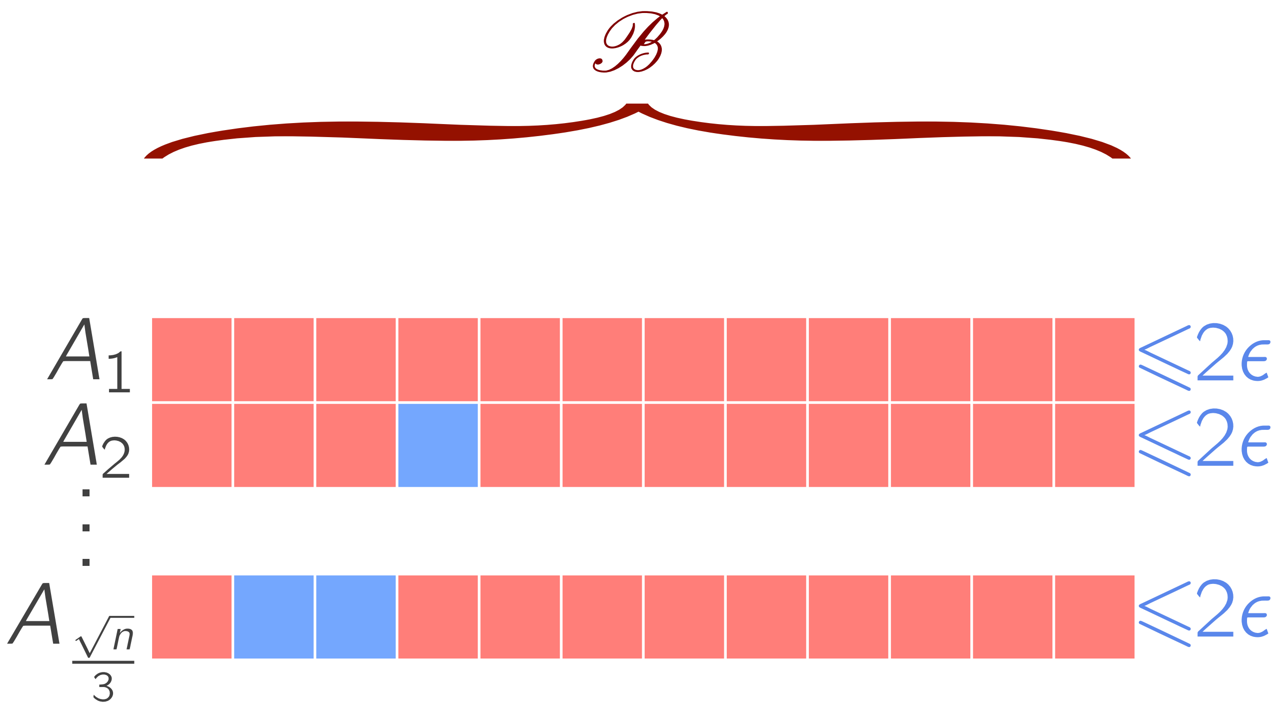

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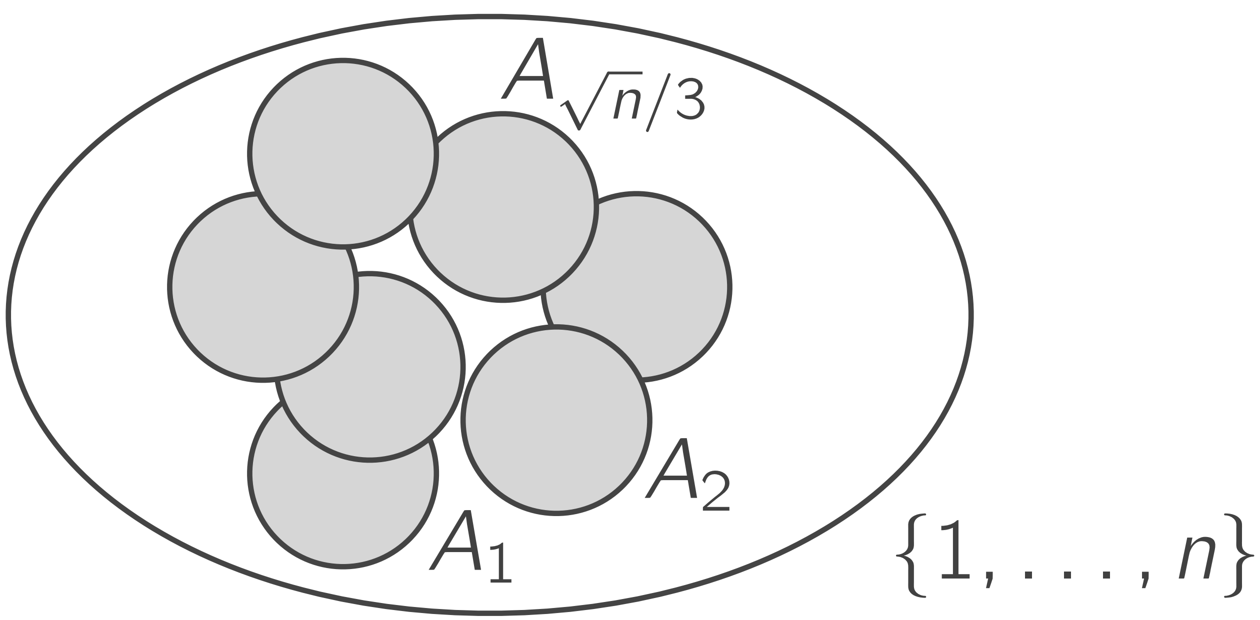

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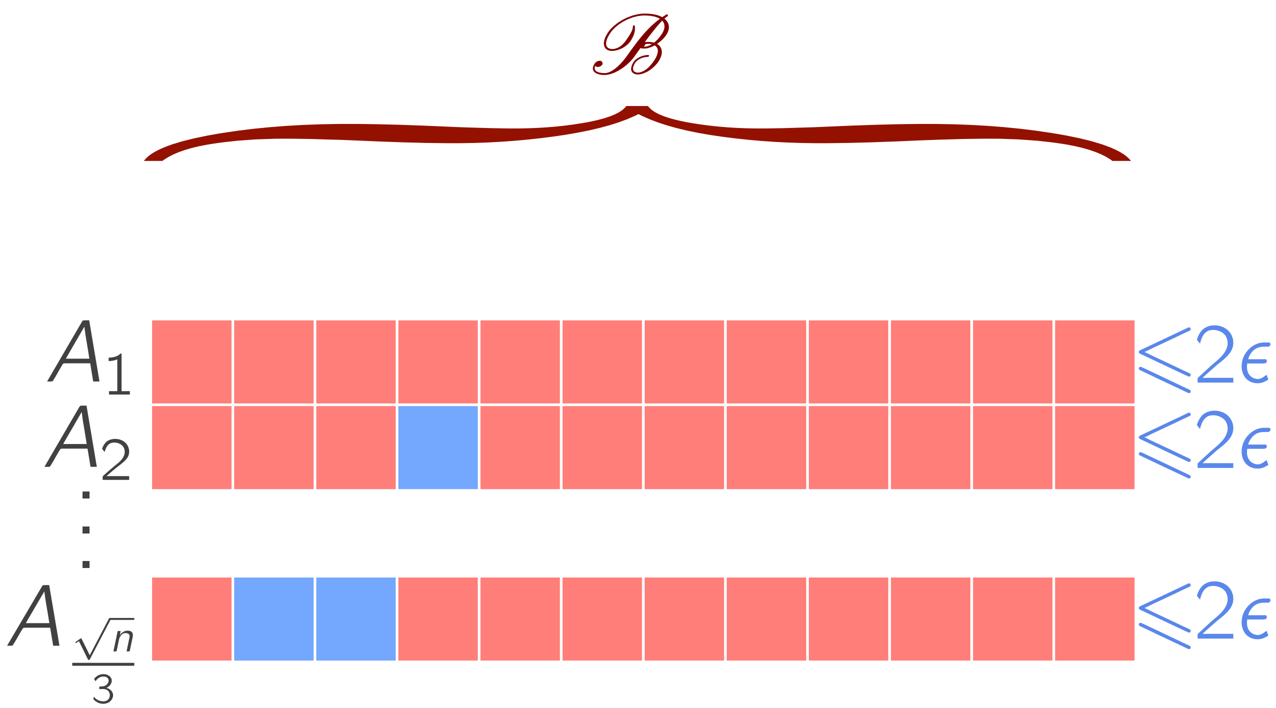

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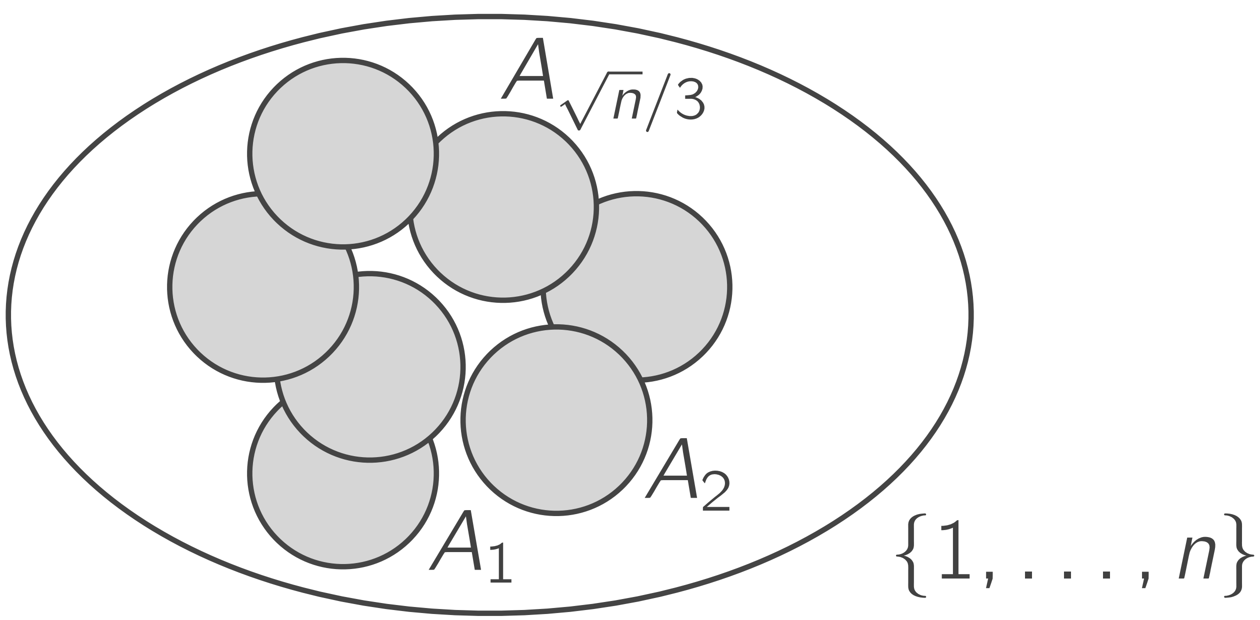

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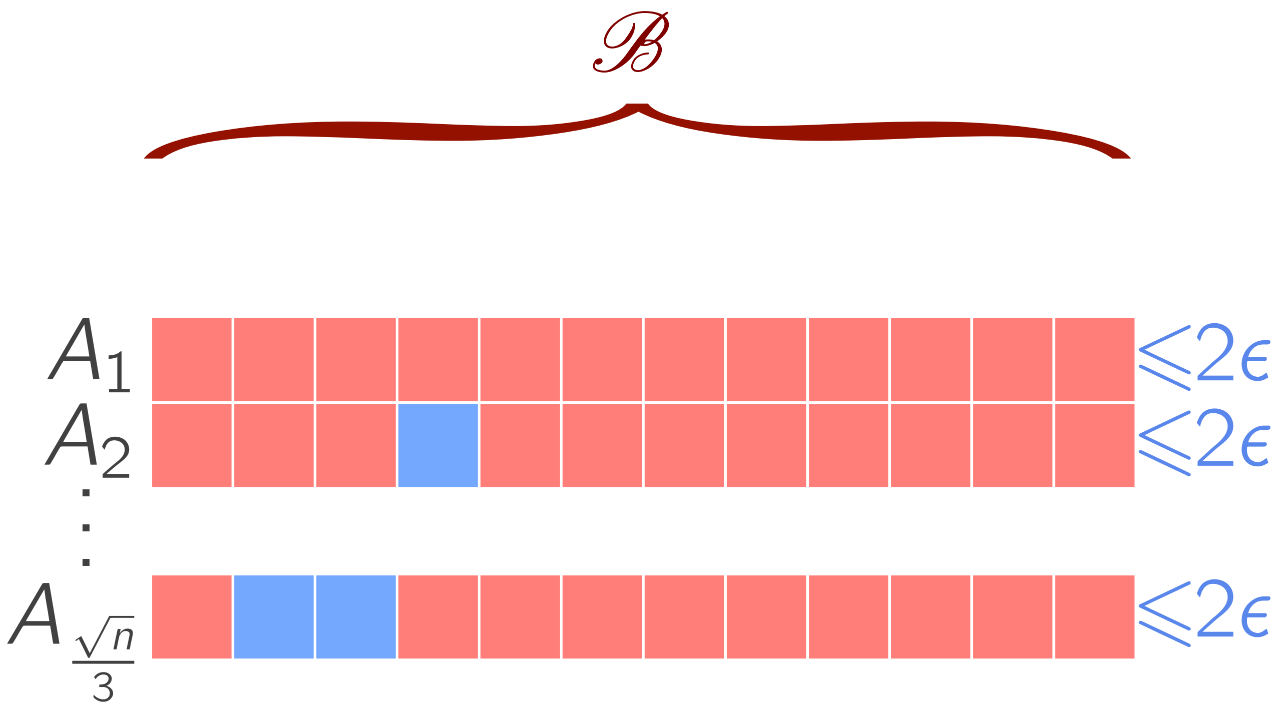

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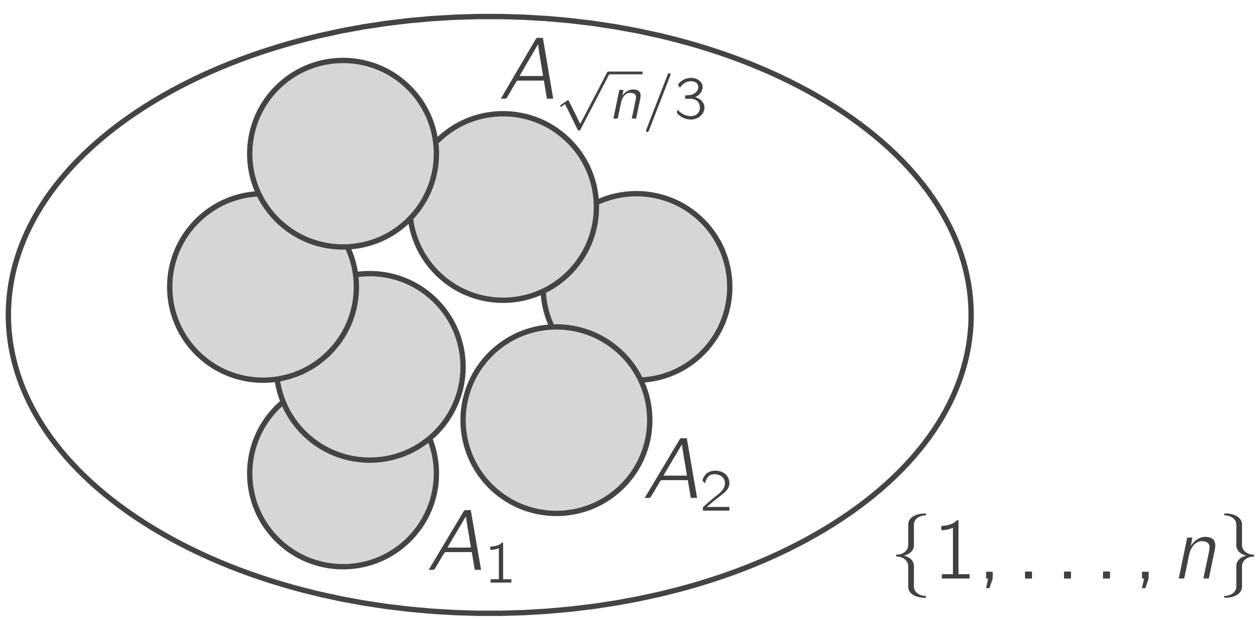

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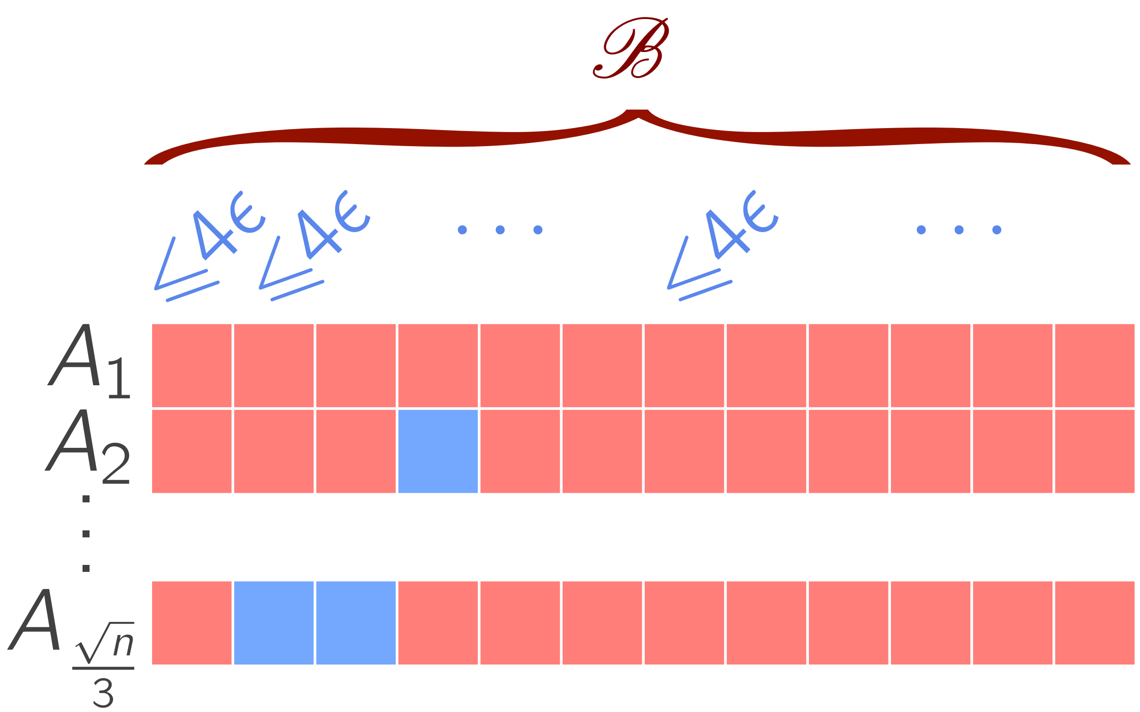

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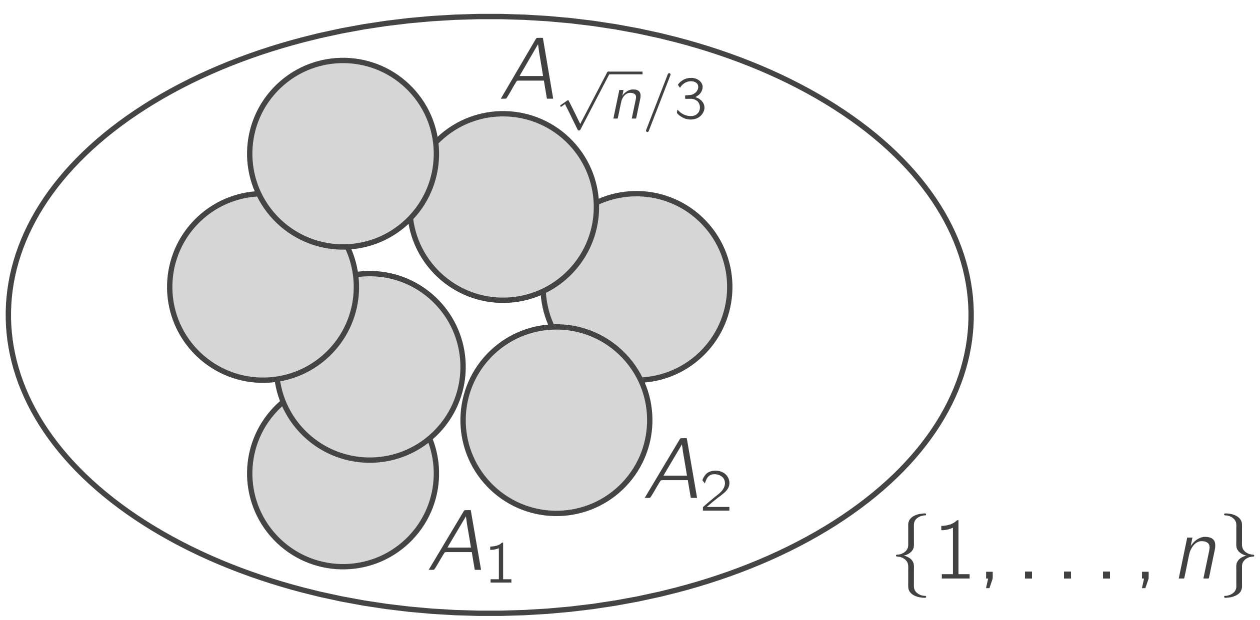

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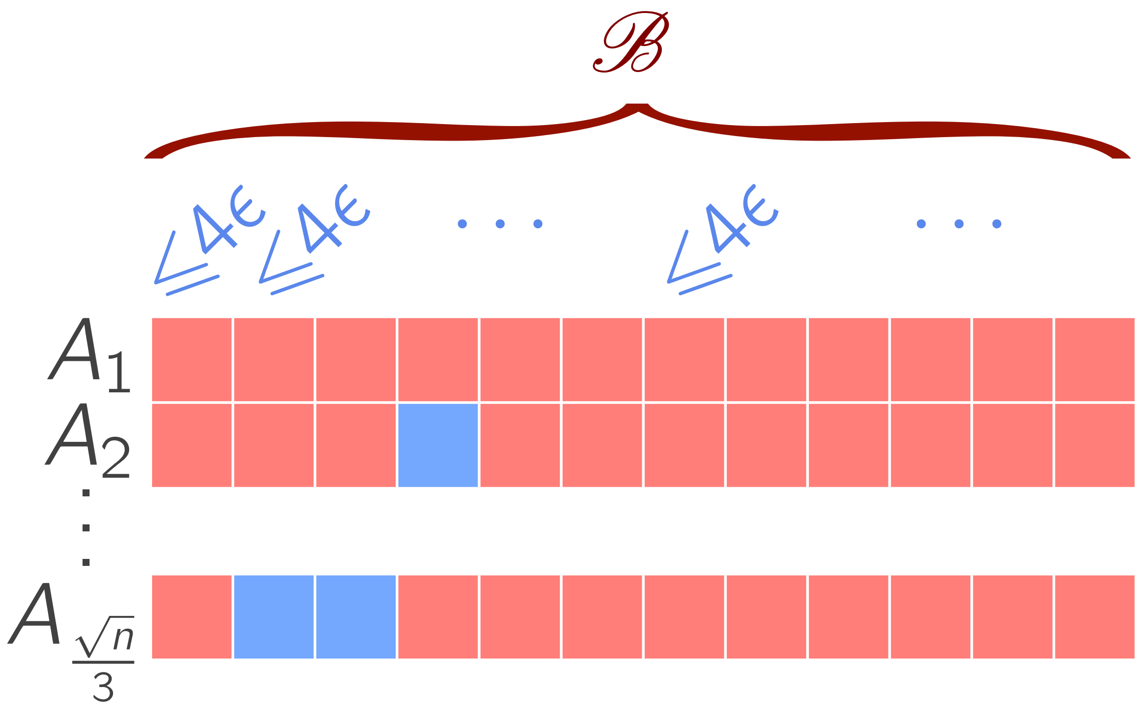

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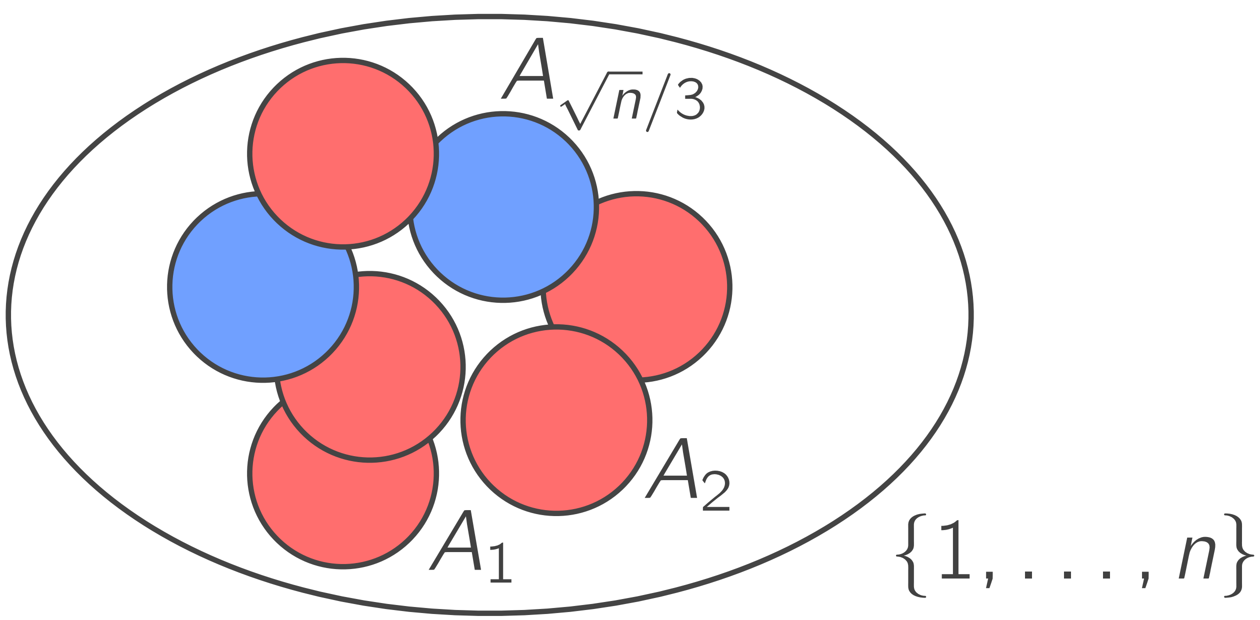

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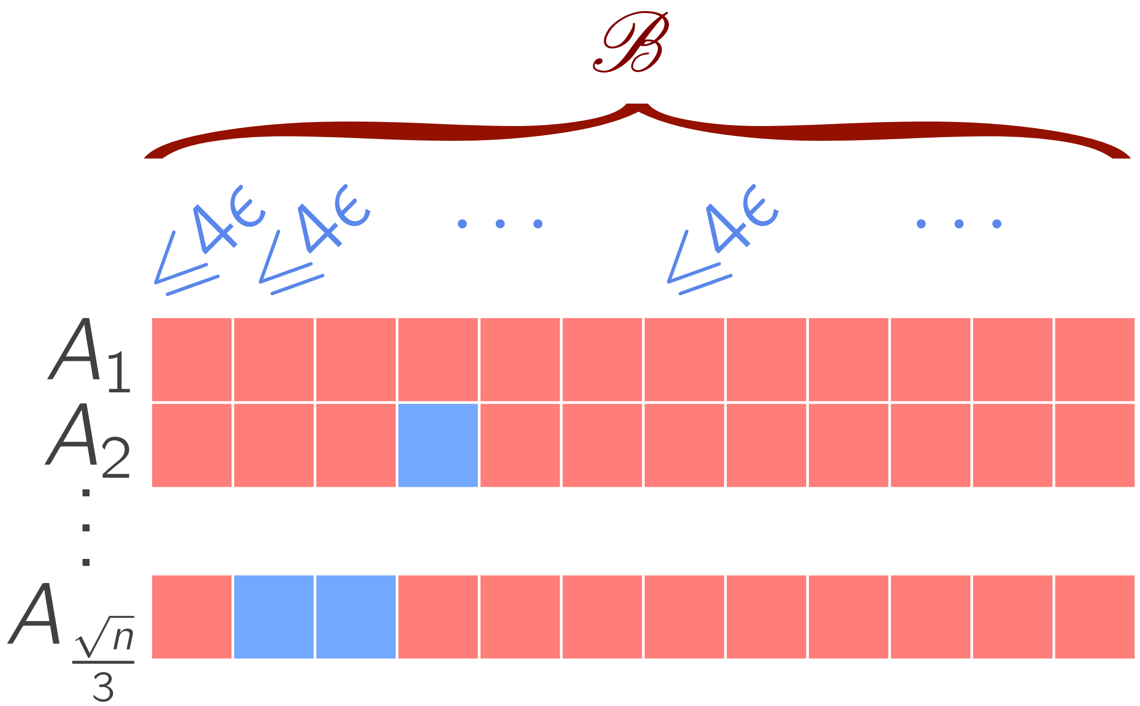

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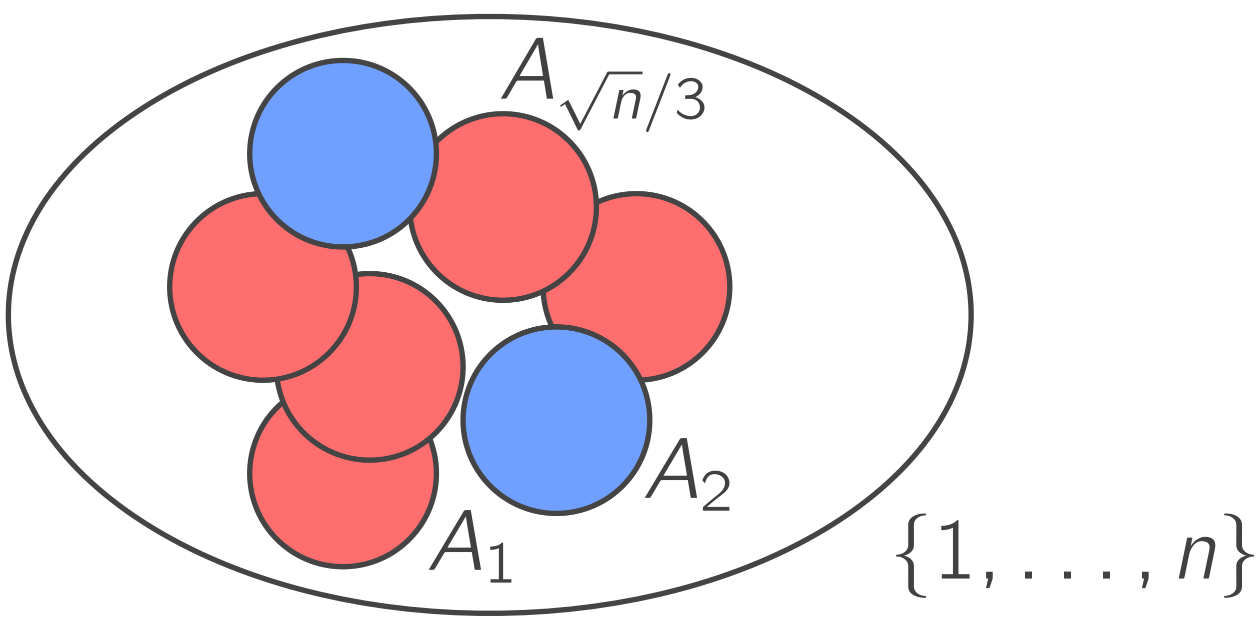

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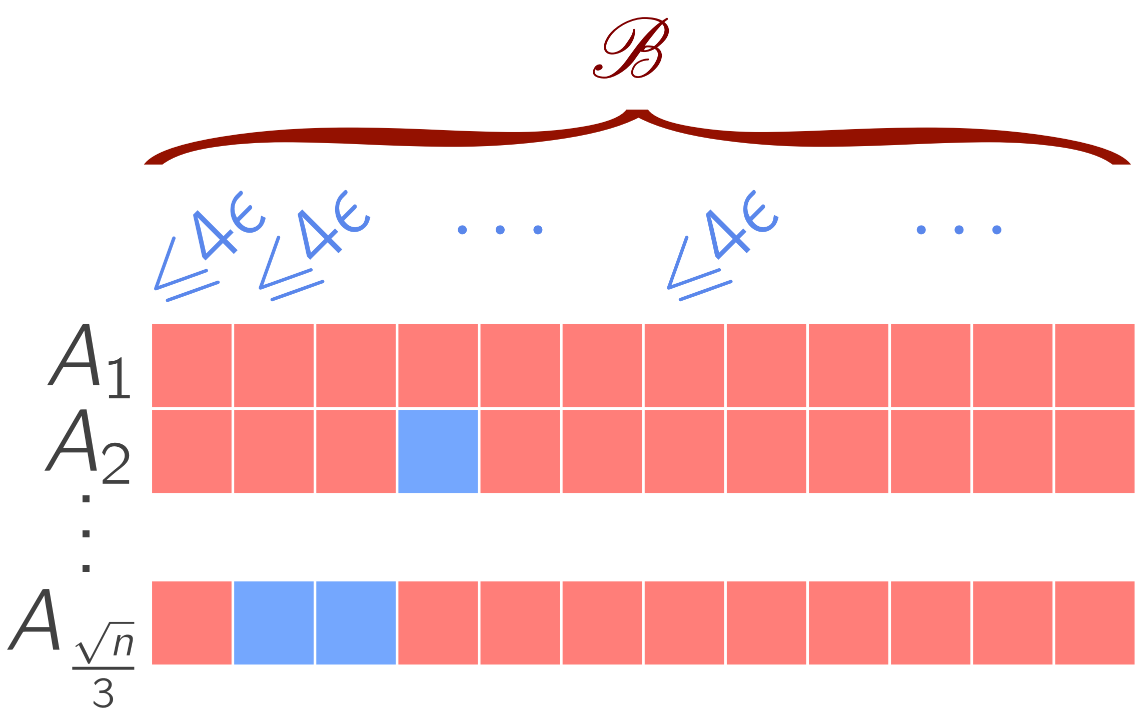

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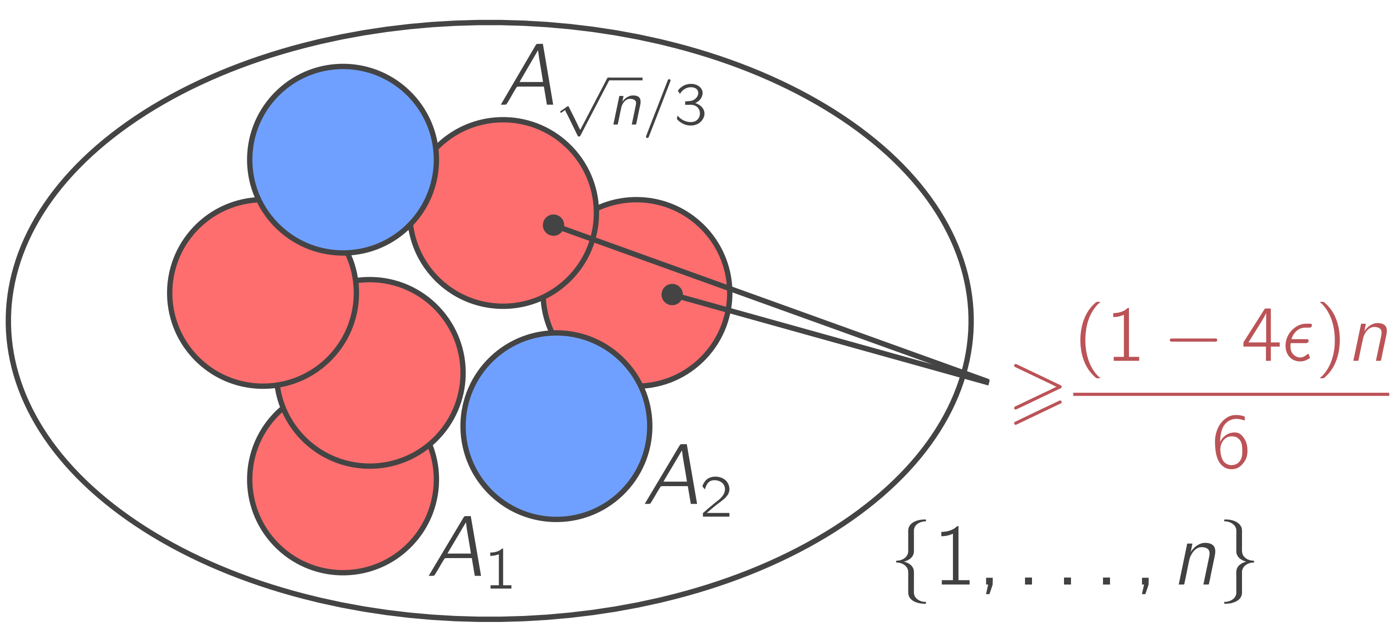

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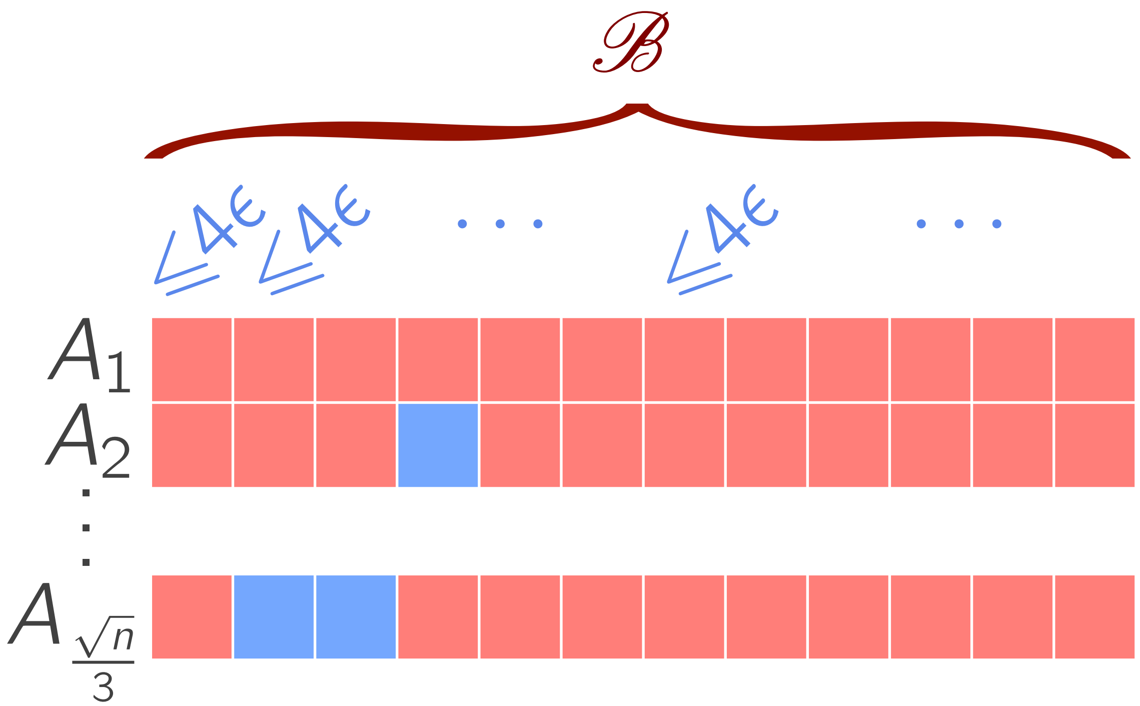

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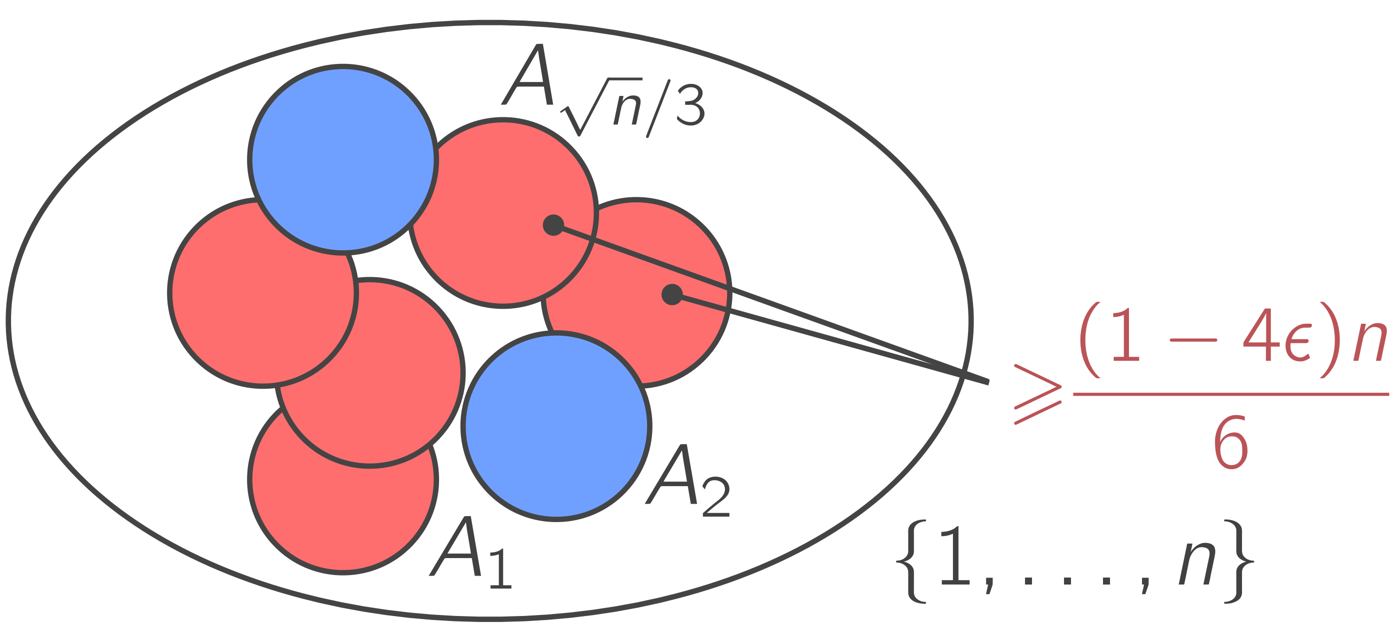
$$\therefore\ |\mathscr{B}| \leq \binom{\sqrt{n}/3}{4\epsilon\sqrt{n}/3} \times \binom{n - (1-4\epsilon)n/6}{\sqrt{n}}$$

$\blacksquare$



Set disjointness

Theorem (Babai, Frankl, Simon 1986).

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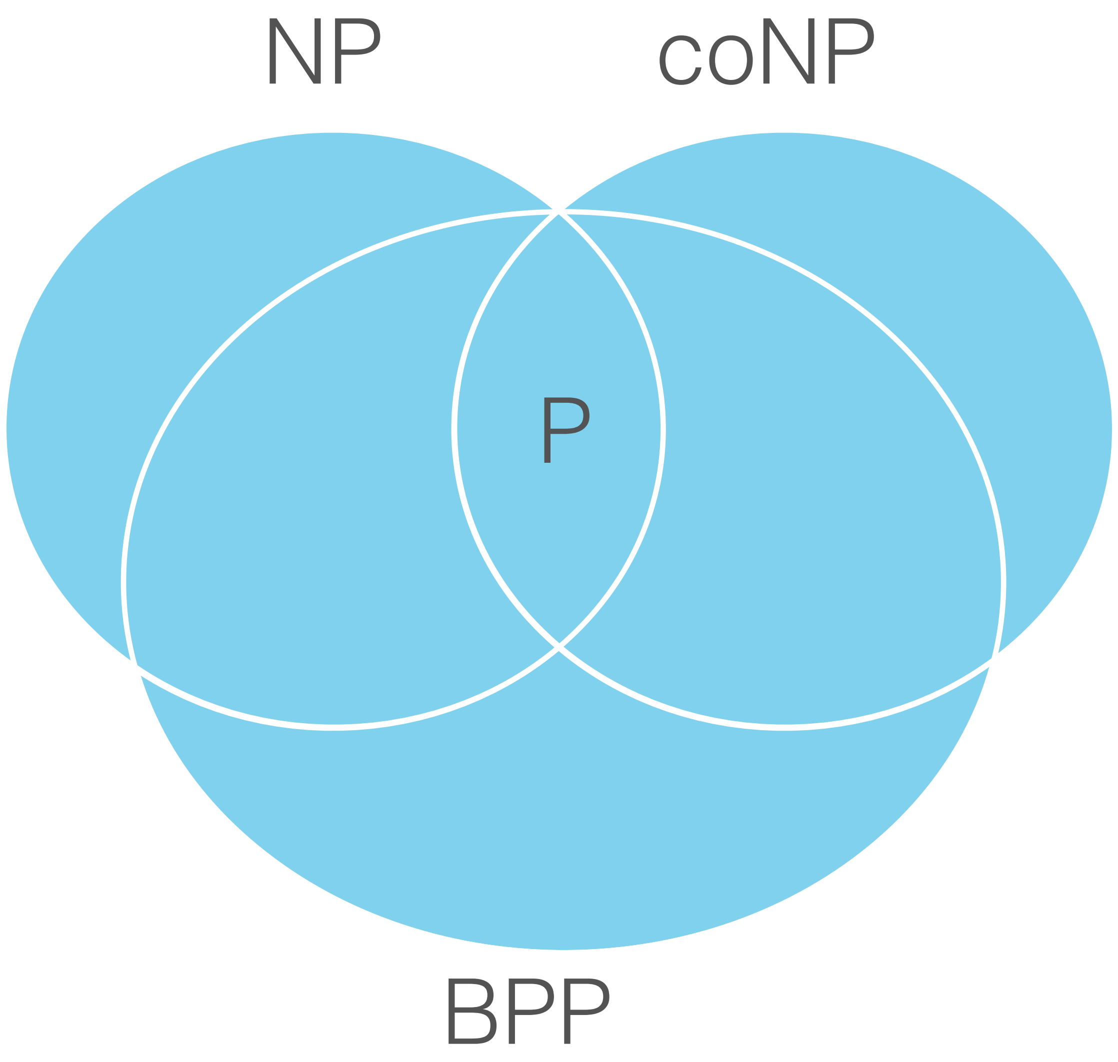
Complexity classes revisited



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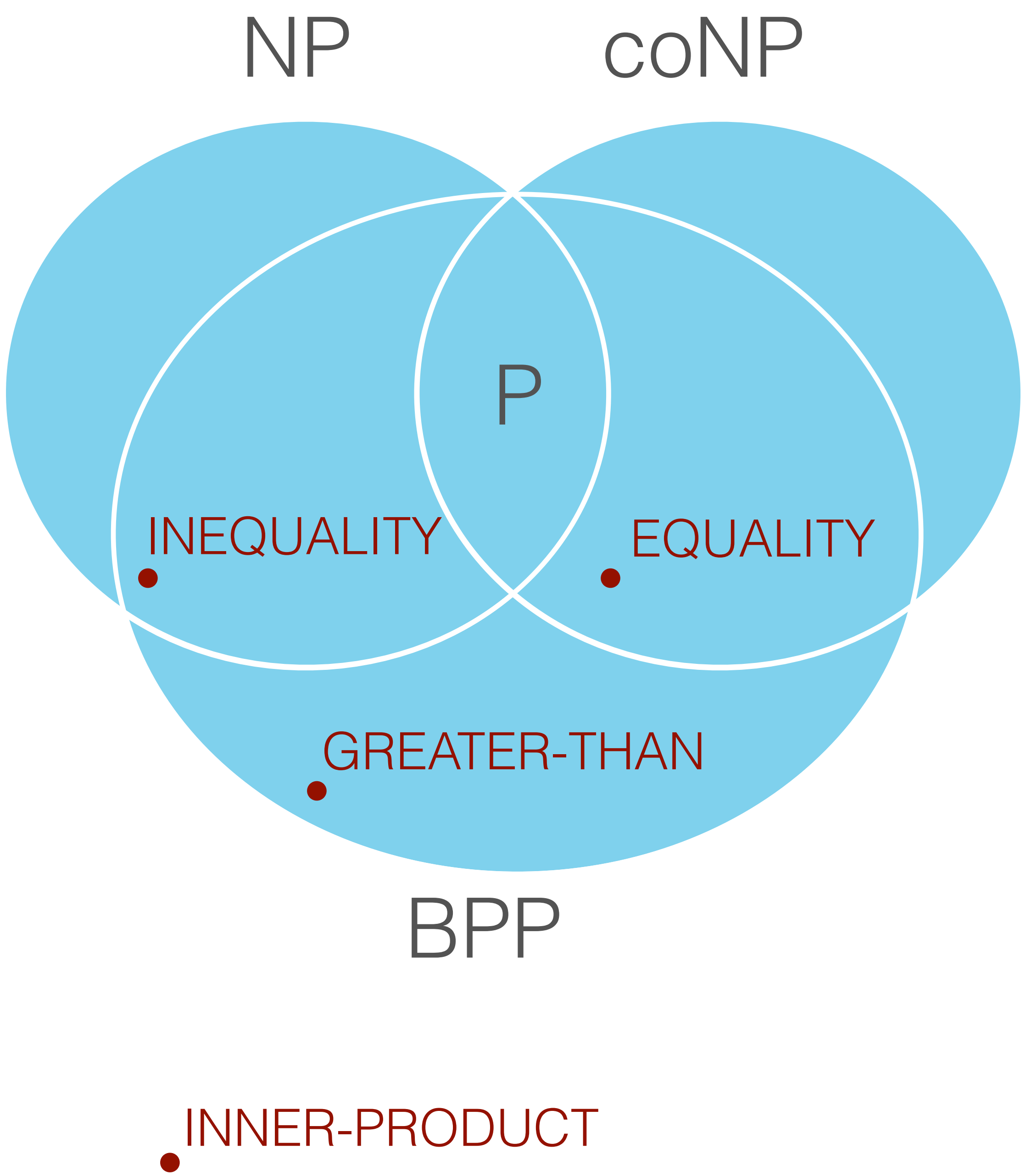

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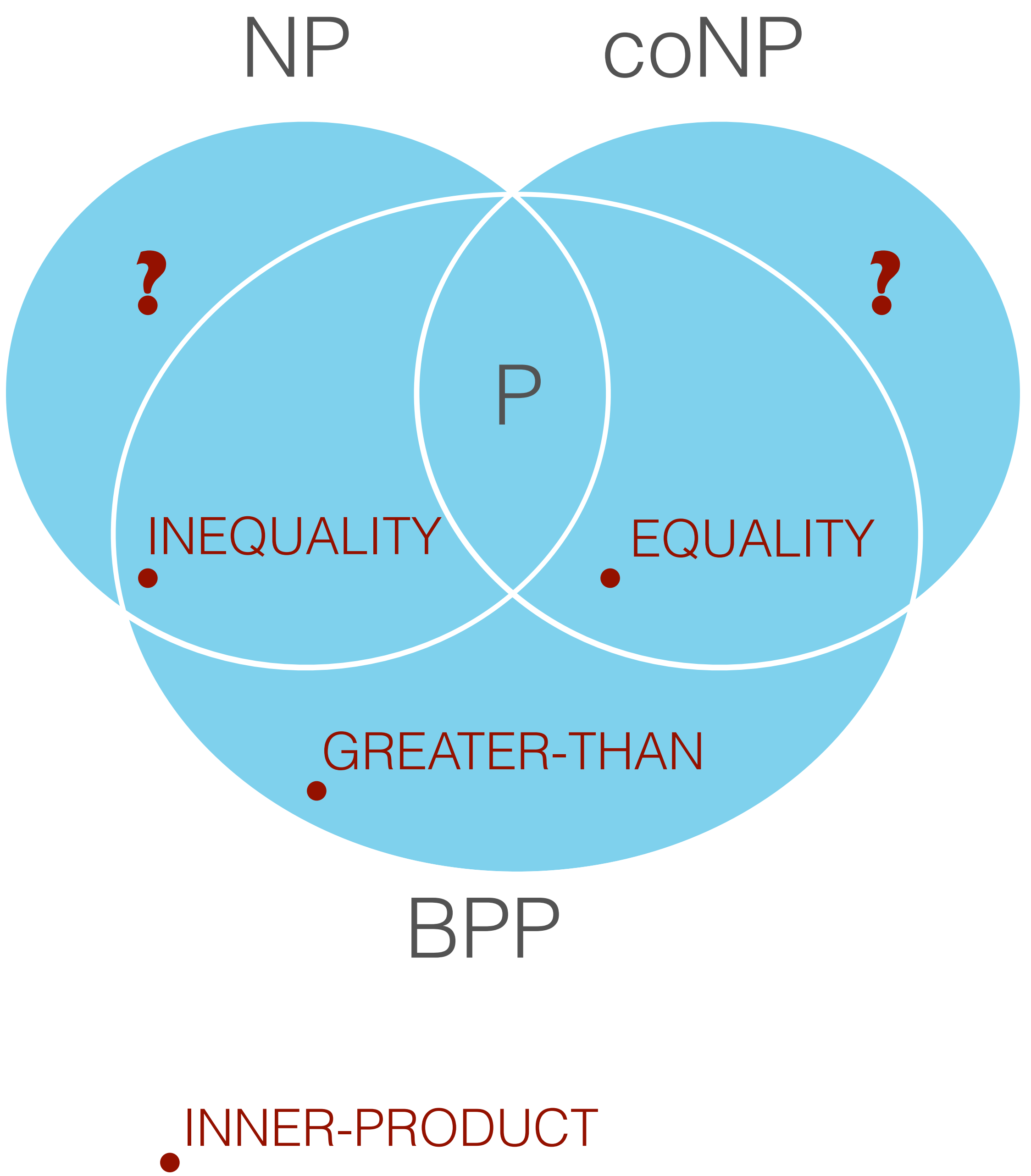

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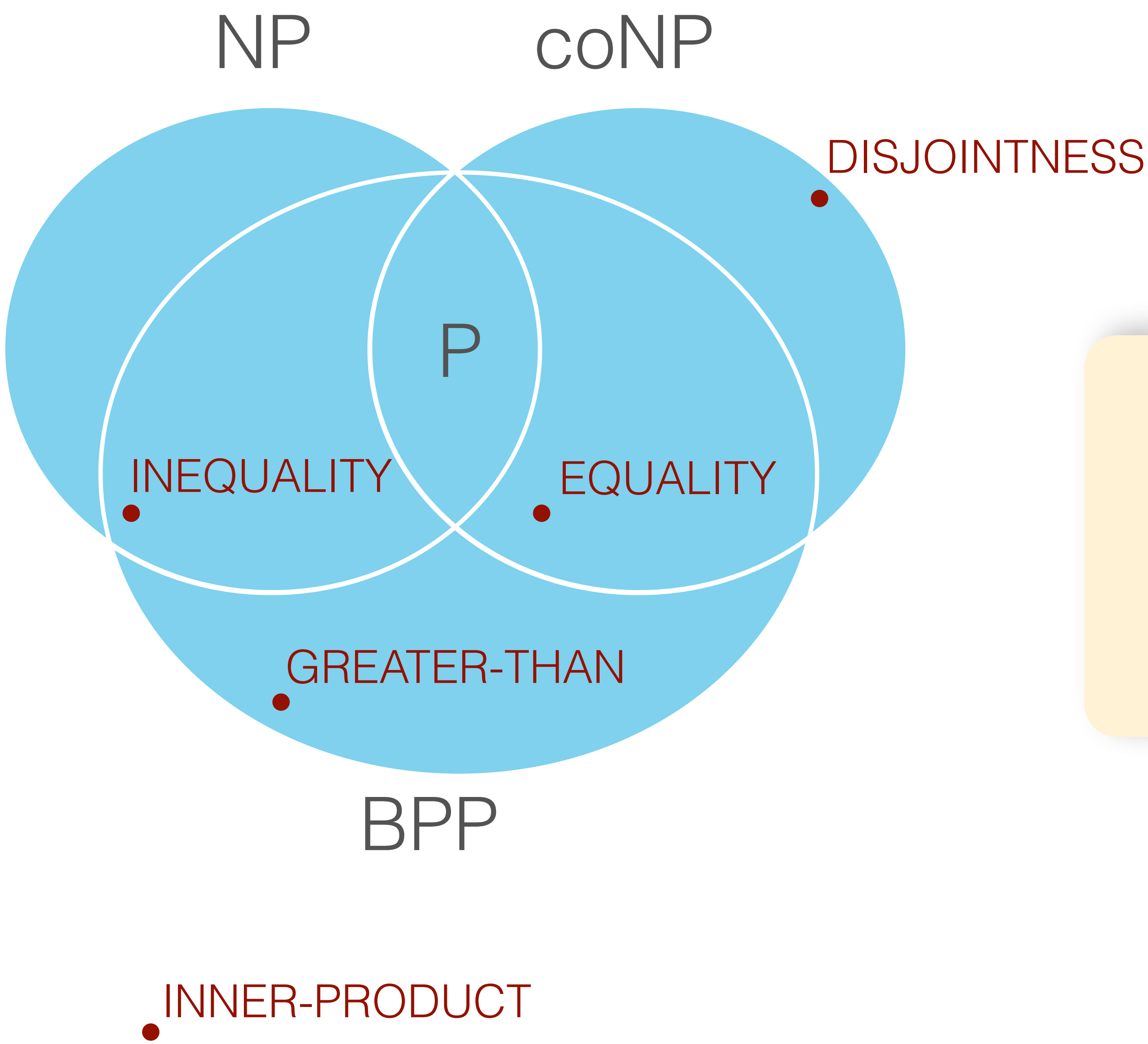

Theorem (Babai et al.)

$$R(\neg \mathsf{DISJ}_n) = \Omega(\sqrt{n})$$
$$N(\neg \mathsf{DISJ}_n) = \log n$$

Questions?