Trapping Sets of **Iterative Decoders** for Quantum Low-Density Parity Check Codes

Bane Vasić
Nithin Raveendran

Error Correction Laboratory
Center for Quantum Networks
Superconducting Quantum Materials and Systems Center
Department of ECE
University of Arizona

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More in our recent paper

N. Raveendran and B. Vasic, "Trapping Sets of Quantum LDPC Codes," Quantum 5, 562, Oct. 2021. also at arXiv:2012.15297 [cs.IT]

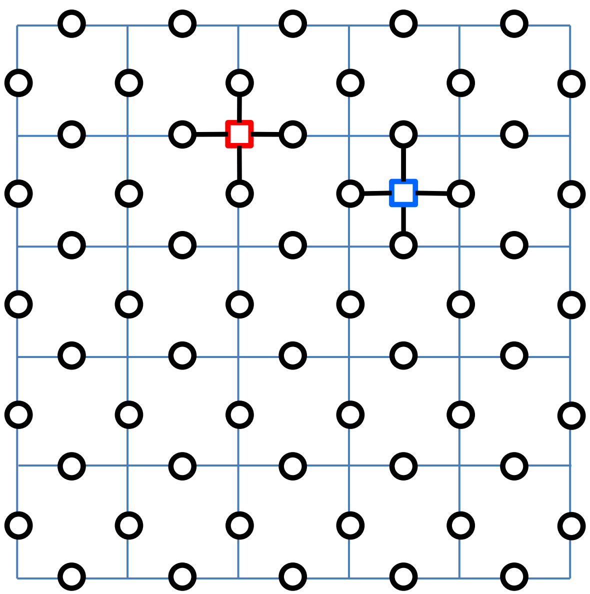
NITHIN RAVEENDRAN

XIN XIAO

ANKUR RAINA

NARAYANAN RENGASWAMY

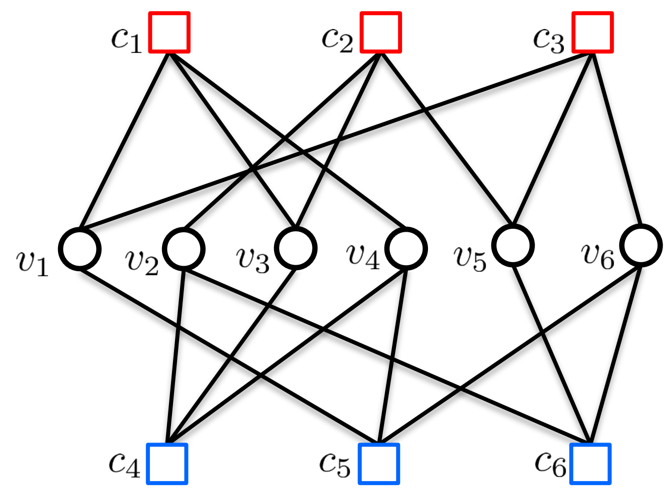
Surface codes and LDPC codes



- vertex checks (H_X)
- plaquette checks (H_Z)

$$H_X = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$H_Z = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$



Why quantum LDPC codes?

- Renewed interest in quantum LDPC (QLDPC) codes
  - Promises fault tolerant computation with constant overhead [15-17]
- Decoded efficiently using low-complexity iterative decoding [18]
- Involve stabilizer (parity) checks of bounded and low weight

[15] D. Gottesman, "Fault-Tolerant Quantum Computation with Constant Overhead," arXiv:1310.2984, 2014.

[16] A. Kovalev and L. Pryadko, "Improved quantum hypergraph-product LDPC codes," in 2012 IEEE ISIT, July 2012, pp. 348–352.

[17] O. Fawzi, A. Grospellier, and A. Leverrier, "Constant overhead quantum fault-tolerance with quantum expander codes," arXiv:1808.03821, 2018

[18] O. Fawzi, A. Grospellier, and A. Leverrier, "Efficient decoding of random errors for quantum expander codes," arXiv:1711.08351, 2017

Why quantum LDPC codes?

- Finite (nonzero) asymptotic rate
  - Surface codes: As code length $n \rightarrow \infty$, surface code rate $\rightarrow 0$.
- Minimum distance scaling better than square root of the code length [19-21]. Better than surface codes.

[19] P. Panteleev and G. Kalachev, "Quantum LDPC codes with almost linear minimum distance," arXiv:2012.04068, 2020.

[20] N. P. Breuckmann and J. N. Eberhardt, "Balanced Product Quantum Codes", arXiv: 2012.09271, 2020.

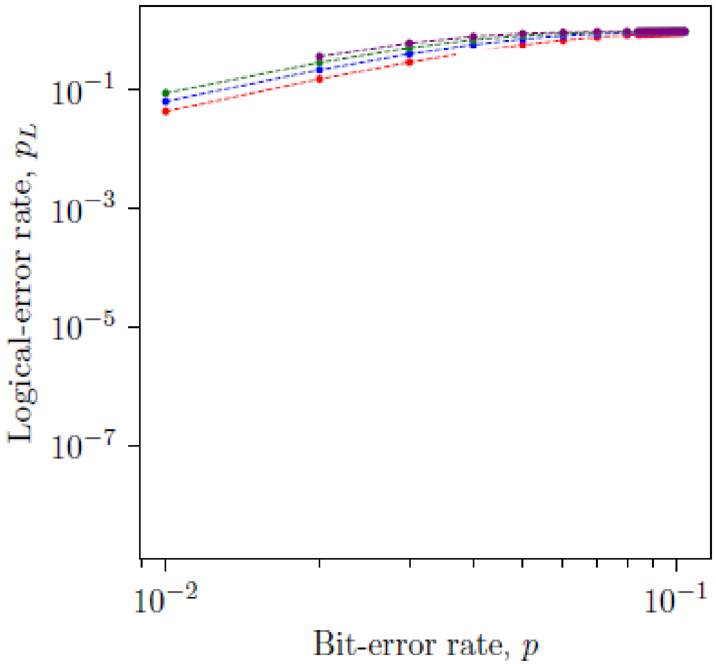
[21] P. Panteleev and G. Kalachev, "Asymptotically Good Quantum and Locally Testable Classical LDPC Codes", arXiv:2111.03654, 2021.

QLDPC decoding problem is still open

- The existing QLDPC code literature primarily focuses on:
  - constructing asymptotically good code families with improved d_{min} scaling with the block length n and higher code rates R
  - designing better iterative decoding algorithms
- QLDPC codes implemented in practical QEC systems will be of finite length and their iterative decoders will propagate finite-precision messages.
  - Performance degradation due to convergence issues
- The convergence failure manifests itself as an error floor of the decoding probability of error at low physical error rate levels.
  - Error floor is observed in all state-of-the-art iterative message-passing decoders: bit-flipping, belief propagation (BP), min-sum algorithm (MSA) and their variants

Example

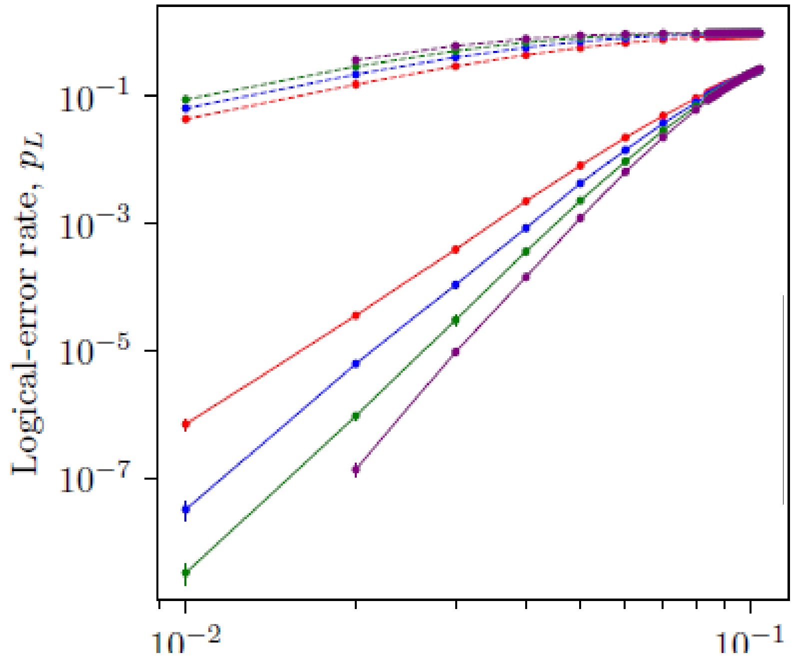
- BP performance (logical error rate) on Toric codes with $k = 2$ and d_{min}: 9, 11, 13, 15.
- Major drawback of BP is inability to handle small row weight of the parity-check matrix present due to the stabilizer commutativity constraint.



Roffe et al. May 2020. arXiv:2005.07016 [quant-ph]

BP with post-processing

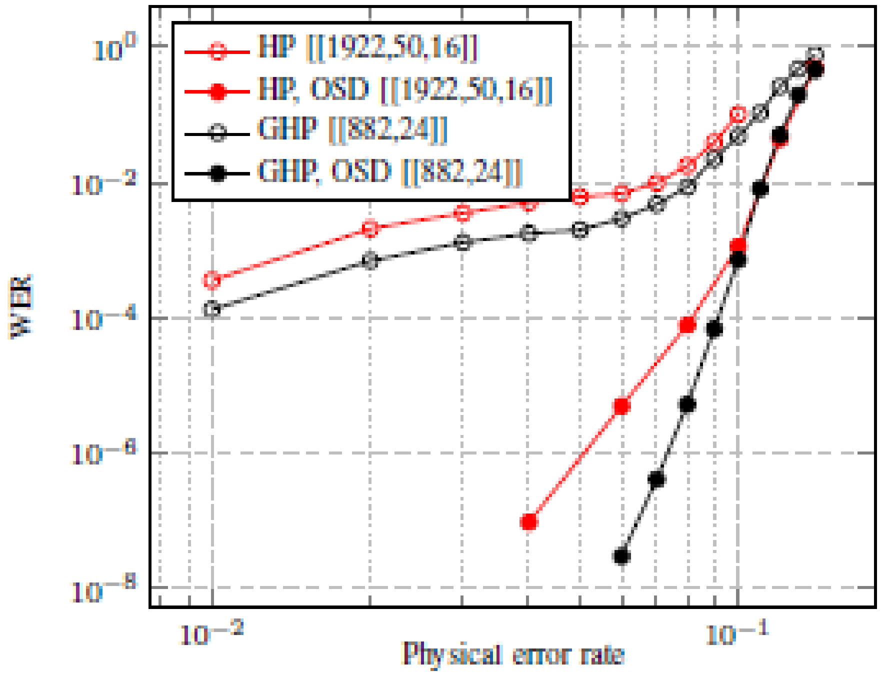
- A typical way to deal with error floor is post-processing with the Ordered Statistics Decoding (OSD).



Roffe et al. May 2020. arXiv:2005.07016 [quant-ph]

BP with post-processing

- However, the complexity of OSD is exponential in the code dimension k. In addition to the BP complexity.



P. Panteleev and G. Kalachev, arXiv:1904.02703, 2019.

Our general objective

- Achieve BP-OSD performance but use message-passing, i.e. only local processing.

General objectives of our research

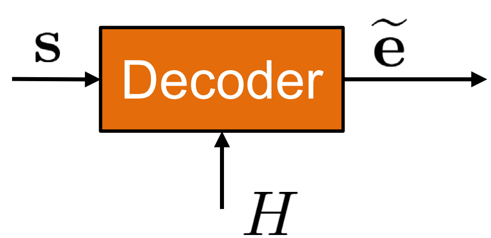
- For a given QLDPC code, design a decoder that will guarantee correction of all error patterns up to weight t.
- For a given decoder, construct QLDPC code that guarantees correction of all error patterns up to weight t.
- Understand precisely what causes error floor:
  - *Degenerate errors*: errors with the same non-zero syndrome
  - *Short cycles: in the Tanner graph: quantum code imposes s*ymplectic product/commutativity constraints on Tanner graph
- Other constraints
  - Convergence
  - Complexity
  - Fault tolerance

Our current projects

- Classical projects:
  - NSF CCF-2100013 Small: Learning To Correct Errors
  - NSF ECCS/CCSS-2027844: Neural Network Nonlinear Iterative LDPC Decoders with Guaranteed Error Performance and Fast Convergence
  - NSF-CCSS-2052751: Collaborative Research: Secure and Efficient Post-quantum Cryptography: from Coding Theory to Hardware Architecture

The problem setting

- A $[\![n, \ell, d]\!]$ stabilizer code given by the $m \times n$ binary parity check matrix $H = \begin{pmatrix} H_\mathrm{X} & H_\mathrm{Z} \end{pmatrix}$, where
$$H_\mathrm{X} H_\mathrm{Z}^\mathrm{T} + H_\mathrm{Z} H_\mathrm{X}^\mathrm{T} = \mathbf{0}$$
- Channel is a depolarizing channel. On each qubit, flips occur independently with probability α , resulting in the error pattern $\mathbf{e} = (\mathbf{e}_\mathrm{X}, \mathbf{e}_\mathrm{Z})$.
- Syndrome measurement of the quantum state $E\,|\psi\rangle$ results in the syndrome $\mathbf{s}$.
- Decoder is a quantum syndrome decoder.



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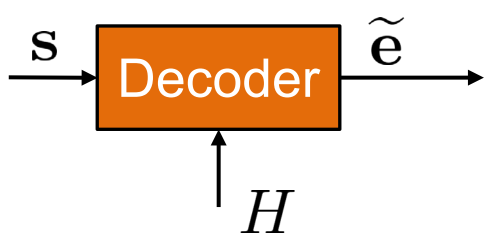
Decoders for CSS codes

- The parity check matrix is in the form (the rows are linearly independent)

$$H = \begin{pmatrix} H_{\mathrm{X}} & 0 \\ 0 & H_{\mathrm{Z}} \end{pmatrix} \qquad H_{\mathrm{X}} H_{\mathrm{Z}}^{\mathrm{T}} = 0$$

- The errors $\mathbf{e}_{\mathrm{X}}$ and $\mathbf{e}_{\mathrm{Z}}$ are independent, thus two decoders - one operating on H_{Z} to correct $\mathbf{e}_{\mathrm{X}}$ and one operating on H_{X} to correct $\mathbf{e}_{\mathrm{Z}}$ can be independently run.



CSS codes with H_X=H_Z

- The two codes can be chosen to be the same

$$H_{\mathrm{X}} = H_{\mathrm{Z}} = H$$

- The condition $HH^{\mathrm{T}} = 0$ means that the code $\mathcal{C}$ is dual-containing $\mathcal{C}^{\perp} \subset \mathcal{C}$.
- Codewords of $\mathcal{C}$ include the rowspace of *H* but also 2^{ℓ} other codewords.

$$\ell = \dim(\mathcal{C}) - \dim(\mathcal{C}^{\perp})$$

$$G = \begin{pmatrix} H_{(n-k)\times n} \\ L_{\ell\times n} \end{pmatrix}$$

Example (Steane Code)

$\mathcal{C}$

0 0 0 0 0 0 0
1 0 1 0 1 0 1
0 1 1 0 0 1 1
1 1 0 0 1 1 0
0 0 0 1 1 1 1
1 0 1 1 0 1 0
0 1 1 1 1 0 0
1 1 0 1 0 0 1
1 1 1 0 0 0 0
0 1 0 0 1 0 1
1 0 0 0 0 1 1
0 0 1 0 1 1 0
1 1 1 1 1 1 1
0 1 0 1 0 1 0
1 0 0 1 1 0 0
0 0 1 1 0 0 1

$$\longleftarrow \quad G = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$H = G^{\perp} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \longrightarrow$$

$\mathcal{C}^{\perp}$

0 0 0 0 0 0 0
1 0 1 0 1 0 1
0 1 1 0 0 1 1
1 1 0 0 1 1 0
0 0 0 1 1 1 1
1 0 1 1 0 1 0
0 1 1 1 1 0 0
1 1 0 1 0 0 1

$\text{rowspace}(H)$

Example (Steane Code) – dual code

$\mathcal{C}$

0 0 0 0 0 0 0
1 0 1 0 1 0 1
0 1 1 0 0 1 1
1 1 0 0 1 1 0
0 0 0 1 1 1 1
1 0 1 1 0 1 0
0 1 1 1 1 0 0
1 1 0 1 0 0 1
1 1 1 0 0 0 0
0 1 0 0 1 0 1
1 0 0 0 0 1 1
0 0 1 0 1 1 0
1 1 1 1 1 1 1
0 1 0 1 0 1 0
1 0 0 1 1 0 0
0 0 1 1 0 0 1

$$\longleftarrow \quad G = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$H = G^{\perp} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \longrightarrow$$

$\mathcal{C}^{\perp}$

0 0 0 0 0 0 0
1 0 1 0 1 0 1
0 1 1 0 0 1 1
1 1 0 0 1 1 0
0 0 0 1 1 1 1
1 0 1 1 0 1 0
0 1 1 1 1 0 0
1 1 0 1 0 0 1

$\mathrm{rowspace}(H)$

Example (Steane Code) - cosets

$\mathcal{C}$

0 0 0 0 0 0 0
1 0 1 0 1 0 1
0 1 1 0 0 1 1
1 1 0 0 1 1 0
0 0 0 1 1 1 1
1 0 1 1 0 1 0
0 1 1 1 1 0 0
1 1 0 1 0 0 1
1 1 1 0 0 0 0
0 1 0 0 1 0 1
1 0 0 0 0 1 1
0 0 1 0 1 1 0
1 1 1 1 1 1 1
0 1 0 1 0 1 0
1 0 0 1 1 0 0
0 0 1 1 0 0 1

logical 0

$(0000000) + \text{rowspace}(H)$

logical 1

$(1110000) + \text{rowspace}(H)$

$\mathcal{C}^{\perp}$

0 0 0 0 0 0 0
1 0 1 0 1 0 1
0 1 1 0 0 1 1
1 1 0 0 1 1 0
0 0 0 1 1 1 1
1 0 1 1 0 1 0
0 1 1 1 1 0 0
1 1 0 1 0 0 1

$\text{rowspace}(H)$

Quantum decoder is a coset decoder

- Let $\mathbf{e}$ be a non-zero error vector, resulting in a syndrome $\mathbf{s}$

$$\mathbf{s} = \mathbf{e}H^{\mathrm{T}} \neq \mathbf{0}$$

- As opposed to a *classical syndrome decoder* that tries to find $\mathbf{e}$ for a given observed syndrome, a valid output of a *quantum* decoder is any one of the vectors

$$\widetilde{\mathbf{e}} = \mathbf{e} + \mathbf{h}, \mathbf{h} \in \mathrm{rowspace(H)}$$

- When $\mathbf{e} + \widetilde{\mathbf{e}} \neq \mathbf{0}$, but

$$(\mathbf{e} + \widetilde{\mathbf{e}})H^{\mathrm{T}} = \mathbf{0}$$

$$\mathbf{e} + \widetilde{\mathbf{e}} \in \mathrm{rowspace}(L)$$

then the correction vector $\mathbf{e} + \widetilde{\mathbf{e}}$ is applied to flip bits in the (unobservable) quantum codeword resulting in another codeword, and a logical, undetectable, error occurs.

Checklist of statements to clarify

- What is a *depolarizing* channel, what the X and Z components of an error pattern $\mathbf{e} = (\mathbf{e}_{\mathrm{X}}, \mathbf{e}_{\mathrm{Z}})$ mean?
- Why the parity check matrix has 2*n* columns?

$$H = \begin{pmatrix} H_{\mathrm{X}} & 0 \\ 0 & H_{\mathrm{Z}} \end{pmatrix}$$

- Quantum origin of the condition $H_{\mathrm{X}} H_{\mathrm{Z}}^{\mathrm{T}} + H_{\mathrm{Z}} H_{\mathrm{X}}^{\mathrm{T}} = \mathbf{0}$
  - Discuss LDPC code constructions that satisfy such *commutativity* condition, construction of CSS codes too.
  - Constraints on row weights of *H* due implementation, d_{min}
- Why $\widetilde{\mathbf{e}} = \mathbf{e} + \mathbf{h}, \mathbf{h} \in \mathrm{rowspace(H)}$ is a valid decoding output, why quantum codewords are defined as cosets?
- Misscorrections in the general case $H = \begin{pmatrix} H_{\mathrm{X}} & H_{\mathrm{Z}} \end{pmatrix}$

Quantum questions

- How is it possible to obtain a syndrome without disturbing the quantum state corresponding to a codeword?
- How do we measure a syndrome?
- What are stabilizer codes?
- How do we construct quantum LDPC codes from classical ones?

Summary of a problem statement

- Given the generator matrix

$$G = \begin{pmatrix} H_{(n-k)\times n} \\ L_{\ell\times n} \end{pmatrix}$$

  and the syndrome *s*, find any error pattern such that

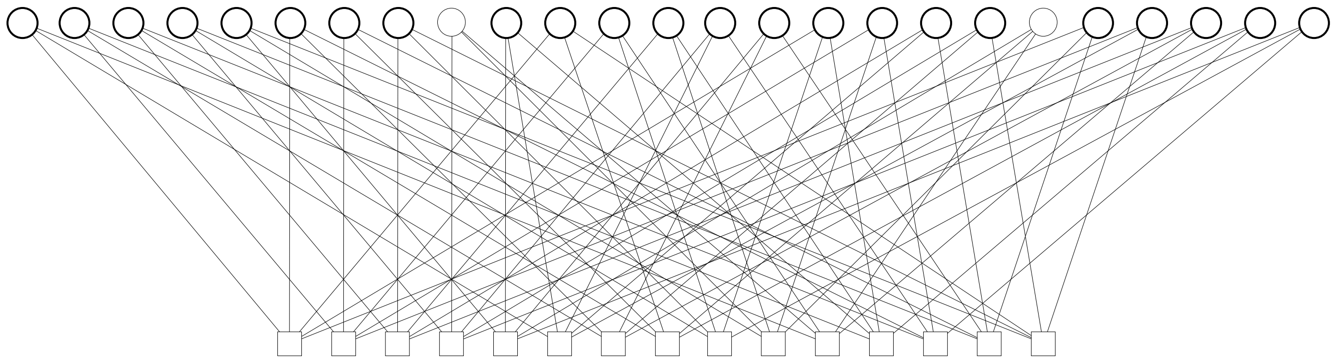
$$\mathbf{e} + \widetilde{\mathbf{e}} \in \mathrm{rowspace}(H) \text{ and } \mathbf{e} + \widetilde{\mathbf{e}} \notin \mathrm{rowspace}(L)$$

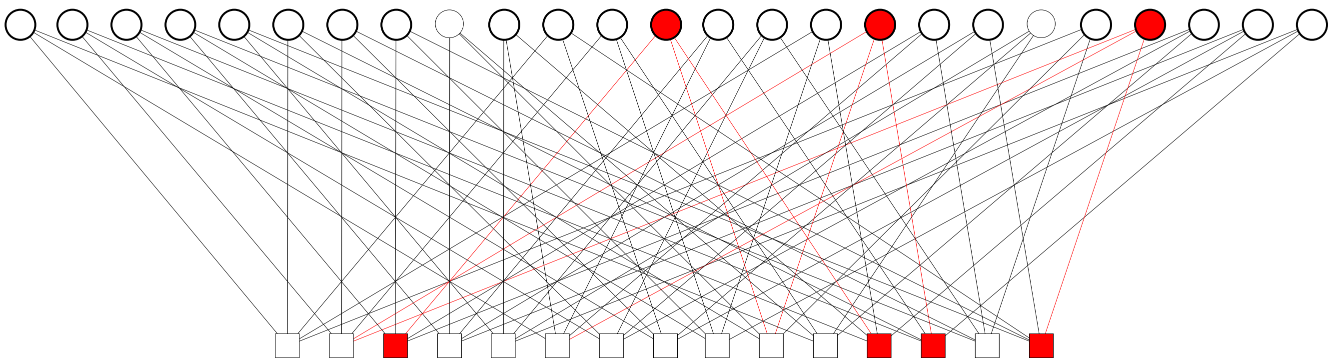
- H is a parity check matrix of an LDPC code
- Decoding is iterative, message passing decoding

Quantum syndrome decoding

- Whether a decoding algorithm $\mathcal{D}$ produces a valid correction vector depends on $\mathrm{wt}(\mathbf{e})$ and $d_{\min}$ of the code, but also on $\mathbf{e}$ and on $\mathcal{D}$.
- For *codes on graphs* (such as QLDPC codes, surface codes, etc.), and ability of a given *iterative* decoder $\mathcal{D}$ to correct an error pattern is determined by specific topologies of subgraphs found in $\mathcal{G} = \mathcal{G}(H)$, the Tanner graph of H.
- We refer to these subgraphs as trapping sets.
- Error floor is attributed to these dense subgraphs present in the Tanner graph.

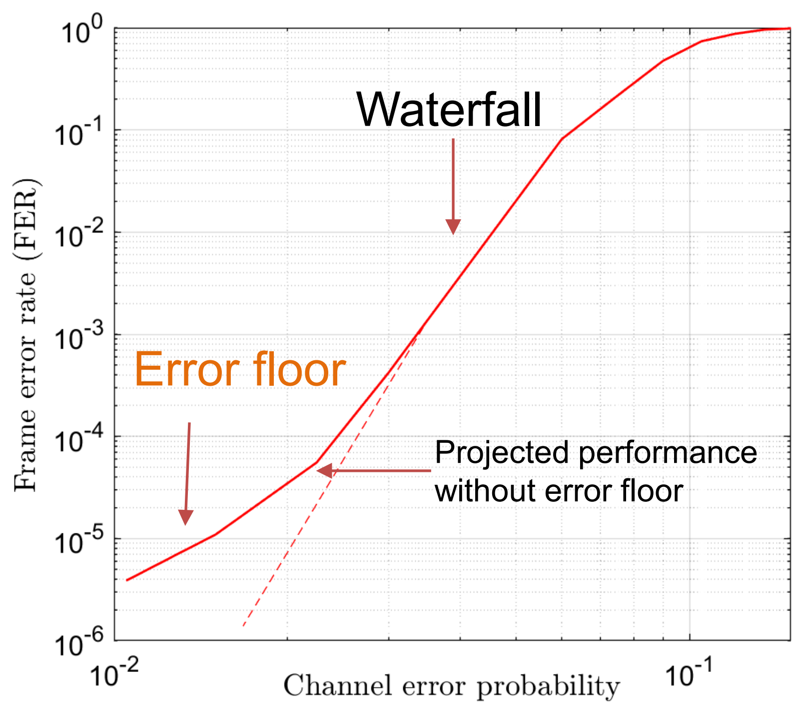
Tanner graph of H and syndrome matching





Typical performance curve of LDPC codes under iterative decoding

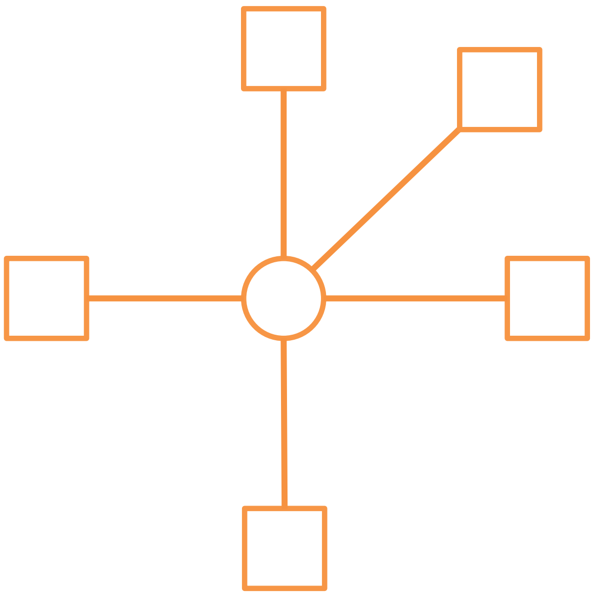
- Waterfall – Frame error rate *drops significantly* with decreasing noise
- Error floor - Abrupt degradation of frame error rate at low channel noise region - curve tends to *floor/flatten.*
  - Dense sub-graphs in the Tanner graph cause iterative decoder to fail for low-weight error patterns: trapping sets.

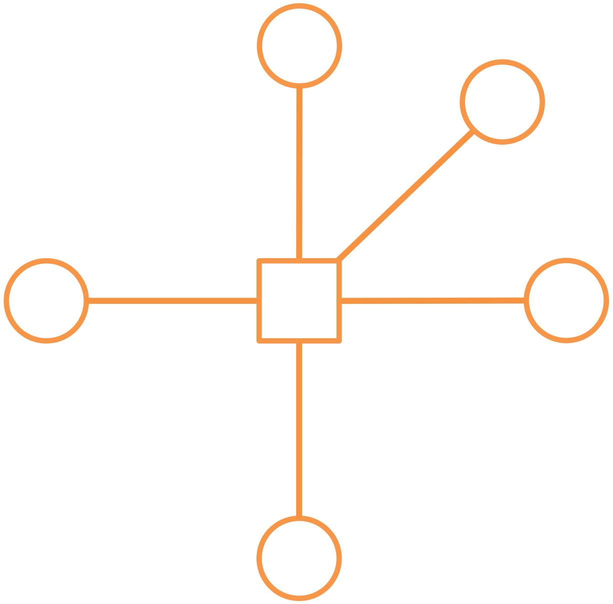


Outline of the talk

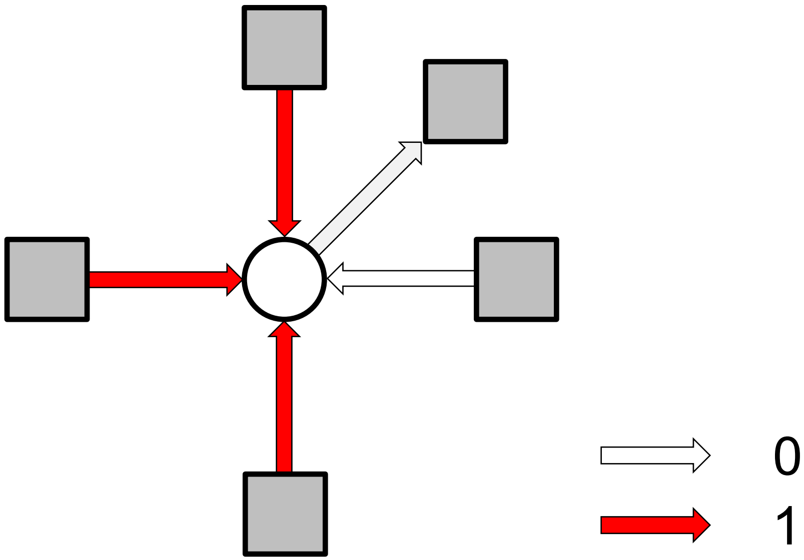
- Finite message precision iterative decoders
  - Gallager B decoder
- Understanding trapping sets in syndrome decoding
- Two types of trapping sets in QLDPC codes
  - Classical-like trapping sets
  - Trapping sets imbedded in symmetric stabilizers
- Using trapping sets to design better QLDPC codes and better decoders
- Enumeration of trapping sets in some known code families: bicycle codes and hypergraph product codes
- An efficient trapping set search algorithm

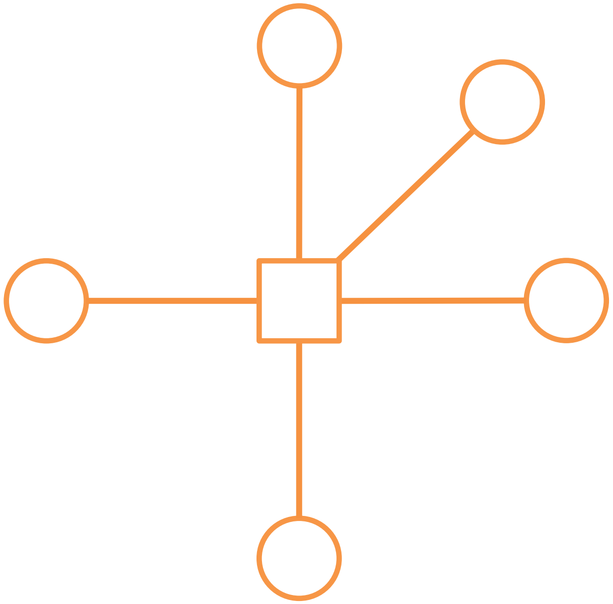
Gallager B algorithm





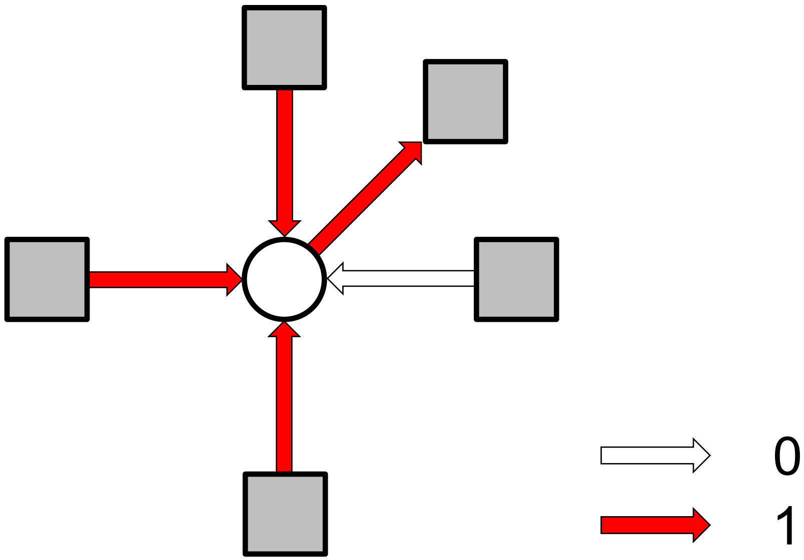
Gallager B
variable message update

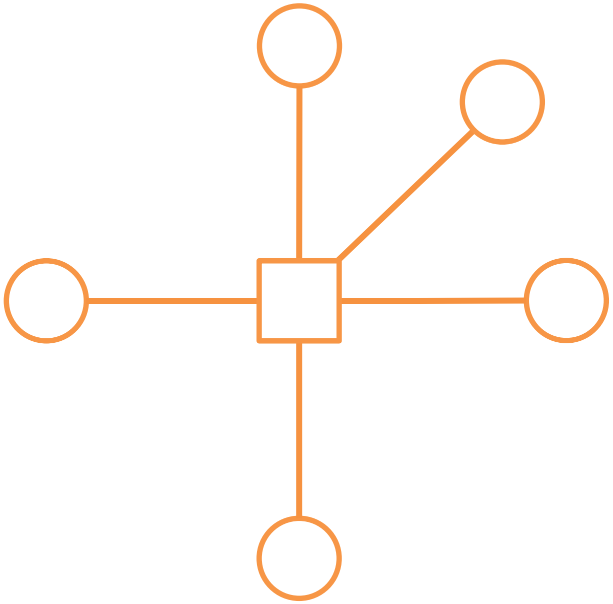




when there is majority

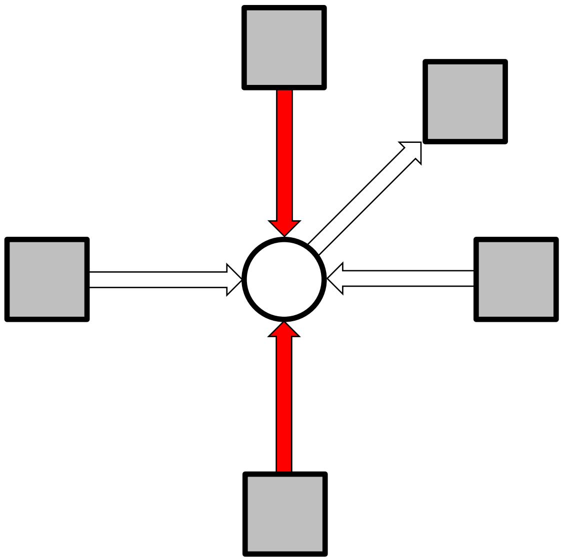
Gallager B
variable message update

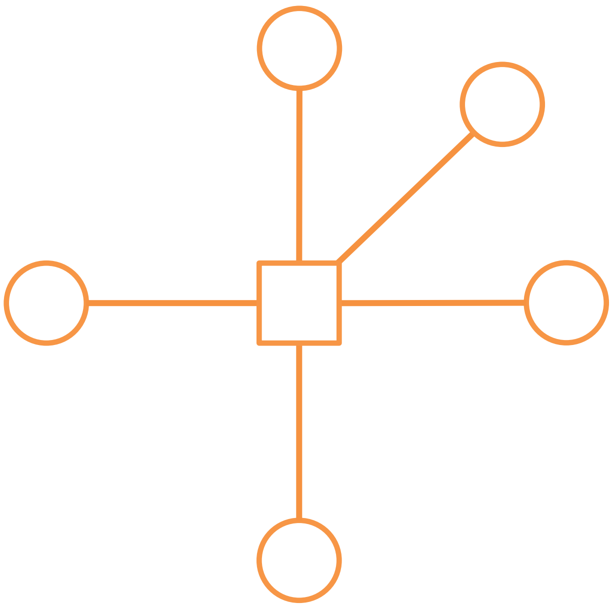




when there is majority, send the majority value

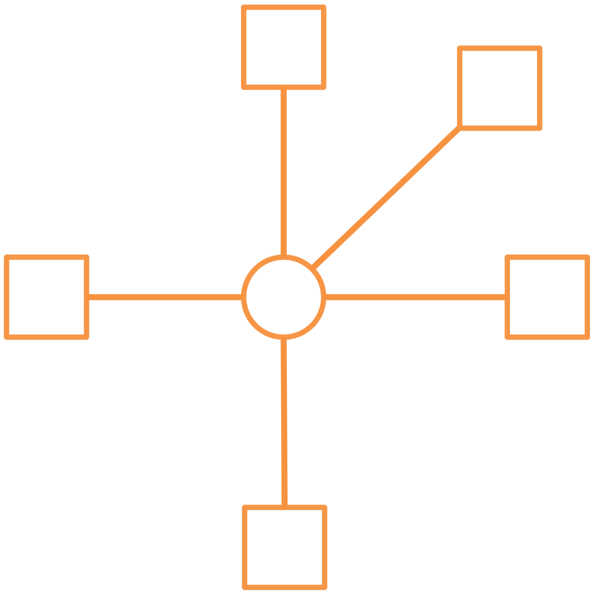
Gallager B
variable message update

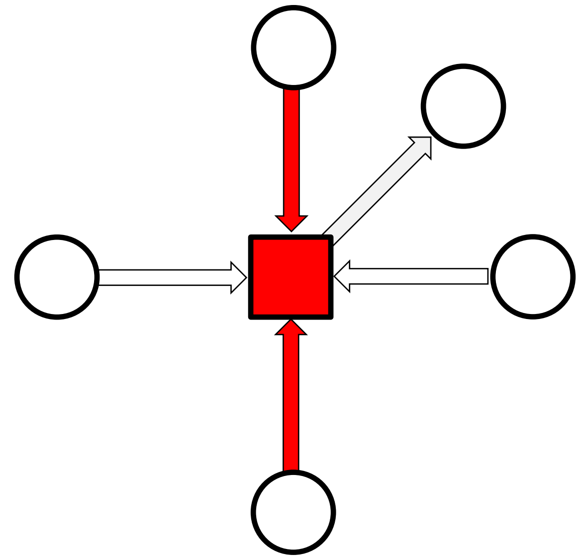




when there is a tie, send the zero value

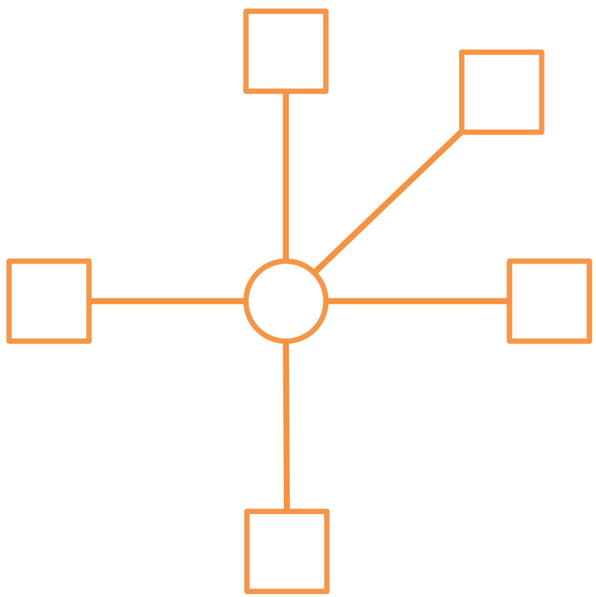
Gallager B
check message update

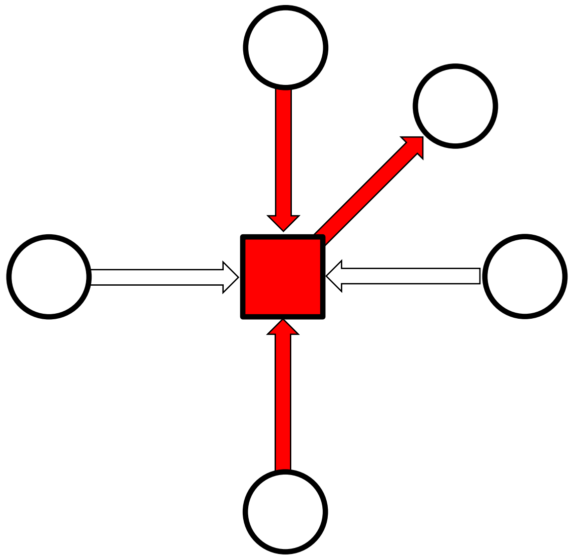




XOR the incoming messages and the check value

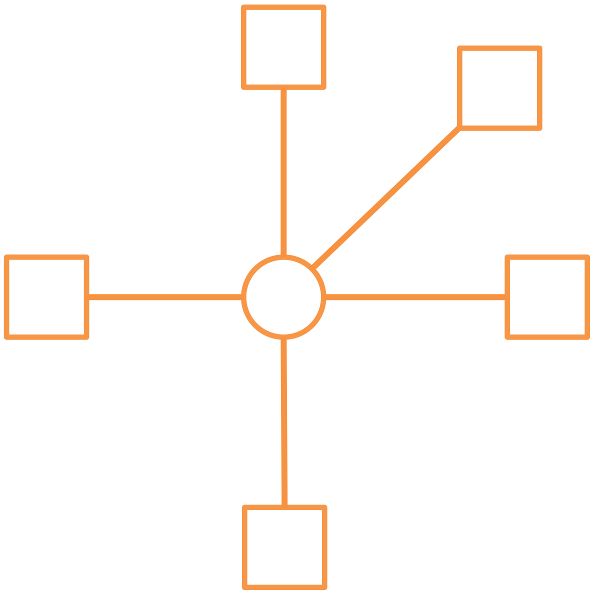
Gallager B
check message update

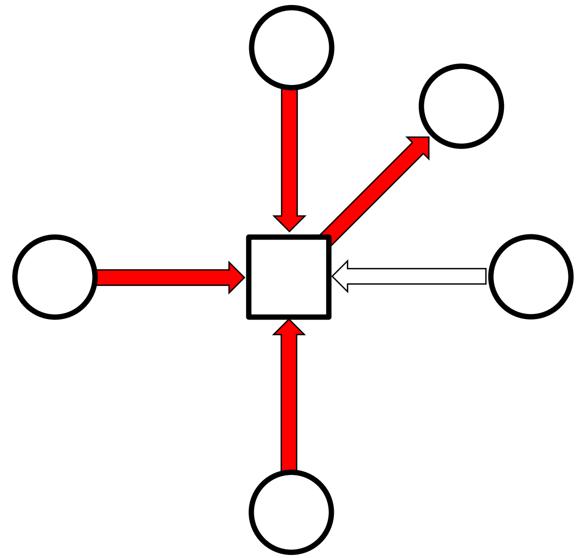




XOR the incoming messages and the check value
- make the number of red colors even

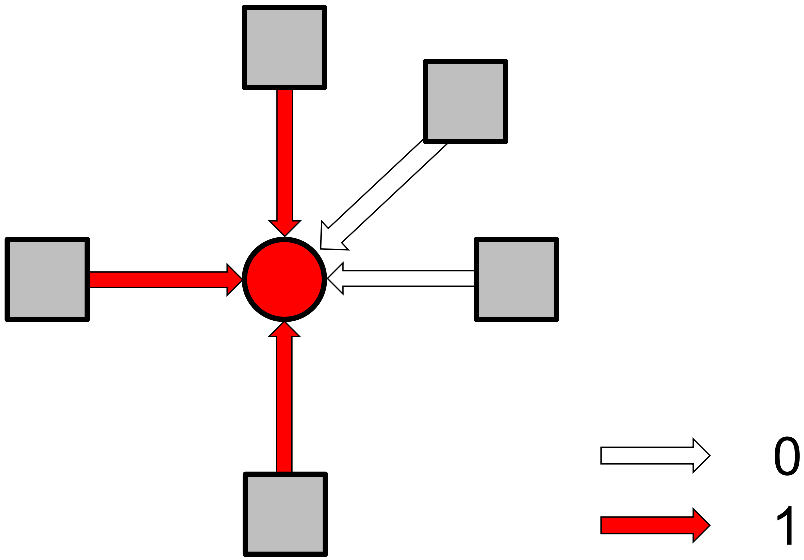
Gallager B
check message update

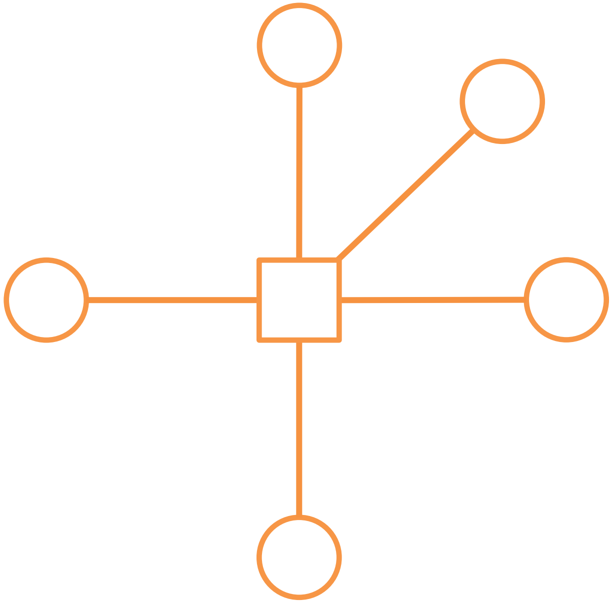




XOR the incoming messages and the check value
- make the number of red colors even

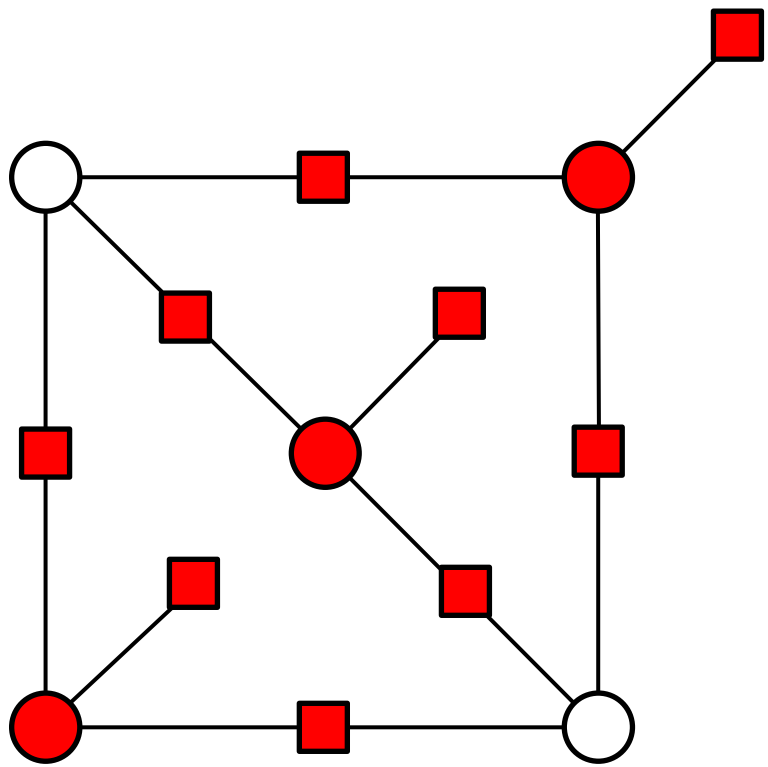
Gallager B decision

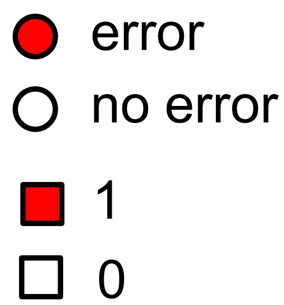




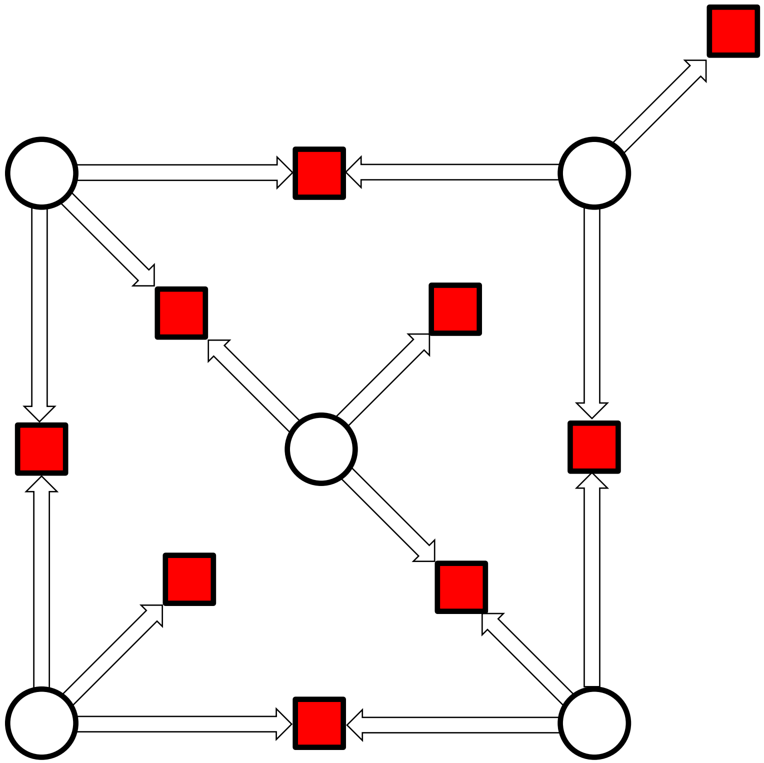
the bit value is decided as
the majority of **all** incoming messages

Trapping set illustration

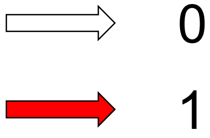




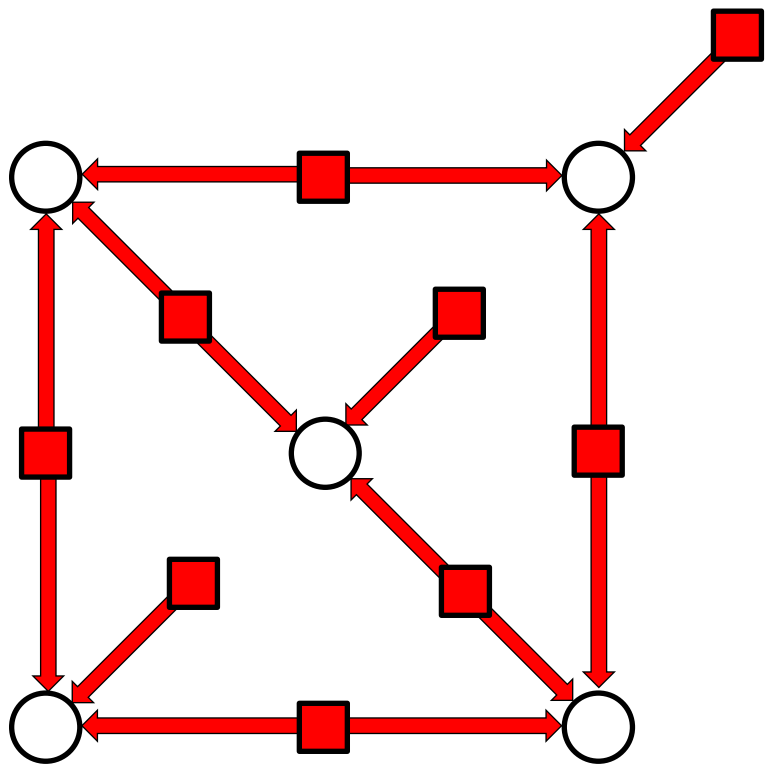
Trapping set illustration



iteration 1 – initialization
all variables send zero

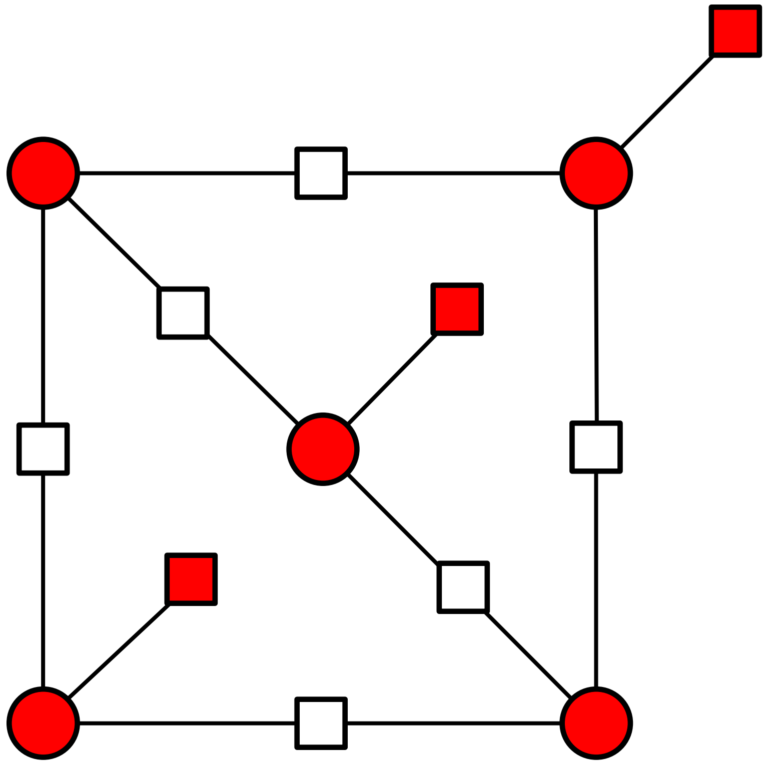


Trapping set illustration



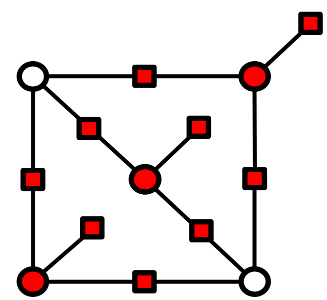
iteration 1 – the second half

Trapping set illustration

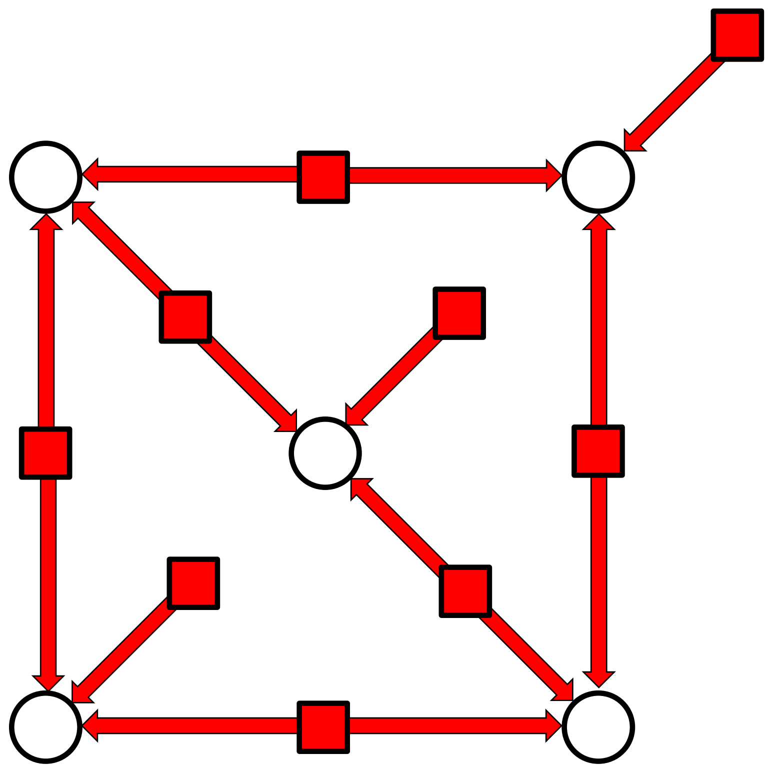


Iteration 1 - decison

syndrome mismatch

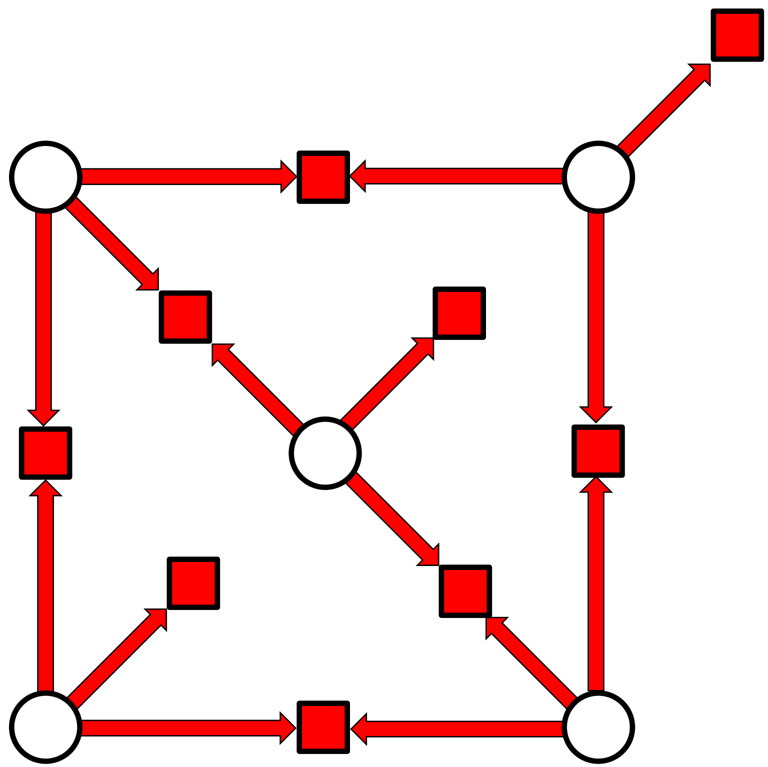


Trapping set illustration

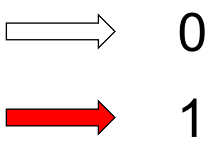


recall what messages were sent to variable nodes

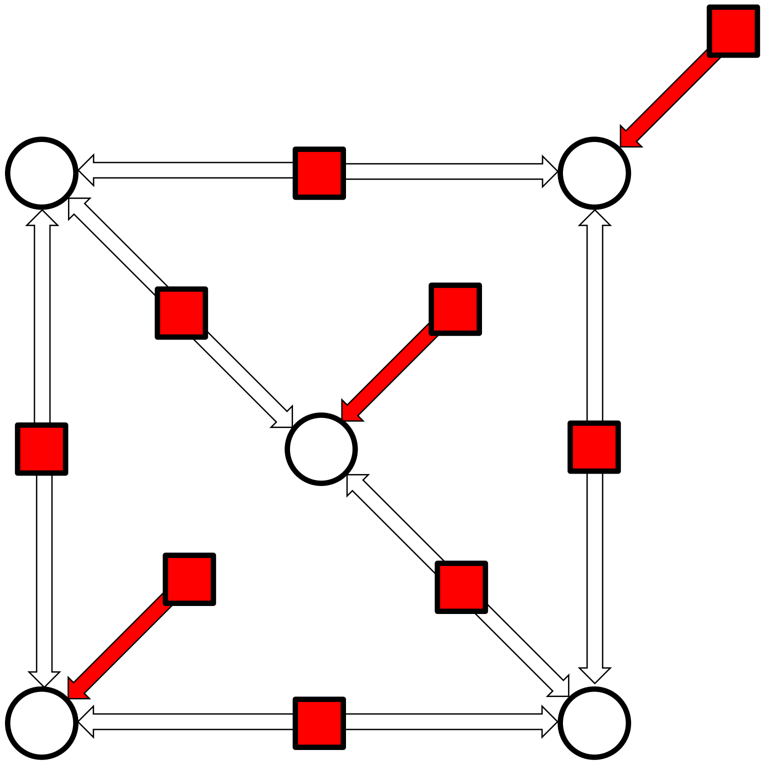
Trapping set illustration



iteration 2 – first half
variables send the majority
of incoming messages

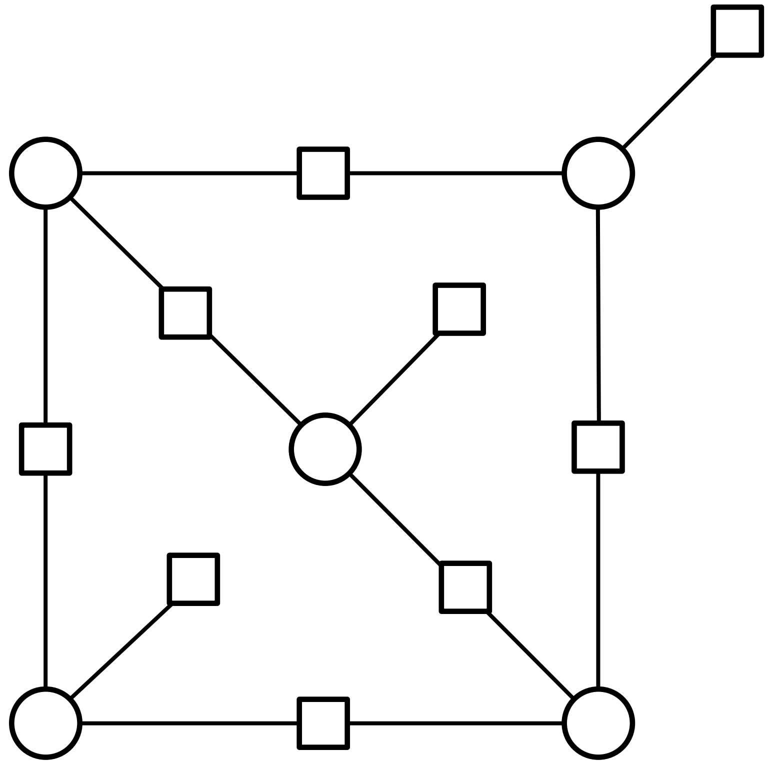


Trapping set illustration

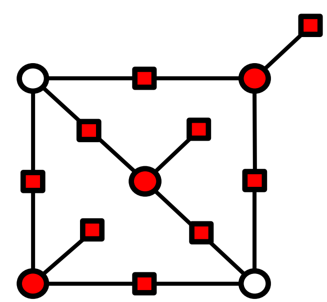


iteration 2 – second half

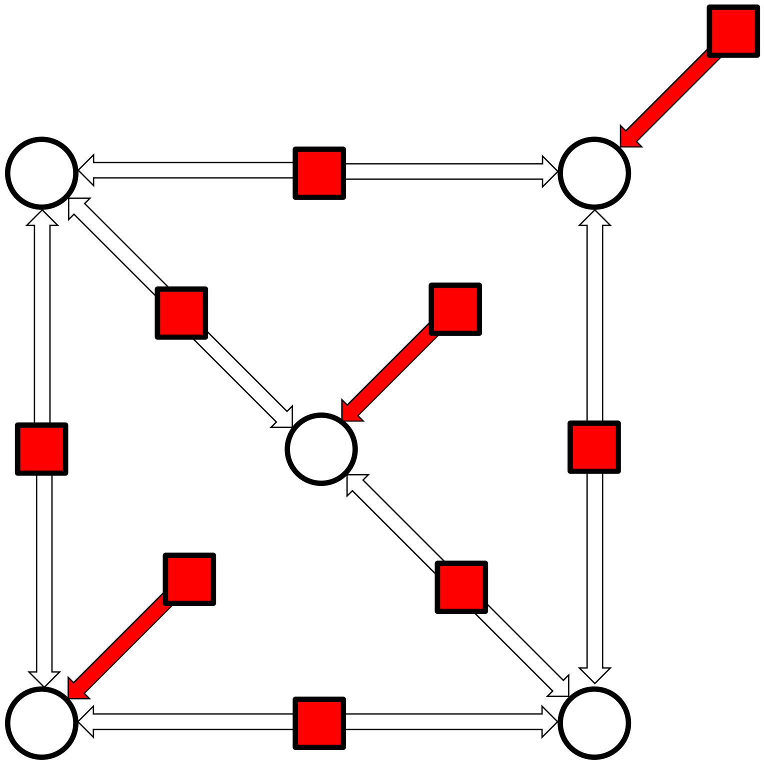
Trapping set illustration



iteration 2 - decision
syndrome mismatch

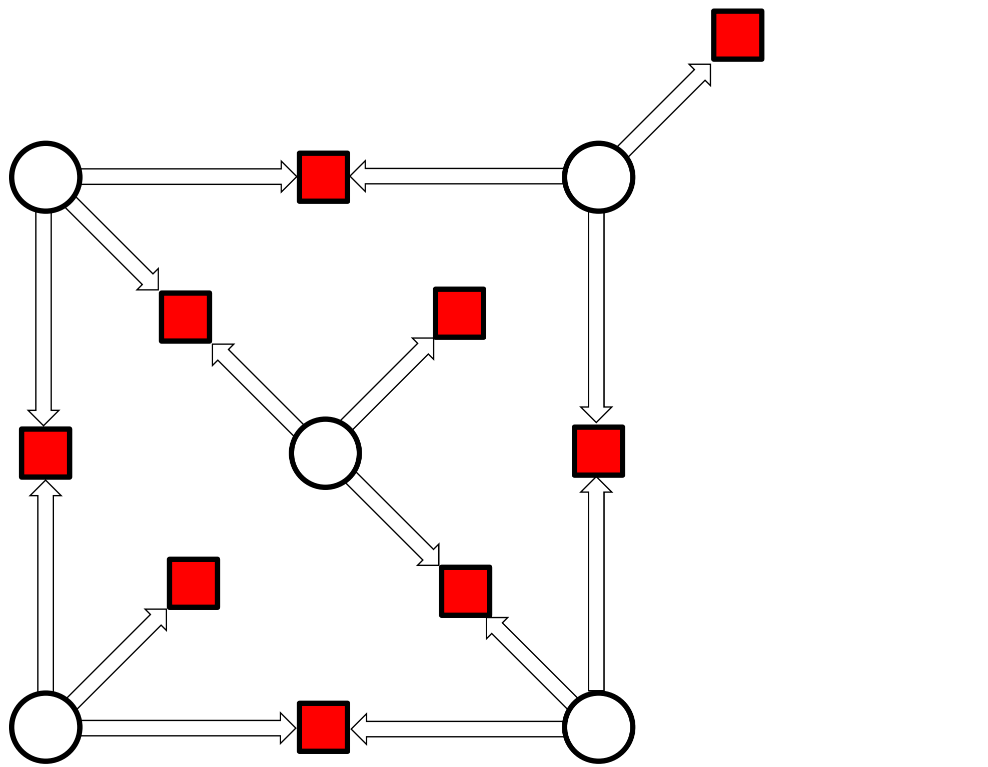


Trapping set illustration



messages sent to variable nodes in previous iteration

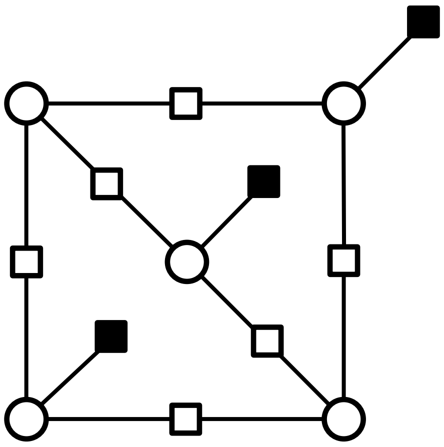
Trapping set illustration



iteration 3 – first half
as when we started

Quantum trapping sets

- Classical definition: An (a, b) trapping set $\mathcal{T}_c$ for an iterative decoder $\mathcal{D}$ is a non-empty set not eventually correct variable nodes in a Tanner graph of size a, inducing a subgraph $\mathcal{G}(\mathcal{T})$ with b odd degree check nodes.
- In a syndrome-based iterative decoders, we keep the (a, b) notation, but the situation is more complex.

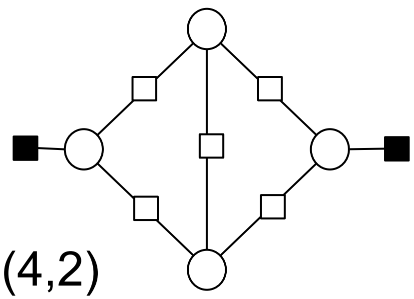


(a, b)= (5,3) trapping set

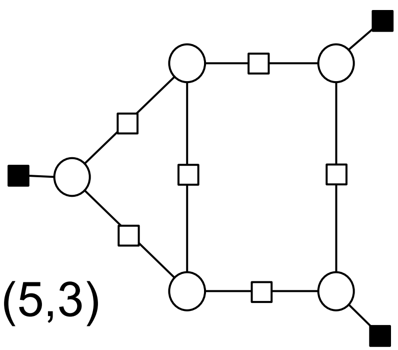
Summary of our findings

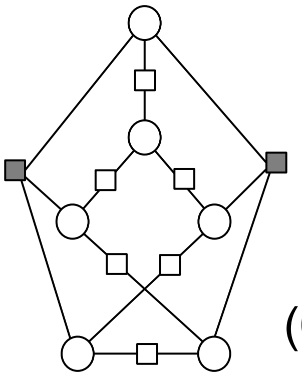
- QLDPC codes have two classes of trapping sets:
- Classical-looking trapping set due to dense subgraphs

(4,2)



(5,3)



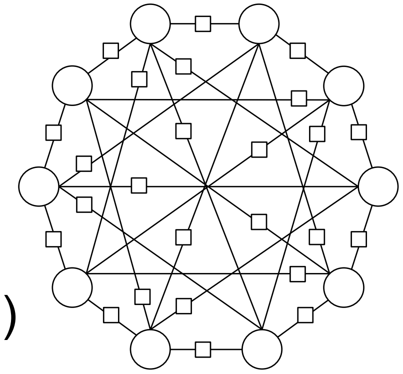


(6,2)

in [[900,36,10]] code

- Inherently-quantum trapping sets due to the symmetry of stabilizers

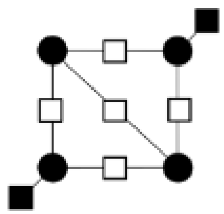
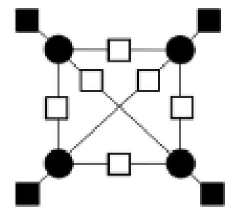
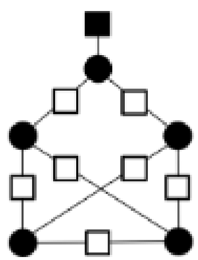
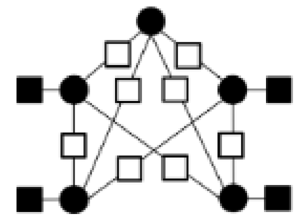
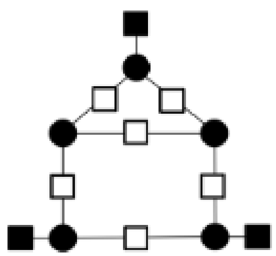
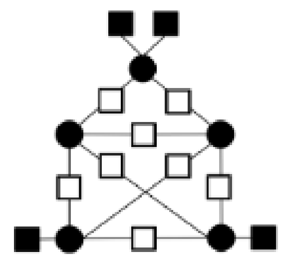
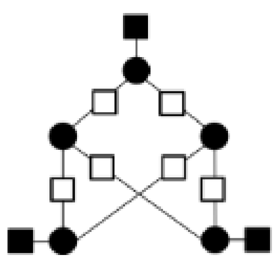
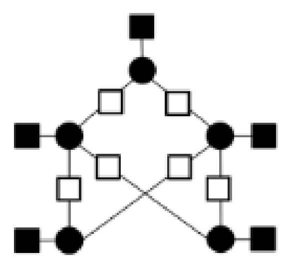
(10,0)



in [[254,28]] code,
“A1” code of Panteleev and Kalatchev

THE UNIVERSITY OF ARIZONA. TUCSON ARIZONA

Trapping sets in [[900,36,10]] HP Code

Quantum TS	Parameters [1] (a,b) CYC(x) Count	Quantum TS	Parameters (a,b) CYC(x) Count
	(4,2) $2x^6 + x^8$ 720		(4,4) $4x^6 + 3x^8$ 72
	(5,1) $2x^6 + 3x^8 + 2x^{10}$ 240		(5,4) $4x^6 + 5x^8 + 4x^{10}$ 36
	(5,3) $x^6 + x^8 + x^{10}$ 4080		(5,4) $5x^6 + 5x^8 + 2x^{10}$ 90
	(5,3) $3x^8$ 360		(5,5) $3x^8$ 5184

Trapping sets due to degeneracy

- Recall: the minimum distance of a code is the minimal weight of codewords in $\mathrm{rowspace}(L)$.
- Code is called degenerate if the minimum distance is much greater than the row weight of *H*.
  - Degenerate errors have weight much smaller than the minimum distance.
- Degenerate errors: Errors $\boldsymbol{e}$ and $\boldsymbol{f}$ that differ by an element in the stabilizer group.

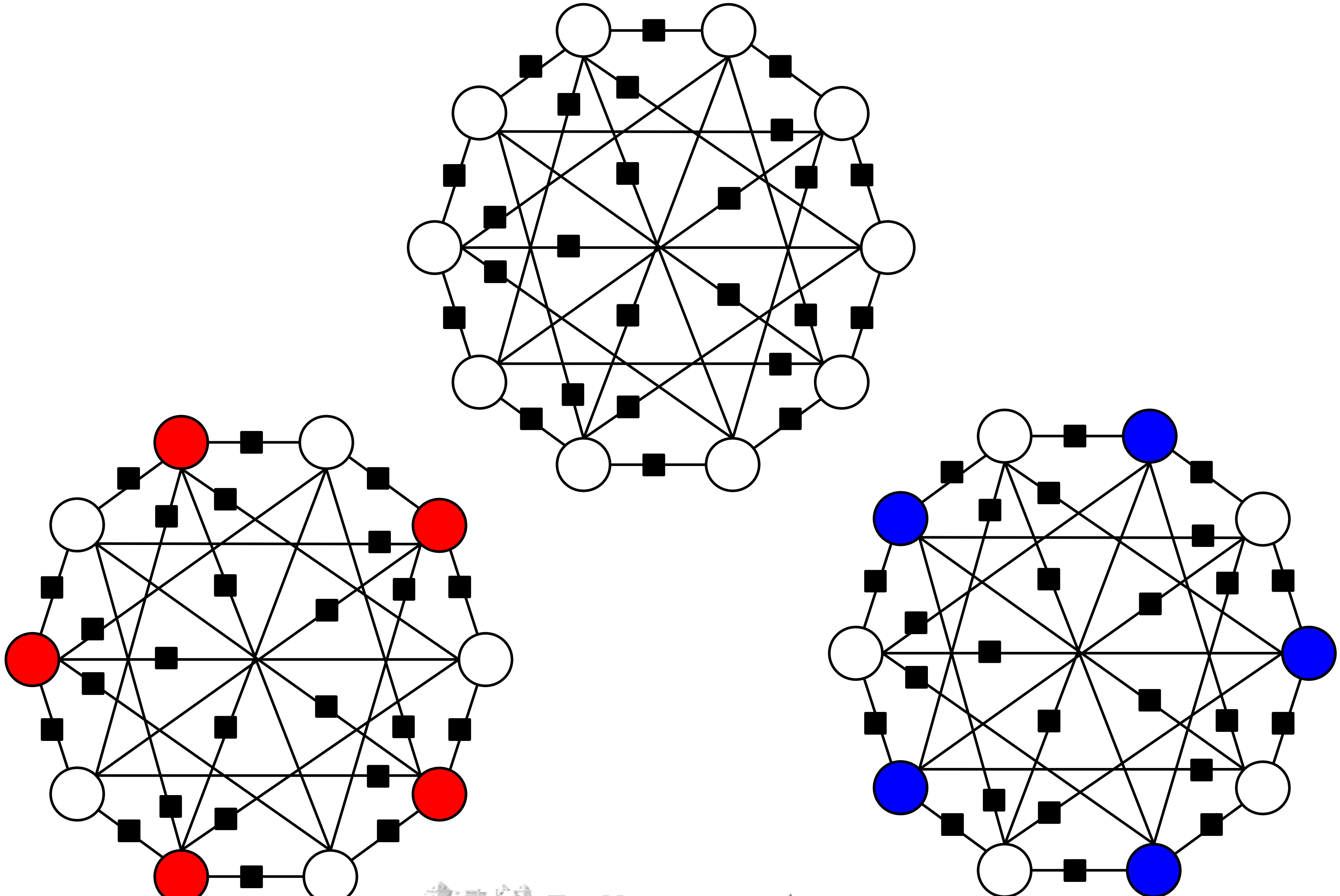
$$\boldsymbol{e} = \boldsymbol{f} + \boldsymbol{h}, \quad \boldsymbol{h} \in \mathrm{rowspace}(H).$$

- Impossible to tell them apart using the syndrome.

Impacts of degeneracy

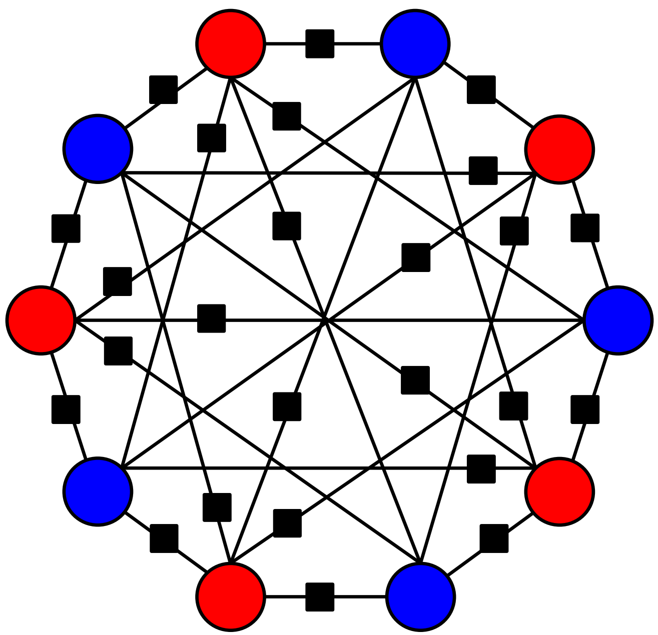
- We typically look for errors of lowest weight.
- H is sparse. If $\boldsymbol{e}$ is a low weight error pattern, then $\boldsymbol{e}+\boldsymbol{h}$, $\boldsymbol{h} \in \mathrm{rowspace}(H)$ may remain low-weight error pattern.
- Typically, there are many low-weight error patterns giving the same syndrome.
- Iterative decoder fails when degeneracy is combined with symmetry!

An example of a degenerate error



An example of a degenerate error

- Error patterns $\boldsymbol{e}$ and $\boldsymbol{f}$ induce a subgraph of a codeword.
- Iterative decoder attempts to converge to both $\boldsymbol{e}$ and $\boldsymbol{f}$ simultaneously leading to decoder failure.

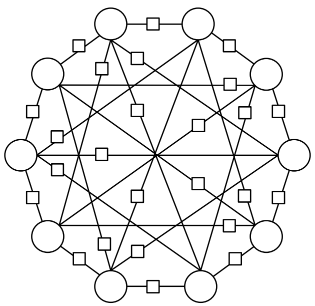


Symmetric stabilizer trapping sets are bad

- Symmetric degenerate errors are harmful for <u>all</u> iterative decoders, even “strong” decoders.
  - The sum and product operations in the sum-product algorithm are symmetric functions, thus messages in a symmetric graph are all equal, and decoder instead of $\boldsymbol{e}$ or $\boldsymbol{f}$ outputs $\boldsymbol{e}+\boldsymbol{f}$.

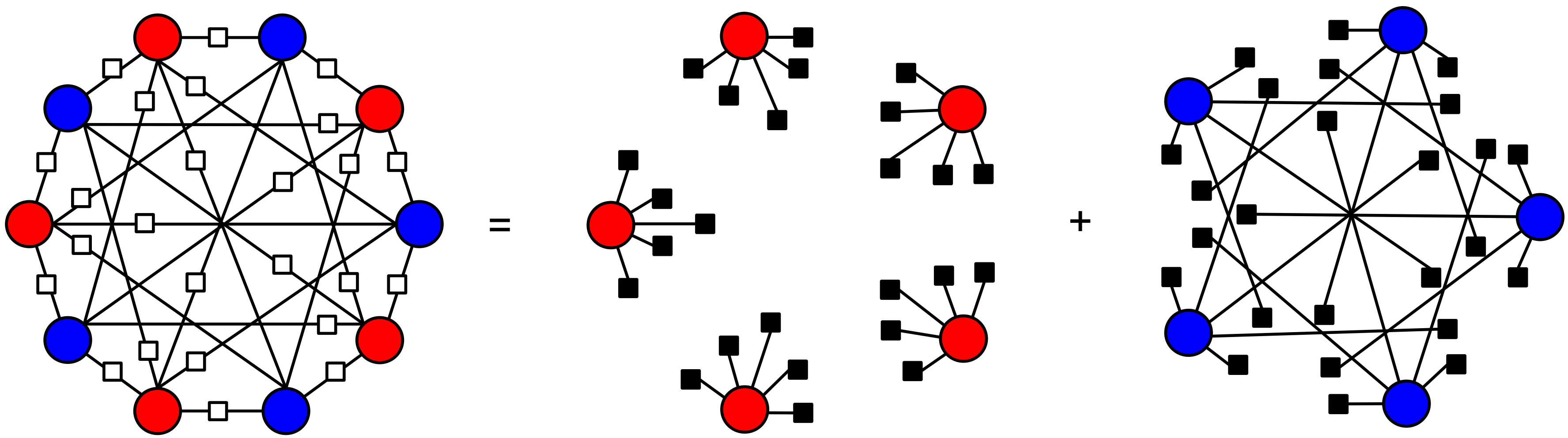
Harmfulness of symmetric stabilizers

- Different from classical-type TSs
- If an iterative decoder can correct half of the error patterns, degeneracy takes care of the rest!
- How degeneracy can be exploited in decoding!



- Lemma: For an $(a, 0)$ symmetric stabilizer TS with any iterative decoder which can correct up to $\frac{a}{2} - 1$ error patterns in the symmetric stabilizer, no error pattern on $\frac{a}{2} + 1$ nodes or more on the symmetric stabilizer is a failure configuration. In Trapping Sets of Quantum LDPC Codes: arxiv:2012.15297

Topology of Symmetric Stabilizer TSs



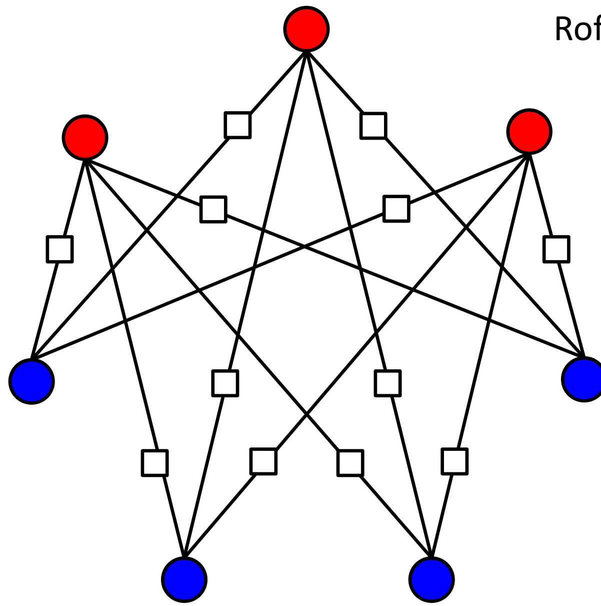
- This specific symmetric stabilizer is present in the A1 code - [[254,28]] code with circulant size $\ell = 127$.
- Non-classical behavior! A decoder fails only for <u>exactly</u> five errors, higher-weight errors are corrected.

P. Panteleev and G. Kalachev, arXiv:1904.02703, 2019.

Many stabilizers are asymmetric

- Asymmetric stabilizer in the [[900,36,10]] HP code

Roffe et al. May 2020. arXiv:2005.07016



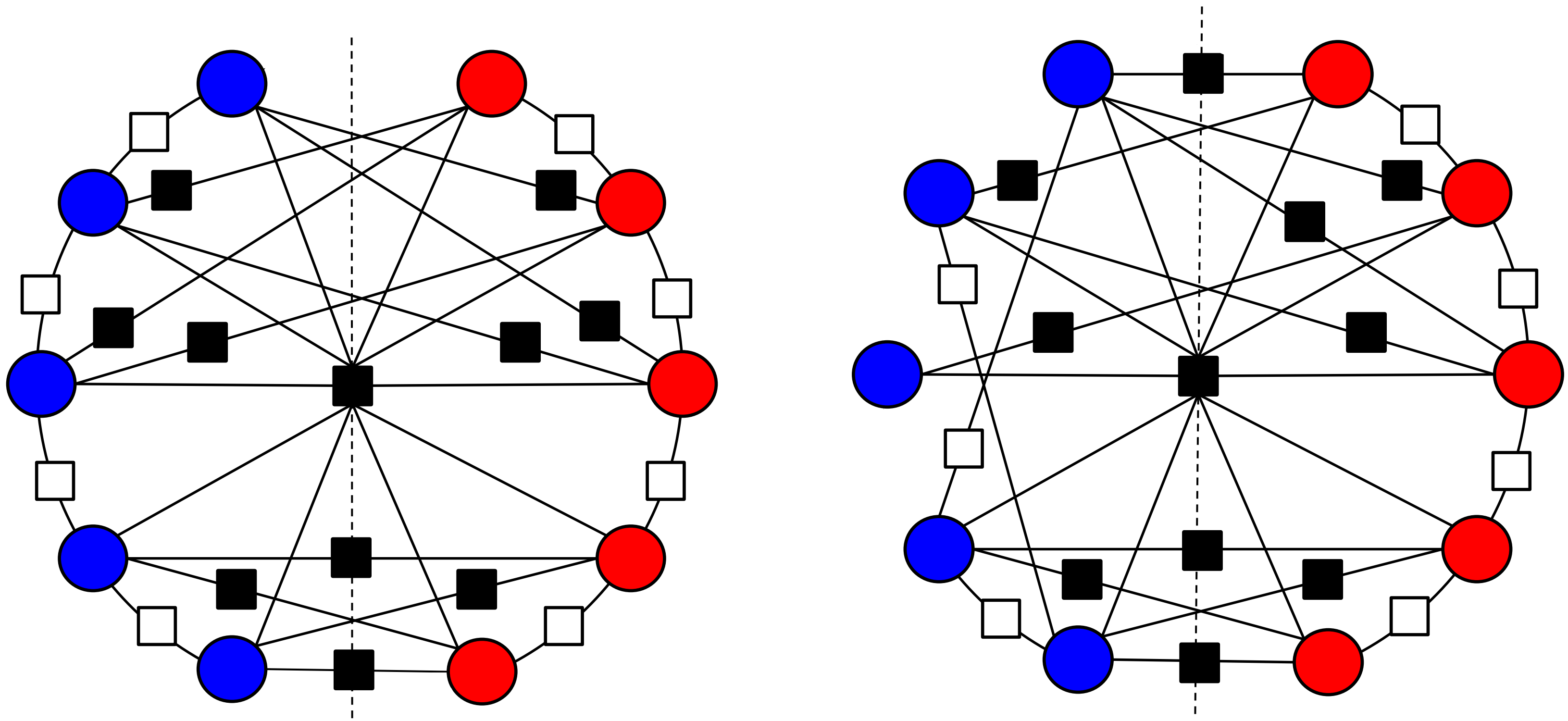
- When a QLDPC code is constructed from classical codes, trapping sets in classical codes remain and multiply in the Tanner graph of the quantum code.
- More structure and elegance, more symmetric stabilizers.

Summary

- Iterative decoders on QLDPC codes fail due to presence of trapping sets - dense subgraphs of specific structure:
  - classical looking (but quite different message dynamic)
  - symmetric stabilizers
- We present the methodology to identify and enumerate trapping sets.
- In code design, increasing d_{min} is not sufficient, the Tanner graph must be also free of small trapping sets.
- BP has a fundamental flaw and fails on dense graphs, but message passing algorithms can be designed using the knowledge of trapping sets.

Implications

- Knowledge of trapping set helps to design better codes and better decoders
- Method I: modify stabilizers to make them asymmetric.



Elimination of small TSs

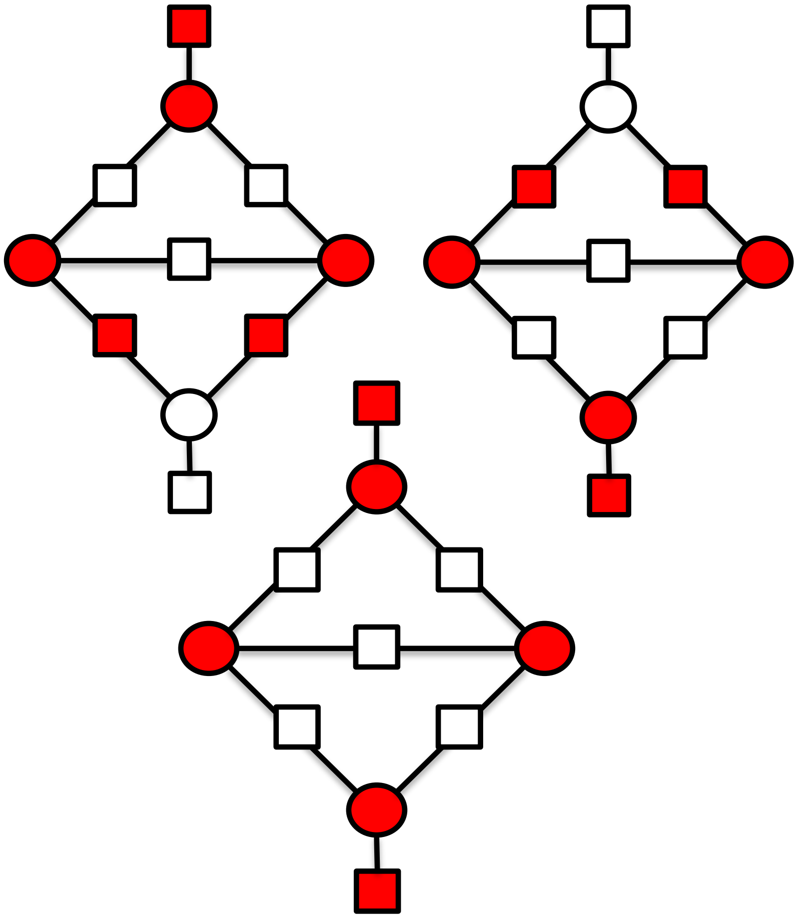
- Method II: Eliminate small trapping sets in code construction.
  - Eliminating TS in constituent codes in HP code construction automatically eliminates them in global code (picture of a HP code here)
- But, which one are more dangerous than the others?
- The answer in the recent Nithin's paper:

N. Raveendran, D. Declercq, and B. Vasić, "A Sub-Graph Expansion-Contraction Method for Error Floor Computation," *IEEE Transactions on Communications*, 2020.

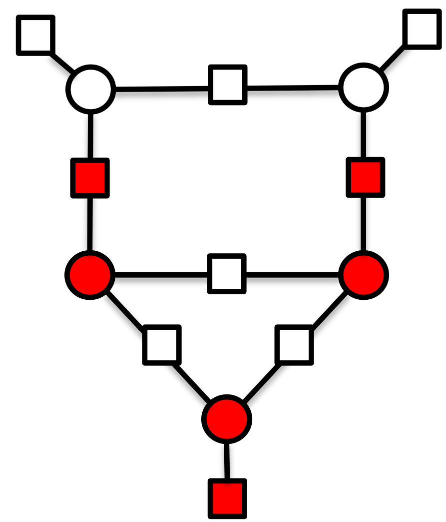
- It is not the size of trapping set that determines harmfulness, it is a critical number and strength of a trapping set.

Harmful syndrome/error patterns: min-sum

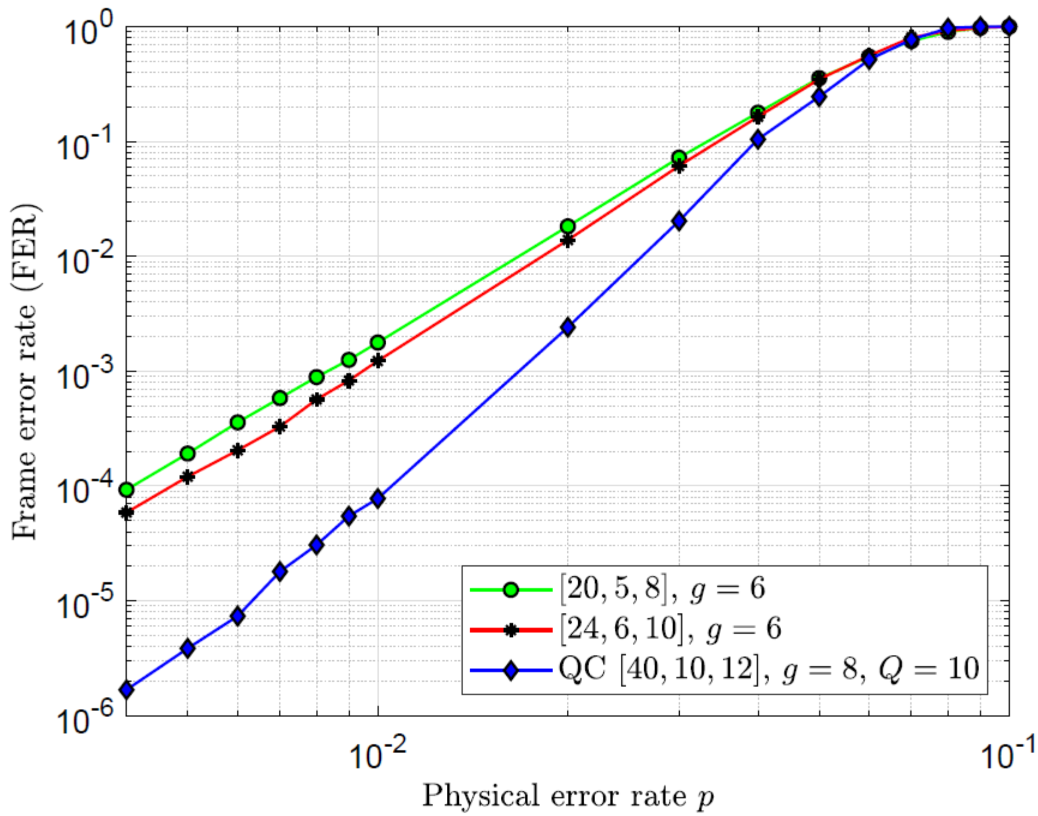
- (4,2) TS



Critical error pattern: (5,3) TS



TS-aware code construction

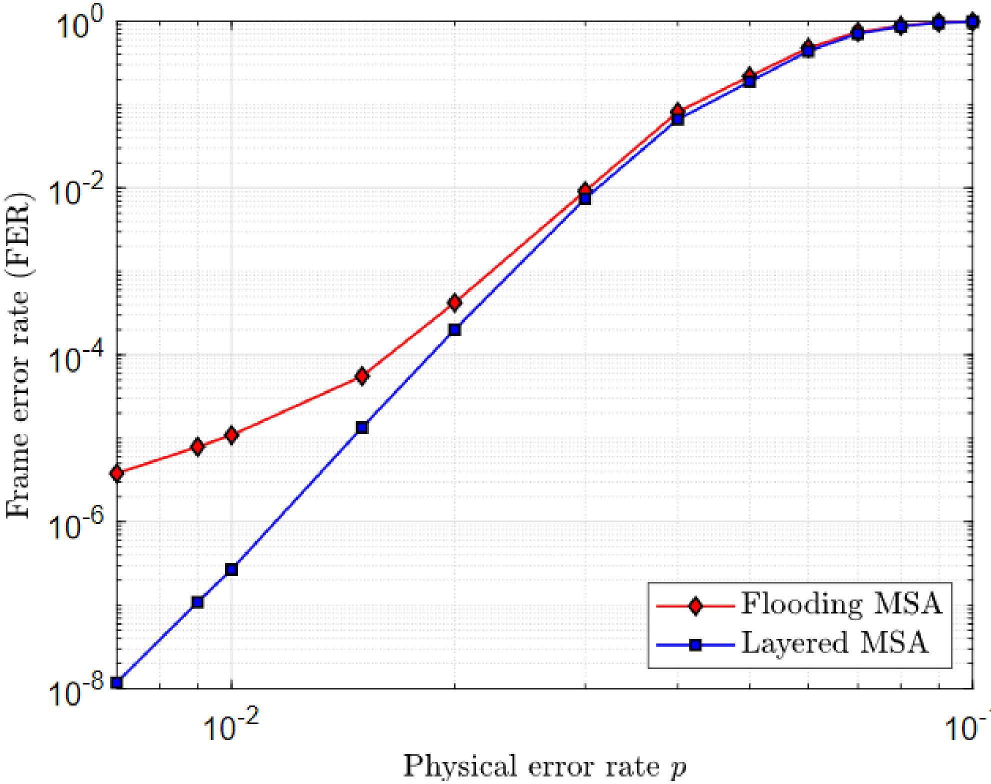


symmetric HP codes constructed using random constituent codes [20,5,8] and [24,6,10] from Roffe et al.
HP code constructed using a trapping set aware QC[40,10,12] code

Roffe et al. May 2020. arXiv:2005.07016 [quant-ph]

Better decoders

- A1 [[254,28]] code decoded by the min-sum algorithm (MSA) for two different schedules:
  - The layered schedule corrects **all** symmetric stabilizer TSs and numerous classical-type TSs.
  - Large unexplored area with potentially big impact.



Thank you!

N. Raveendran and B. Vasic, “Trapping Sets of Quantum LDPC Codes,” arXiv:2012.15297 [cs.IT]

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Quantum bit, quantum state

- Qubit: $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1$$

- Unlike classical bit having 'state' either 0 or 1, a qubit can be in a linear superposition of states.
- For N–qubit state

$$|\psi\rangle = a_0\,|00\ldots0\rangle + a_1\,|00\ldots1\rangle + \ldots + a_{2^N-1}\,|11\ldots1\rangle$$

$$a_i \in \mathbb{C}, \quad \sum_i |a_i|^2 = 1$$

$$|x_1 x_2 \ldots x_N\rangle \triangleq |x_1\rangle \otimes |x_2\rangle \ldots \otimes |x_N\rangle\,, |x_i\rangle \in \{|0\rangle\,, |1\rangle\}$$

Pauli group and its properties

- Pauli group
  - $\mathcal{G}_1$ with Pauli matrices and multiplicative factors $\pm 1, \pm i,$ closed under matrix multiplication-on single qubit.
  - $\mathcal{G}_n$ is an *n*-fold tensor product of $\mathcal{G}_1$-on *n* qubits.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Pauli matrices either commute or anti commute - with eigenvalues ± 1 - and have self inverse:
- Eg. $XZ = -ZX$

  $X^2 = I$

$$X^2 = I \quad Y^2 = I \quad Z^2 = I$$

$$XY = iZ \quad YX = iZ$$

$$YZ = iX \quad ZY = iX$$

$$ZX = iY \quad XZ = iY$$

Pauli operators on *n* qubits

$$X|0\rangle = |1\rangle\,, X|1\rangle = |0\rangle\,, X(a|0\rangle + b|1\rangle) = a|1\rangle + b|0\rangle$$
$$Z|0\rangle = |0\rangle\,, Z|1\rangle = -|1\rangle\,, Z(a|0\rangle + b|1\rangle) = a|0\rangle - b|1\rangle$$

$$Y|0\rangle = -i(XZ)|0\rangle = -iX|0\rangle = -i|1\rangle$$
$$Y|1\rangle = -i(XZ)|1\rangle = -iX(-|1\rangle) = i|0\rangle$$

$$A = (X \otimes Z \otimes I \otimes I \otimes Y \otimes Z)$$
$$(X \otimes Z \otimes I \otimes I \otimes Y \otimes Z)|000000\rangle = i|100010\rangle$$

$$(X \otimes Z \otimes I \otimes I \otimes Y \otimes Z)|111111\rangle = -i|010001\rangle$$

Commuting operators (example)

$$AB = (X \otimes Z)(Z \otimes X) = \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right)\left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & -1 & \end{array}\right)$$

$$AB = \left(\begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & 1 & & \\ -1 & & & \end{array}\right)$$

$$BA = (Z \otimes X)(X \otimes Z) = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & -1 & \end{array}\right)\left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right)$$

$$BA = \left(\begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & 1 & & \\ -1 & & & \end{array}\right)$$

$$A = (X \otimes Z)$$

$$B = (Z \otimes X)$$

$$\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathrm{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathrm{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \mathrm{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Commuting Operators

$$
\begin{array}{cccccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
A & = & (X \otimes & Z \otimes & I \otimes & I \otimes & Y \otimes & Z) \\
| & & | & \vdots_1 & | & | & \vdots_2 & | \\
B & = & (X \otimes & X \otimes & X \otimes & Y \otimes & Z \otimes & Z)
\end{array}
$$

$$
\begin{array}{cccccccccccccc}
 & & 1_X & 2_X & 3_X & 4_X & 5_X & 6_X & 1_Z & 2_Z & 3_Z & 4_Z & 5_Z & 6_Z \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbf{a} & = & (1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1) \\
| & & & & & & & & & & & & & \\
\mathbf{b} & = & (1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1)
\end{array}
$$

1 2

$$AB = BA$$

Simplectic product

- Simplectic product

$$\mathbf{a} = (\mathbf{a}_{\mathrm{X}}, \mathbf{a}_{\mathrm{Z}})$$

$$\mathbf{b} = (\mathbf{b}_{\mathrm{X}}, \mathbf{b}_{\mathrm{Z}})$$

$$\mathbf{a}_{\mathrm{X}}\mathbf{b}_{\mathrm{Z}}^{\mathrm{T}} + \mathbf{a}_{\mathrm{Z}}\mathbf{b}_{\mathrm{X}}^{\mathrm{T}}$$

$$(\mathbf{a}_X, \mathbf{a}_Z) \odot (\mathbf{b}_X, \mathbf{b}_Z)^{\mathrm{T}} = \mathbf{a}_X\mathbf{b}_Z^{\mathrm{T}} + \mathbf{a}_Z\mathbf{b}_X^{\mathrm{T}}$$

Entanglement as coding resource

- Superposition versus entangled states

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix} \qquad \underset{\substack{\uparrow \\ \text{superposition}}}{(|\psi\rangle \otimes |\phi\rangle)} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

superposition

$$\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

entanglement

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq |\psi\rangle \otimes |\phi\rangle$$

Quantum code - illustration

0000000
0001111
0110011
0111100
1010101
1011010
1100110
1101001

$$\frac{1}{\sqrt{8}}\left(|0000000\rangle + |0001111\rangle + |0110011\rangle + |0111100\rangle + |1010101\rangle + |1011010\rangle + |1100110\rangle + |1101001\rangle\right)$$

codeword

0010110
0011001
0100101
0101010
1000011
1001100
1110000
1111111

$$\frac{1}{\sqrt{8}}\left(|0010110\rangle + |0011001\rangle + |0100101\rangle + |0101010\rangle + |1000011\rangle + |1001100\rangle + |1110000\rangle + |1111111\rangle\right)$$

another codeword

Stabilizer formalism for quantum codes

- Stabilizer Group $\mathcal{S}$: Subgroup of *n*-qubit Pauli group $\mathcal{G}_n$ that leaves a non-trivial code state invariant.
  - Let $|\psi\rangle$ be any codeword state, then $\forall\ S \in \mathcal{S}$, $S\,|\psi\rangle = |\psi\rangle$.
  - All codewords have eigenvalue $+1$ for stabilizers S in $\mathcal{S}$.
  - All elements of $\mathcal{S}$ commute.
  - $-I \notin \mathcal{S}$.
- An [[*n*,*k*]] stabilizer code (in terms of stabilizers)
  - Vector space $V_{\mathcal{S}}$ stabilized by subgroup $\mathcal{S}$ of $\mathcal{G}_n$ such that $-I \notin \mathcal{S}$
  - $V_{\mathcal{S}}$ is the intersection of the subspaces fixed by each operator in $\mathcal{S}$
  - Example: S={I, Z_1Z_2, Z_2Z_3, Z_1Z_3}.

The subspace fixed by Z_1Z_2 is spanned by $|000\rangle$,$|001\rangle$,$|110\rangle$ and $|111\rangle$

The subspace fixed by Z1Z3 is spanned by $|000\rangle$,$|011\rangle$,$|101\rangle$ and $|111\rangle$

The subspace fixed by Z2Z3 is spanned by $|000\rangle$,$|011\rangle$,$|100\rangle$ and $|111\rangle$

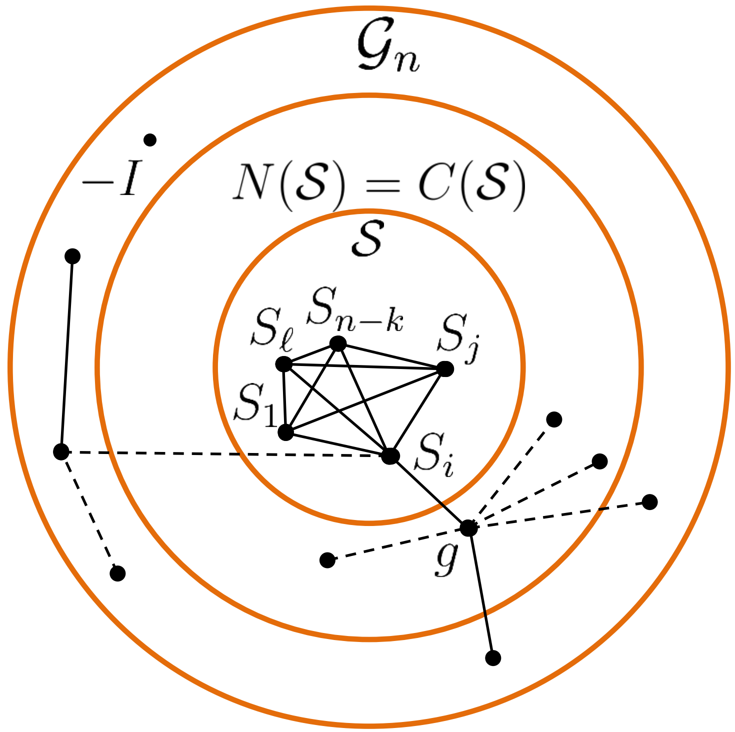
Intersection: $|000\rangle$ and $|111\rangle$

Stabilizer generators

- The code is specified by subspaces stabilized by all operators in the subgroup $\mathcal{S}$.
- $\mathcal{S}$ can be compactly represented by its generators.
- Let $\mathcal{S} = \langle g_1, g_2, \ldots, g_{n-k} \rangle$, where g_i are the generators, then every element of $\mathcal{S}$ can be written as a product of its generators.
- Example. The set of generators of the subgroup

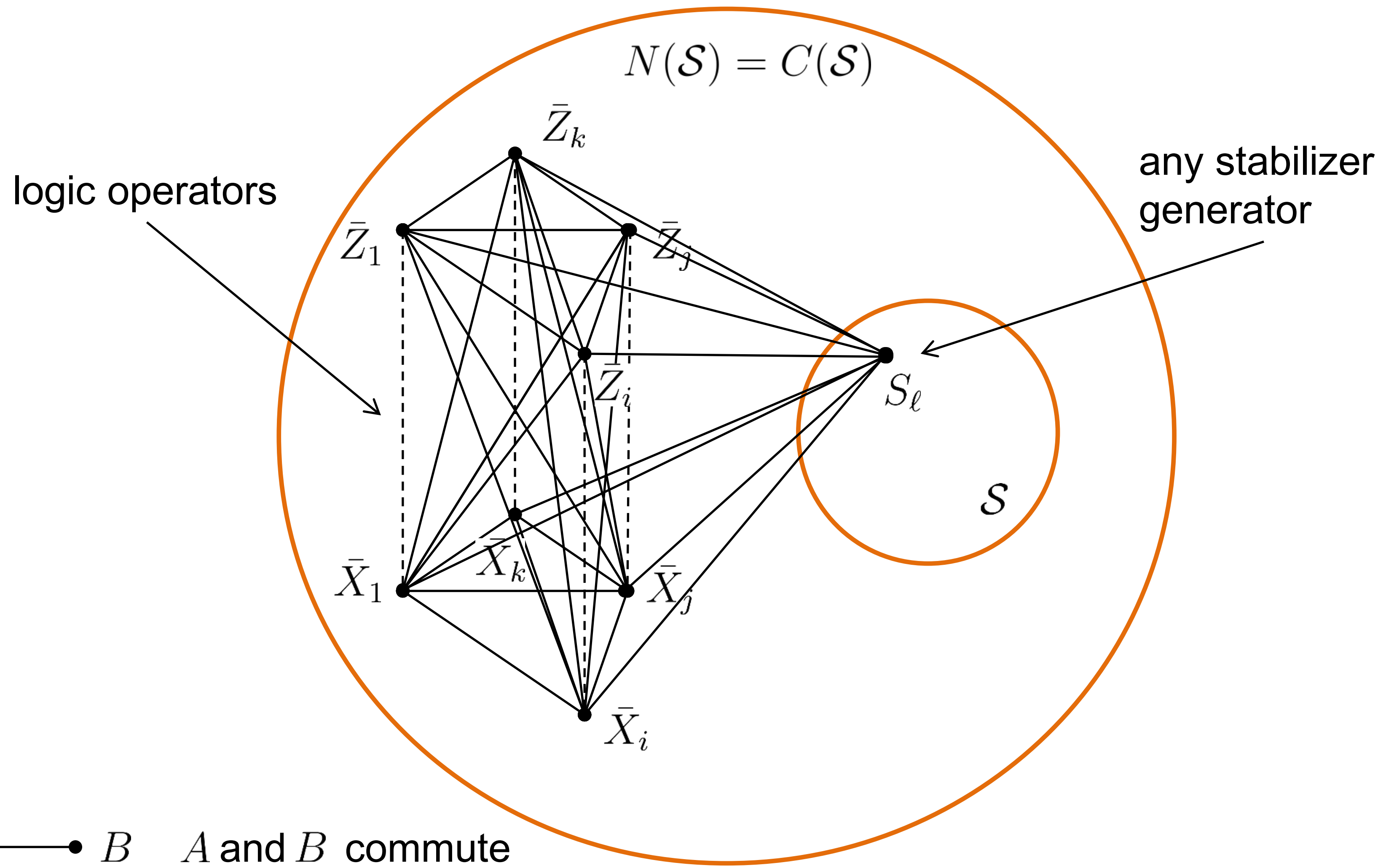
  $S=\{I, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$ is $\langle Z_1Z_2, Z_2Z_3 \rangle$ as $Z_1Z_3=(Z_1Z_2)(Z_2Z_3)$, and $I=(Z_1Z_2)^2$.

Commuting Diagram



A ●——● B	A and B commute
A ●- - -● B	A and B anti-commute

Logical Operators and Normalizer



A •——• B A and B commute

A •- - - -• B A and B anti-commute

Syndrome measurement

- Nature prevents us from learning anything about the probability amplitudes α and β and
- Nature only allows us to measure *observables*.
  - Observable is a Hermitian operator
  - Measurement outcome is one of the eigenvalues of the operator (real number)
  - Quantum state after measurement is eigenvector corresponding to that eigenvalue
- Examples of qubit observables: the Pauli operators X, Y, and Z.
- Measurement – projection to an eigenvector.
- Idea: choose measurements so that encoded state is an eigenvector corresponding to eigenvalue +1.

A Classical Equivalent of Stabilizer Codes

- Recall the definition of a symplectic product $\odot$ of vectors

$$(\mathbf{x},\mathbf{z}) \odot (\mathbf{u},\mathbf{v})^{\mathrm{T}} = \mathbf{x}\mathbf{v}^{\mathrm{T}} + \mathbf{u}\mathbf{z}^{\mathrm{T}}$$

- Since the stabilizer generators commute, any two rows

$$\mathbf{a} = (\mathbf{a}_{\mathrm{X}}, \mathbf{a}_{\mathrm{Z}}) \qquad \mathbf{b} = (\mathbf{b}_{\mathrm{X}}, \mathbf{b}_{\mathrm{Z}})$$

of the parity check matrix $H = \begin{pmatrix} H_{\mathrm{X}} & H_{\mathrm{Z}} \end{pmatrix}$ must satisfy

$$\mathbf{a} \odot \mathbf{b}^{\mathrm{T}} = 0$$

$$(\mathbf{a}_{\mathrm{X}}, \mathbf{a}_{\mathrm{Z}}) \odot (\mathbf{b}_{\mathrm{X}}, \mathbf{b}_{\mathrm{Z}})^{\mathrm{T}} = \mathbf{a}_{\mathrm{X}}\mathbf{b}_{\mathrm{Z}}^{\mathrm{T}} + \mathbf{b}_{\mathrm{X}}\mathbf{a}_{\mathrm{Z}}^{\mathrm{T}} = 0$$

- This leads the condition

$$H_{\mathrm{X}}H_{\mathrm{Z}}^{\mathrm{T}} + H_{\mathrm{Z}}H_{\mathrm{X}}^{\mathrm{T}} = \mathbf{0}$$

CSS Codes

- The parity check matrix is in this form

$$H = \begin{pmatrix} H_{\mathrm{X}} & 0 \\ 0 & H_{\mathrm{Z}} \end{pmatrix}$$

where $H_{\mathrm{X}} H_{\mathrm{Z}}^{\mathrm{T}} = 0.$

- The syndrome has two components

$$\mathbf{e} \odot H^{\mathrm{T}} = (\mathbf{e}_{\mathrm{X}}, \mathbf{e}_{\mathrm{Z}}) \begin{pmatrix} H_{\mathrm{X}}^{\mathrm{T}} & 0^{\mathrm{T}} \\ 0^{\mathrm{T}} & H_{\mathrm{Z}}^{\mathrm{T}} \end{pmatrix} = (\mathbf{s}_{\mathrm{X}}, \mathbf{s}_{\mathrm{Z}})$$

where

$$\begin{aligned} \mathbf{s}_{\mathrm{X}} &= \mathbf{e}_{\mathrm{X}} 0^{\mathrm{T}} + \mathbf{e}_{\mathrm{Z}} H_{\mathrm{X}}^{\mathrm{T}} \\ \mathbf{s}_{\mathrm{Z}} &= \mathbf{e}_{\mathrm{X}} H_{\mathrm{Z}}^{\mathrm{T}} + \mathbf{e}_{\mathrm{Z}} 0^{\mathrm{T}} \end{aligned}$$

thus

$$\begin{aligned} \mathbf{s}_{\mathrm{X}} &= \mathbf{e}_{\mathrm{Z}} H_{\mathrm{X}}^{\mathrm{T}} \\ \mathbf{s}_{\mathrm{Z}} &= \mathbf{e}_{\mathrm{X}} H_{\mathrm{Z}}^{\mathrm{T}} \end{aligned}$$

Quantum channel

- Evolution of a *closed* quantum system is described by a *unitary* transformation E $(EE^\dagger = I)$

$$|\psi\rangle \xrightarrow{E} E\,|\psi\rangle$$

- Discretization of errors:
  - We do not have to separately correct a continuum of possible errors, but only a discrete, finite set of errors.
  - If a code can correct operators in the set $\{E_k\}$, then it can correct any linear sum of these operators.

Error correction

- Must be done without learning the state

$$|\psi\rangle \xrightarrow{E} E\,|\psi\rangle \xrightarrow{\hat{E}^\dagger} \hat{E}^\dagger E\,|\psi\rangle$$

correction

$$\hat{E}$$

error estimation

Syndrome measurement

- Nature prevents us from learning anything about the probability amplitudes α and β and
- Nature only allows us to measure *observables*.
  - Observable is a Hermitian operator
  - Measurement outcome is one of the eigenvalues of the operator (real number)
  - Quantum state after measurement is eigenvector corresponding to that eigenvalue
- Examples of qubit observables: the Pauli operators X, Y, and Z.
- Measurement – projection to an eigenvector.
- Idea: choose measurements so that encoded state is an eigenvector corresponding to eigenvalue +1.

Quantum syndrome decoding

- Let $\mathbf{e}$ be a non-zero error vector, resulting in a syndrome $\mathbf{s}$

$$\mathbf{s} = \mathbf{e} \odot H^{\mathrm{T}} \neq \mathbf{0}$$

$\odot$ symplectic product

- As opposed to a *classical syndrome decoder* that tries to find $\mathbf{e}$ for a given observed syndrome, a valid output of a *quantum* decoder is any one of the vectors

$$\widetilde{\mathbf{e}} = \mathbf{e} + \mathbf{h}, \mathbf{h} \in \mathrm{rowspace(H)}$$

- When $\mathbf{e} + \widetilde{\mathbf{e}} \neq \mathbf{0}$, but

$$(\mathbf{e} + \widetilde{\mathbf{e}}) \odot H^{\mathrm{T}} = \mathbf{0}$$

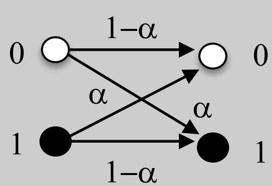
then the correction vector $\mathbf{e} + \widetilde{\mathbf{e}}$ is applied to flip bits in the (unobservable) quantum codeword is also a codeword, and a logical, undetectable, error occurs.

Stabilizer generators

- An [[n,k]] stabilizer code (in terms of stabilizers)
  - Vector space V_S stabilized by subgroup $\mathcal{S}$ of $\mathcal{G}_n$ such that $-I \notin \mathcal{S}$.
  - $\mathcal{S}$ has n-k independent and commuting generators $S_1, \ldots, S_{n-k}$.
  - A codeword is a simultaneous eigenstate of all generators of $\mathcal{S}$ with eigenvalue +1.
- Let $\mathcal{S} = \langle g_1, g_2, \ldots, g_{n-k} \rangle$ be generated by n-k independent generators, and $-I \notin \mathcal{S}$.

  Then there exists $g \in \mathcal{G}_n$ such that

  $$g g_i g^\dagger = -g_i \text{ and } g g_j g^\dagger = g_j \text{ for all } j \neq i .$$
- Since the generators in $\mathcal{S}$ must commute, addition of each generator cuts the dimension of V_S by half.
- Therefore, V_S is 2^k -dimensional.

CLASSICAL ERROR CORRECTION	QUANTUM ERROR CORRECTION	
Linear code $\mathcal{C} = [N, K, d]$	Stabilizer code $\mathcal{Q} = [[N, K, d]]$	
Information bits (K) - $\mathbf{m}$	Logical qubits (K) - $\vert\psi\rangle_L \in \mathbb{C}^{2^K}$	
Coded bits (N) - $\mathbf{x}$	Physical qubits (N) - $\vert\boldsymbol{\psi}\rangle \in \mathbb{C}^{2^N}$	
Parity check matrix H $\mathbf{x}H^{\mathrm{T}} = \mathbf{0}$	Generators of commutative stabilizer group $\mathcal{S}$ Fixes physical qubits/code state $S\vert\boldsymbol{\psi}\rangle = \vert\boldsymbol{\psi}\rangle, \quad \forall S \in \mathcal{S}$	
$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$ for [7,4,3] code	$H_{\mathrm{P}} = \begin{bmatrix} \mathrm{X} & \mathrm{I} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{X} & \mathrm{X} \\ \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{X} & \mathrm{X} & \mathrm{X} & \mathrm{I} \\ \mathrm{I} & \mathrm{I} & \mathrm{X} & \mathrm{I} & \mathrm{X} & \mathrm{X} & \mathrm{X} \\ \mathrm{Z} & \mathrm{I} & \mathrm{I} & \mathrm{Z} & \mathrm{I} & \mathrm{Z} & \mathrm{Z} \\ \mathrm{I} & \mathrm{Z} & \mathrm{I} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z} & \mathrm{I} \\ \mathrm{I} & \mathrm{I} & \mathrm{Z} & \mathrm{I} & \mathrm{Z} & \mathrm{Z} & \mathrm{Z} \end{bmatrix}$ for [[7,1,3]]	
Bit flip error modelled as a BSC(α) 	Bit flip error – X, Phase flip error – Z Bit and phase flip – Y No error - I	Continuous nature of qubit errors discretized to correcting Pauli X, Y, Z errors.

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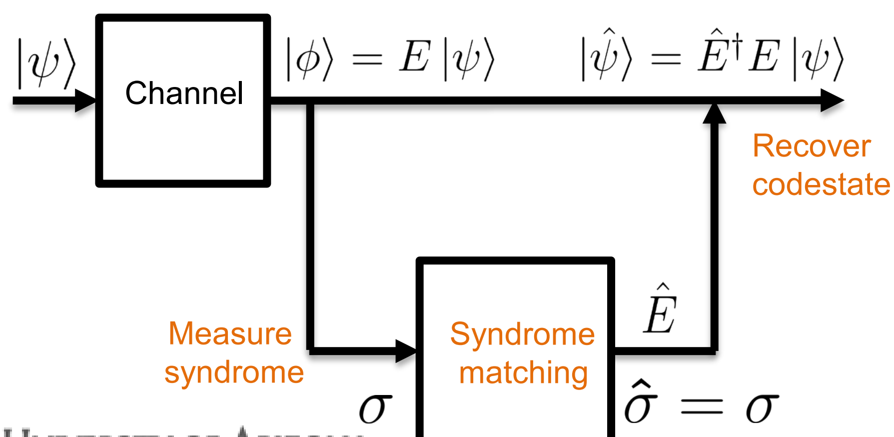
CLASSICAL ERROR CORRECTION	QUANTUM ERROR CORRECTION
Direct access to channel output $\mathbf{r}$	Cannot directly access erroneous code state but can measure syndrome.
Syndrome for error detection For $\mathbf{r} \notin \mathcal{C}$ $\boldsymbol{\sigma} = \{\sigma_1, \dots, \sigma_M\} = \mathbf{r}H^T \neq 0$	Measure syndrome Measure the $M = N - K$ stabilizer generators Eigen value $(-1^{\sigma_i}) \rightarrow$ binary syndrome σ_i $\boldsymbol{\sigma} = \{\sigma_1, \dots, \sigma_M\}$
Error detected when $\boldsymbol{\sigma} \neq \mathbf{0}$	Error detected when $\boldsymbol{\sigma} \neq \mathbf{0}$
Decoding Decoder uses H to recover the correct codeword $\mathbf{x}$ from channel output $\mathbf{r}$.	**Syndrome-based Decoding** 1. Syndrome matching - Find corresponding error that matches the syndrome. 2. Error recovery: Reverse the error to get back the codestate $\vert\psi\rangle$



CLASSICAL ERROR CORRECTION	QUANTUM ERROR CORRECTION
No Classical Analog	**Degenerate error patterns** Errors $\mathbf{E}$ and $\mathbf{F}$ have same non-trivial syndrome $\boldsymbol{\sigma}$ if they differ by a stabilizer (check) $\mathbf{F} = \mathcal{S}\mathbf{E}$, where $\mathcal{S} \in \mathcal{S}$
	Degeneracy property of QLDPC codes allows syndrome based decoders to match the syndrome with $\boldsymbol{E}$ or $\boldsymbol{F}$ as degenerate errors Any error upto a stabilizer $\mathbf{F} = \mathcal{S}\mathbf{E}$, where $\mathcal{S} \in \mathcal{S}$

- Our quantum trapping set study investigates effect of
  - Degeneracy of QLDPC codes
  - Syndrome-based iterative decoding
  - QLDPC code constraints from commutativity of stabilizers

Approach

- We want to find all low weight uncorrectable error patterns of a given iterative decoder $\mathcal{D}$.
- If a decoding algorithm is local, such as a bit-flipping or message passing decoding, then a computationally efficient algorithm for finding all low-weight error patterns exists. Induced subgraphs are trapping sets.

N. Raveendran, D. Declercq, and B. Vasić, "A Sub-Graph Expansion-Contraction Method for Error Floor Computation," *IEEE Transactions on Communications*, 2020.

- Consequences:
  - on a depolarizing channel with probability α, we can accurately compute decoding probability of error for low values of channel error rates.
  - The knowledge of trapping sets allows us to design better codes and better decoders!

Trapping sets due to degeneracy

- Recall: the minimum distance of a code is the minimal weight of operators that commute with all the stabilizers but are not in the stabilizer group. Also, the minimal weight of logical operators.
- Code is called degenerate if the minimum distance is much greater than the weight of the stabilizers.
  - Degenerate errors have weight much smaller than the minimum distance.
- Degenerate errors: Errors $\boldsymbol{e}$ and $\boldsymbol{f}$ that differ by an element in the stabilizer group.

$$\boldsymbol{e} = \boldsymbol{f} + \boldsymbol{h}, \quad \boldsymbol{h} \in \mathrm{rowspace}(H).$$

- Impossible to tell them apart using the syndrome.

Generalized bicycle codes

- $H_X = [A\ B]$ and $H_Z = [B^T A^T]$
  - Commuting binary matrices A and B, i.e, $AB = BA$
  - Introduced by Kovalev and Pryadko, called Kronecker Sum-Product Codes,
  - Generalization of MacKay et. al. Bicycle Codes, where $A = B$
- Stabilizer commutativity satisfied: $H_X H_Z^T = AB + BA = \mathbf{0}$
- Panteleev and Kalachev use circulant matrices.
  - A1 code - [[254,28]] code with circulant size $\ell = 127$
  - $A := a(x) = 1 + x^{15} + x^{20} + x^{28} + x^{66}$,
  - $B := b(x) = 1 + x^{58} + x^{59} + x^{100} + x^{121}$.

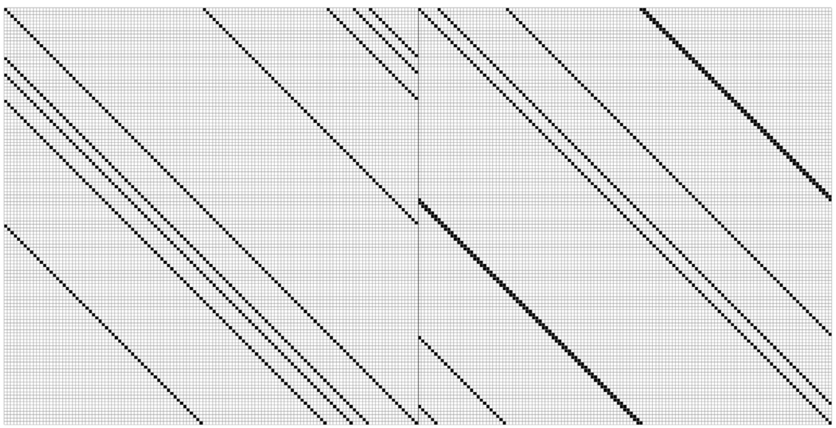
A. A. Kovalev and L. P. Pryadko, Phys. Rev. A 88, 012311, 2013.

P. Panteleev and G. Kalachev, arXiv:1904.02703, 2019.

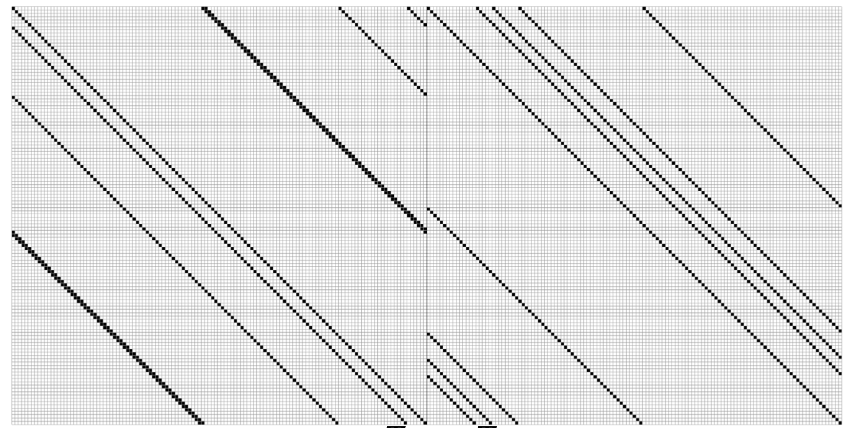
D. J. C. MacKay, et al. , IEEE Trans. Inf. Theory, 50, 10, 2315–2330, 2004.

[[254,28]] generalized bicycle code

- A1 code - [[254,28]], $\ell = 127$
  - $A := a(x) = 1 + x^{15} + x^{20} + x^{28} + x^{66}$,
  - $B := b(x) = 1 + x^{58} + x^{59} + x^{100} + x^{121}$.



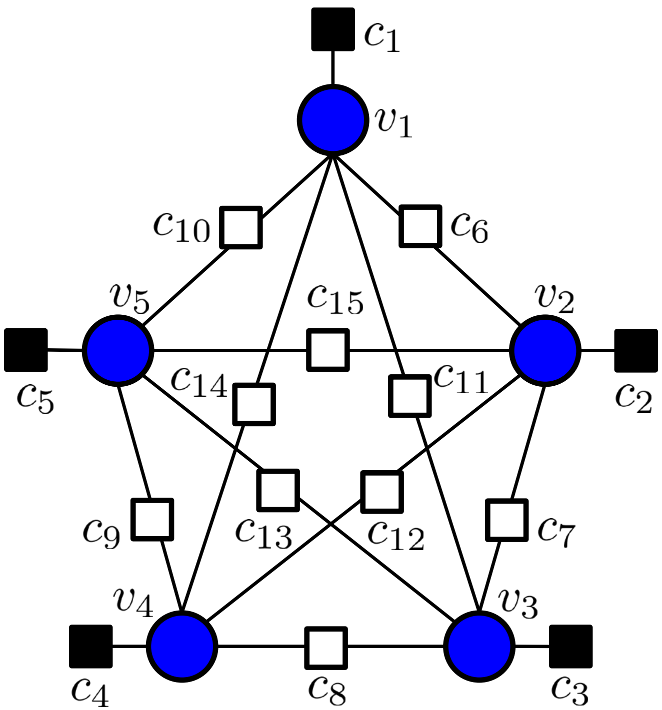
$H_X = [A \mid B]$



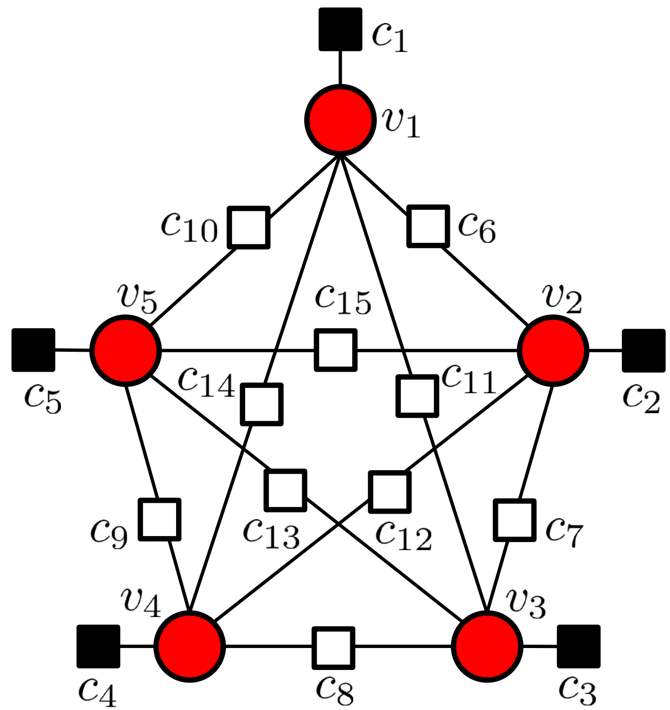
$H_Z = [B^T \mid A^T]$

P. Panteleev and G. Kalachev, arXiv:1904.02703, 2019.

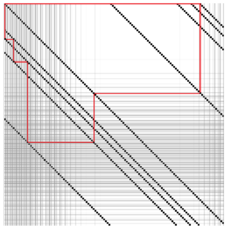
(5,5) TS in A1 code



more harmful - in circulant A

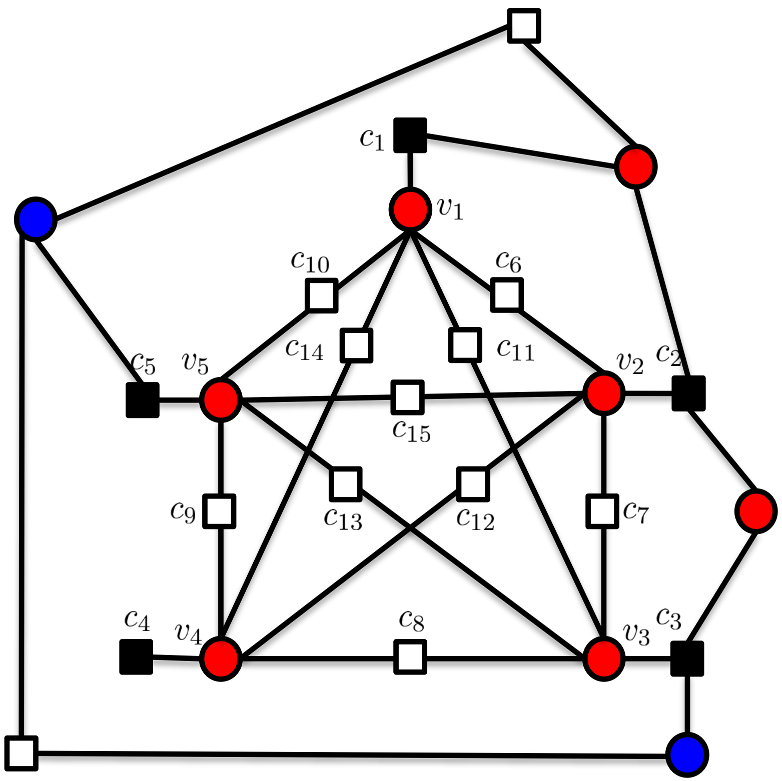


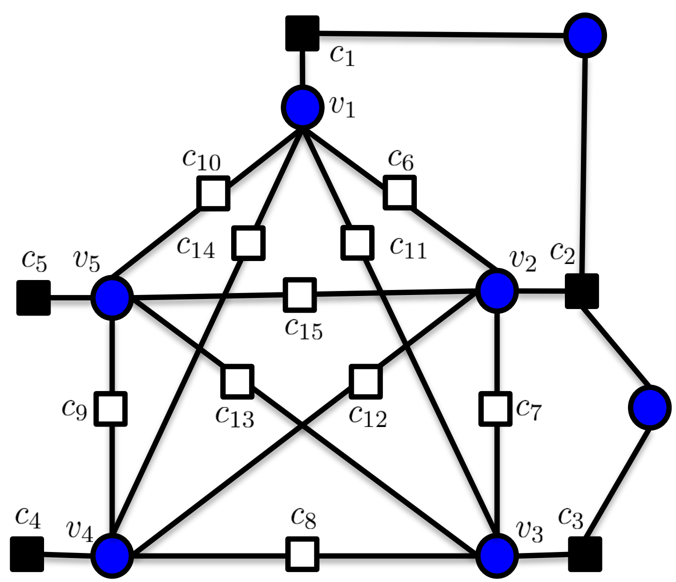
less harmful - in circulant B



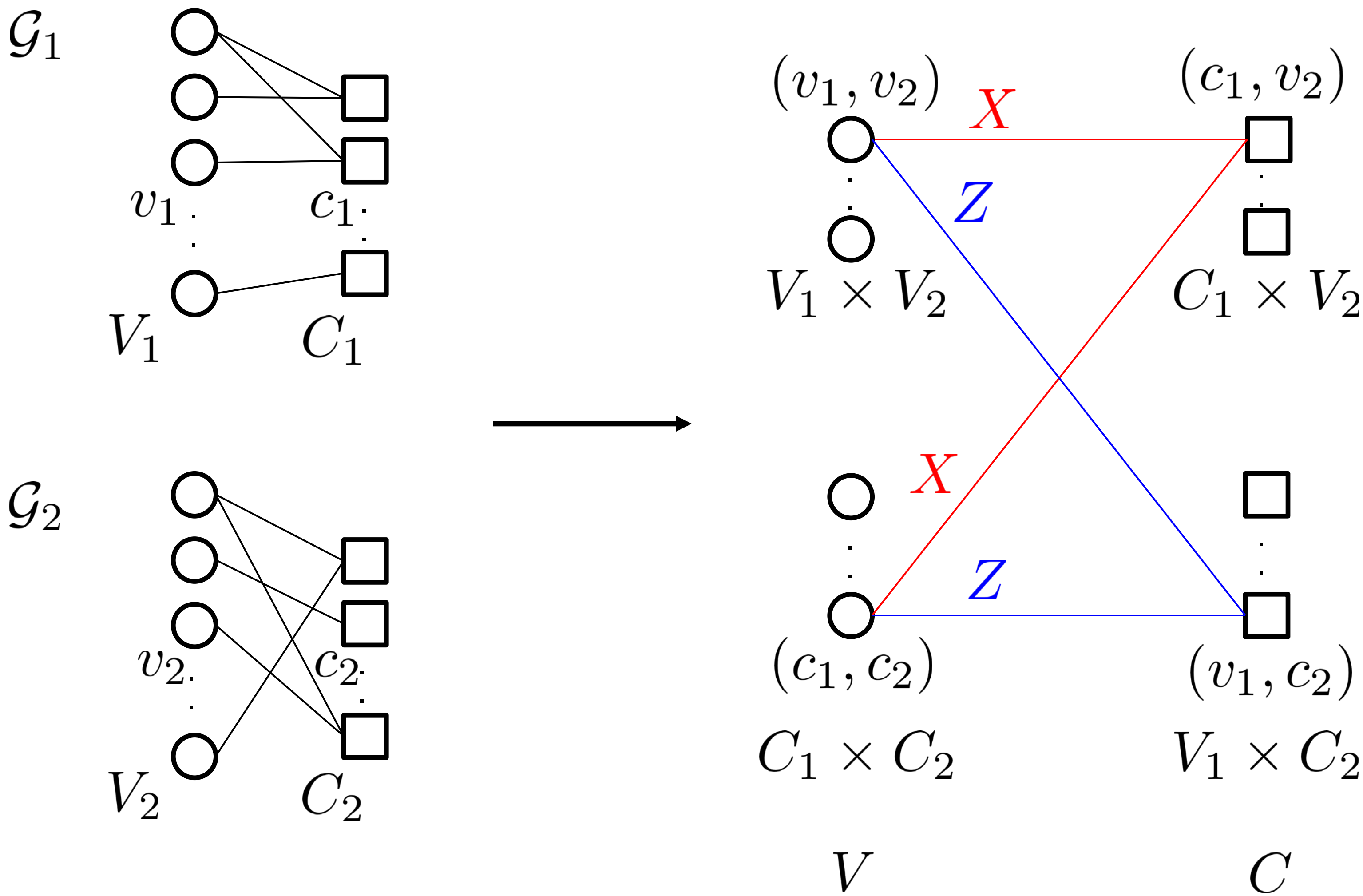
Neighborhood of (5,5)TS is different

- The blue trapping set is more harmful - both floating point min-sum and BP fail on it.

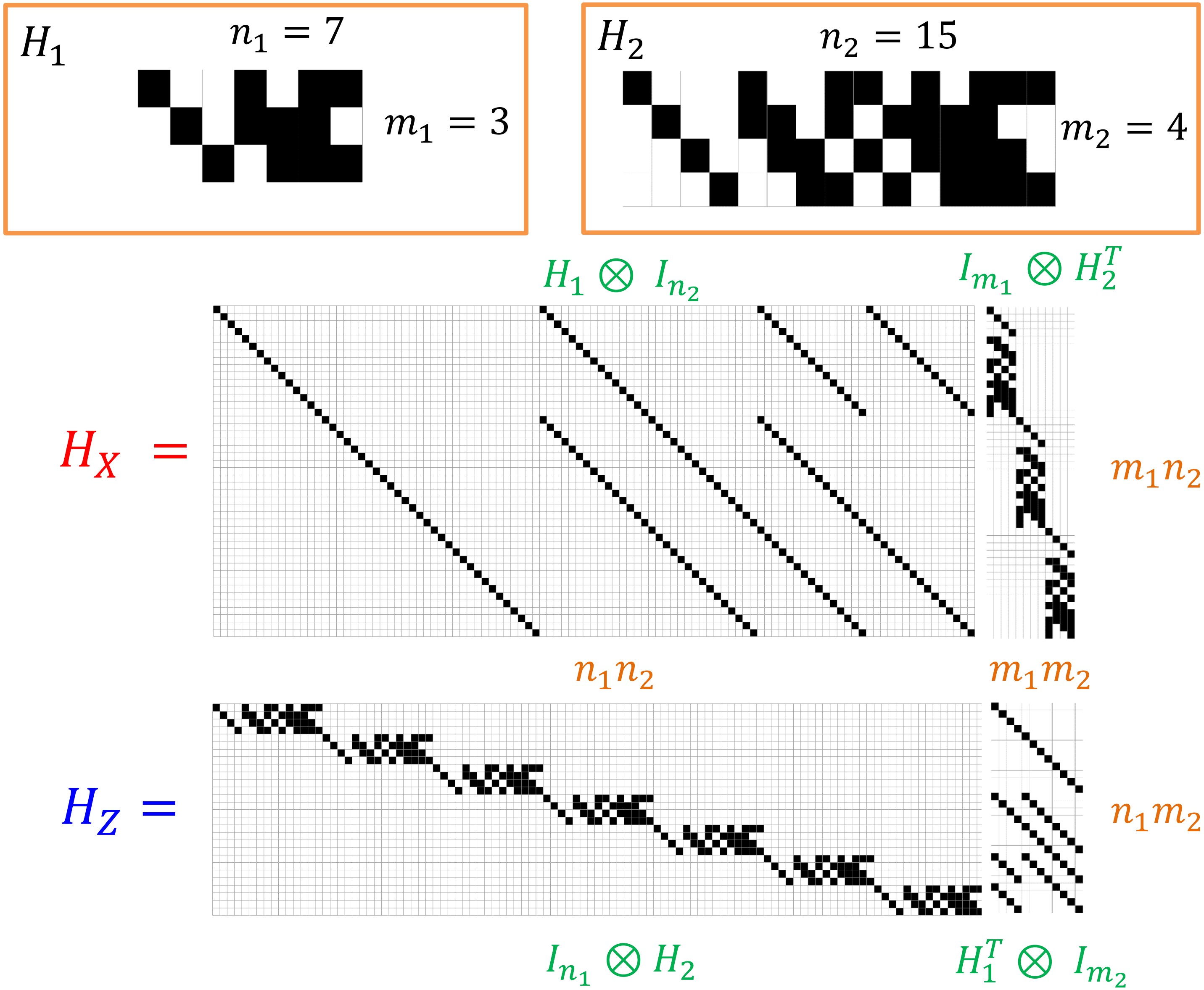




Hyper-graph product (HP) codes



HP code with two classical Hamming codes



Recent example

- Constant rate (4,7) QLDPC code family from (3,4)-LDPC codes

$\mathcal{C}_H$	$\mathcal{C}_H^T$	$\mathcal{HGP}(\mathcal{C}_H)$	$R = k/n$	$\bar{w}$
$[16, 4, 6]$	$[12, 0, \infty]$	$[[400, 16, 6]]$	0.04	7.0
$[20, 5, 8]$	$[15, 0, \infty]$	$[[625, 25, 8]]$	0.04	7.0
$[24, 6, 10]$	$[18, 0, \infty]$	$[[900, 36, 10]]$	0.04	7.0

Roffe et al. May 2020. arXiv:2005.07016 [quant-ph]

Symmetric HP codes

- When C_{H_1} and C_{H_2} are the same - C_H with parameters $[n, k, d]$, the result is a symmetric HP code
  - $H_X = [\, H \otimes \mathrm{I}_n \;\; | \;\; \mathrm{I}_m \otimes H^T]$
  - $H_Z = [\, \mathrm{I}_n \otimes H \;\; | H^T \otimes \mathrm{I}_m]$
  - H is a classical parity check matrix of size $m \times n$.
- $HP(C_H)$ has code parameters:
  - $[[n^2 + m^2 , \; k^2 + (k^T)^2, \min(d, d^T)]]$
  - k^T and d^T are code parameters of the transpose code C_{H^T}
- Commutativity constraint $H_Z \cdot H_X^T = 0$ is satisfied for all binary parity check matrices, thus any classical code to be converted to a quantum code.

J. Tillich and G. Zemor, IEEE Trans. Inf. Theory, 60, 2, 1193–1202, 2014.