On Fano Decoding of Short-Message Convolutional Codes

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April 1, 2022

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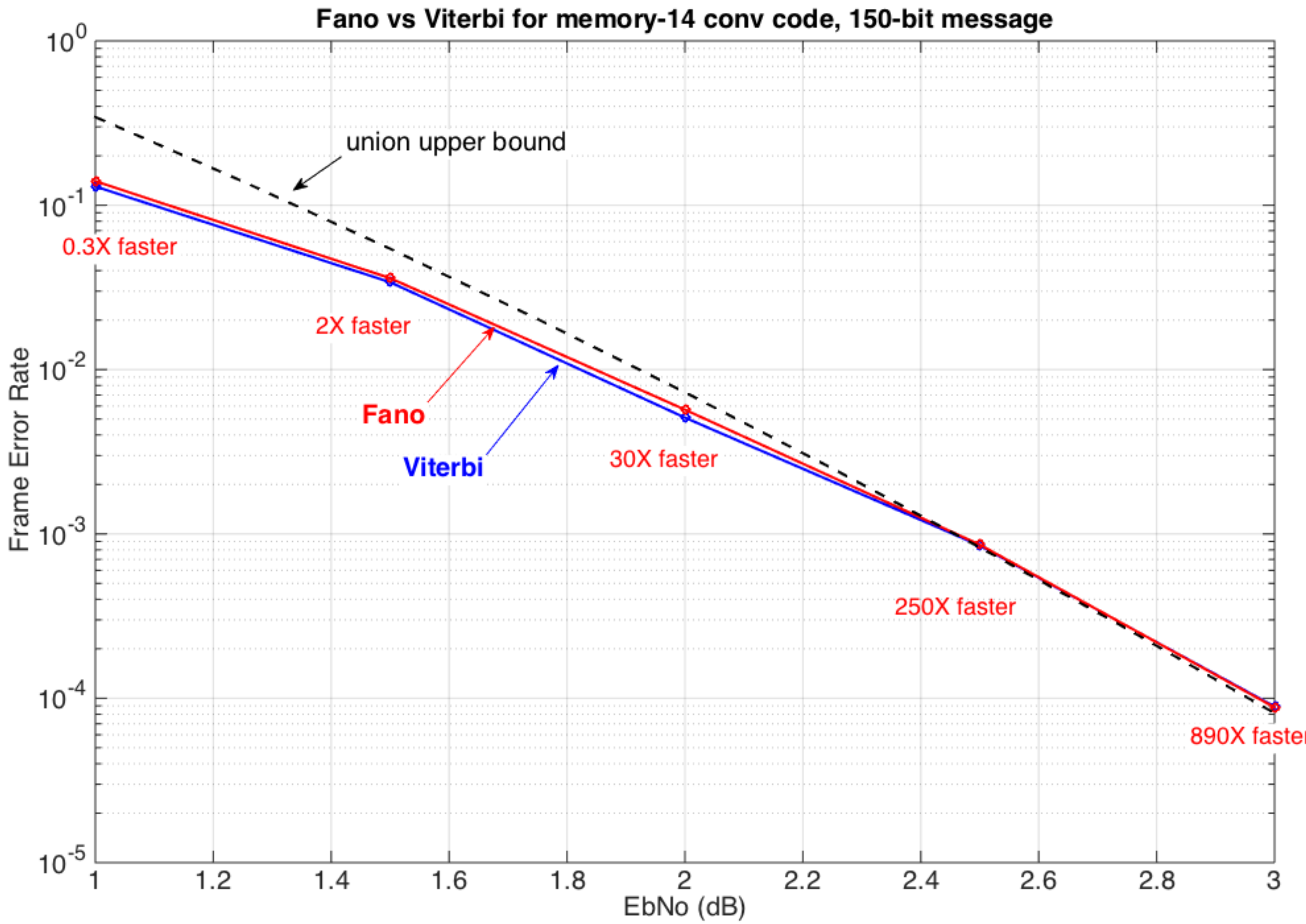
(328, 150) Memory-14, Zero-Terminated Convolutional Code

- Big Viterbi Decoder vs. Fano Decoder: Fano re-sync required for long messages (K. Jordan, 1966)



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ASSUMPTIONS, DEFINITIONS, ACRONYMS

Concerned only with *non-recursive*, *non-systematic* convolutional codes.

v = memory of convolutional code

m = degree of CRC polynomial

(n, k) *code*, k = message block length, n = code block length

$R = k/n$ = code rate

ZTCC = zero-terminated convolutional code
initial and final state are both all-zeros
requires a tail of v zeros after data block

TBCC = tail-biting convolutional code
initial and final state are equal, usually non-zero
requires look-ahead to final v bits in data block

LVA = list Viterbi algorithm

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OUTLINE

1. Notable Recent Results on Short Codes: Rate-1/2, k = 64
2. Unpublished Results on Slightly Longer Codes: Rate ~1/2, k = 150
3. Fano Algorithm Tutorial
4. Wrap-Around Fano Algorithm for TBCCs
5. Toronto Fano Algorithm for TBCCs
6. Open Problems

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Notable Recent Results: *k* = 64 bits

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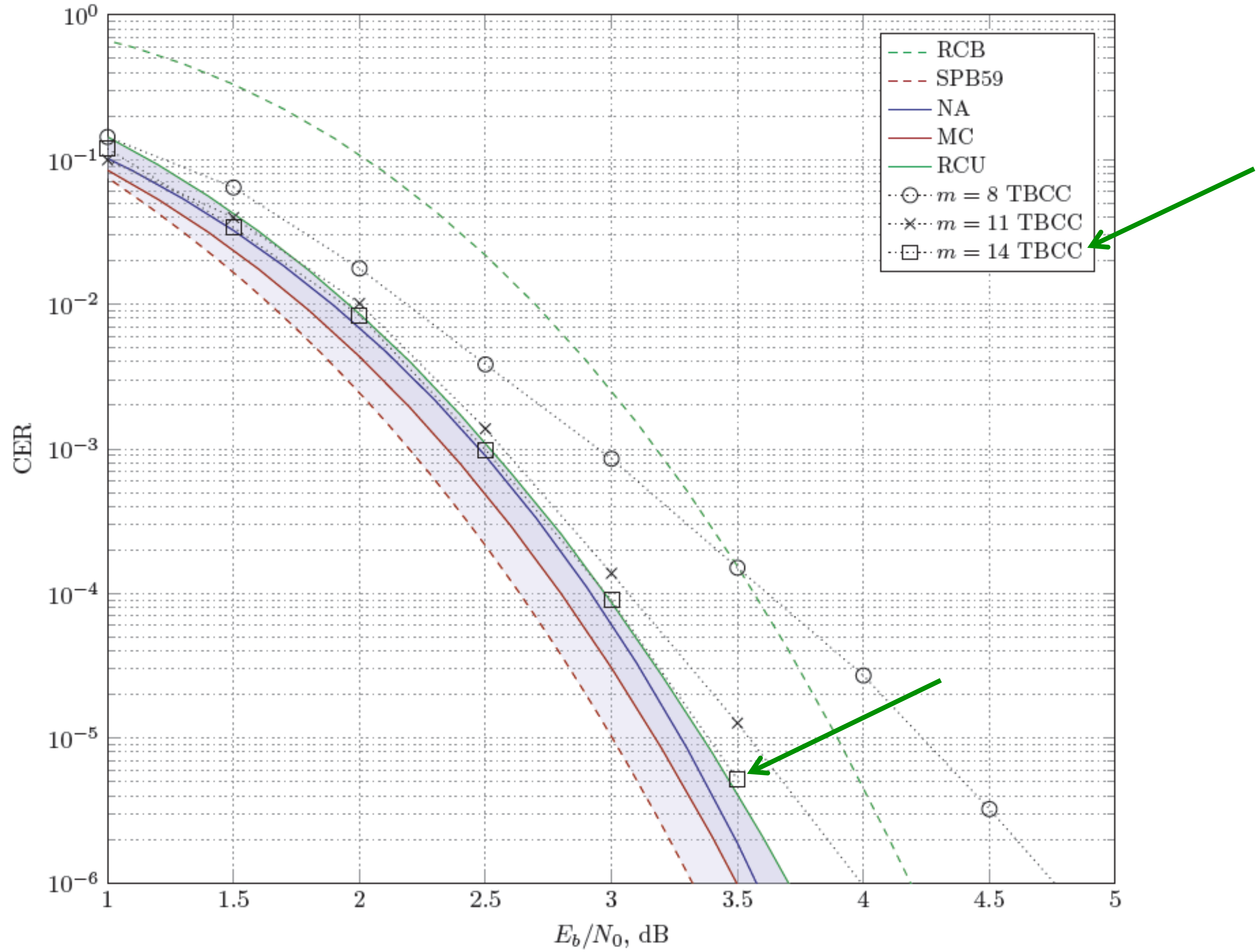
(128, 64) TBCC code

v = 14

WAVA decoder

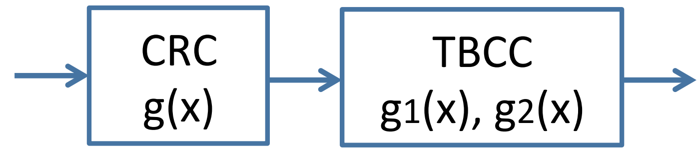
< 0.05 dB gap to RCU at FER = 1e-4



Efficient Error-Correcting Codes in the Short Blocklength Regime

Mustafa Cemil Coşkun[a,c], Giuseppe Durisi[b], Thomas Jerkovits[a], Gianluigi Liva[a], William Ryan[d], Brian Stein[d], Fabian Steiner[c]

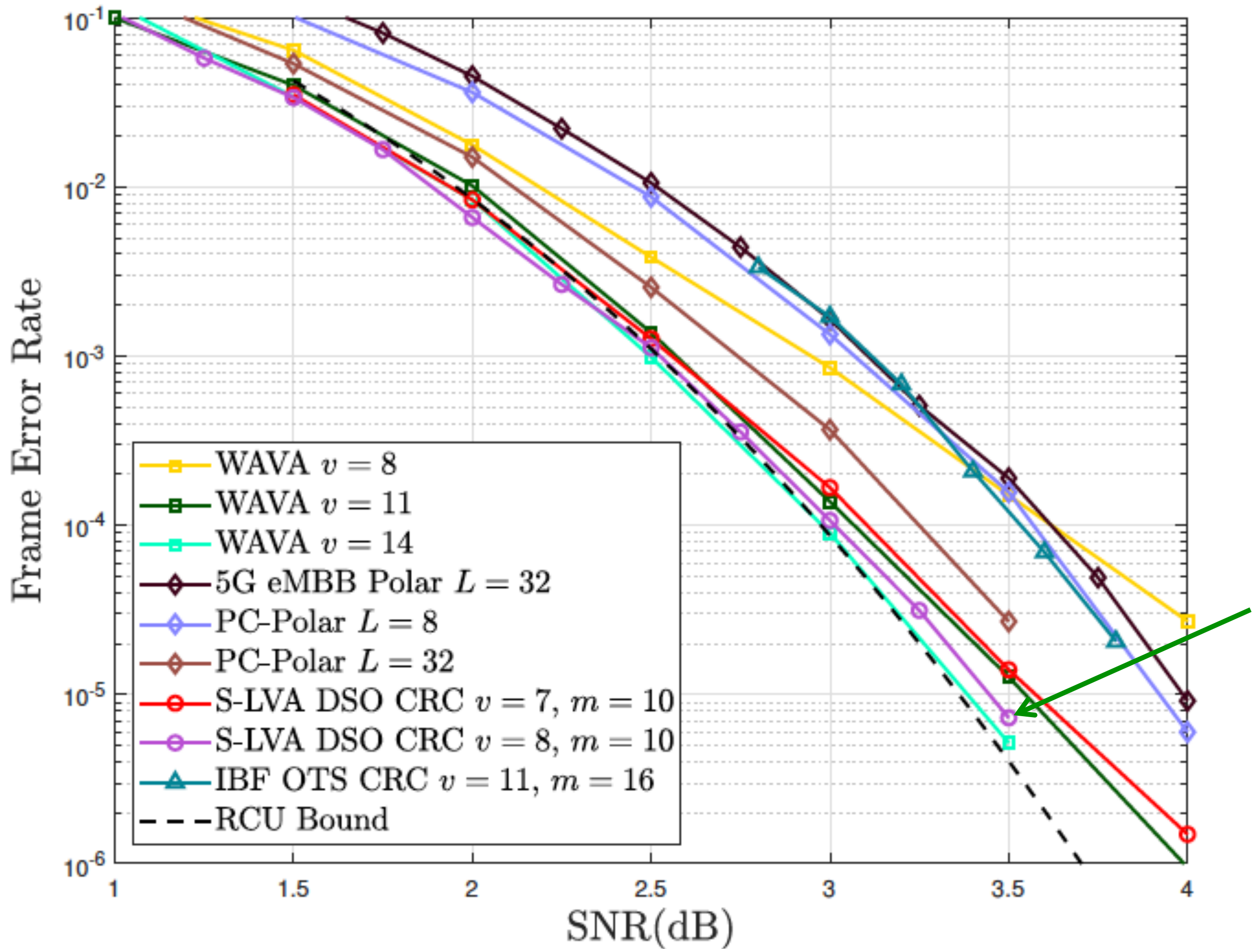
(128, 64) CRC/puncTBCC code



v = 8, CRC10

LVA decoder

< 0.05 dB gap to RCU
at FER = 1e-4.



List-Decoded Tail-Biting Convolutional Codes with Distance-Spectrum Optimal CRCs for 5G

Ethan Liang*, Hengjie Yang*, Dariush Divsalar†, and Richard D. Wesel*

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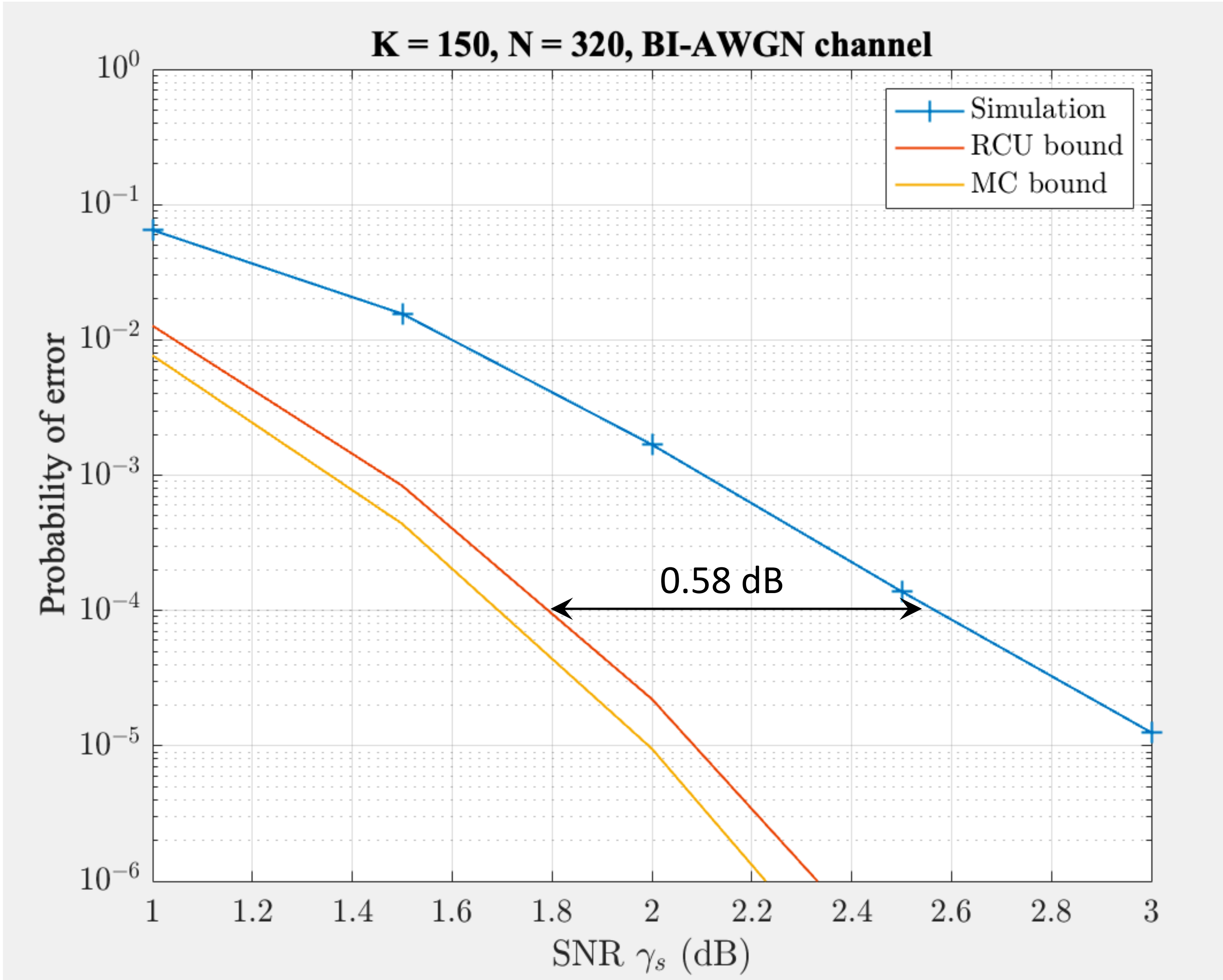
Unpublished Results: k = 150 bits

(320, 150) CRC/TBCC code

v = 8 TBCC, CRC10

LVA decoder

~ 0.6 dB gap to RCU at FER = 1e=3.



H. Yang, R. Wesel, UCLA, Nov. 9, 2021

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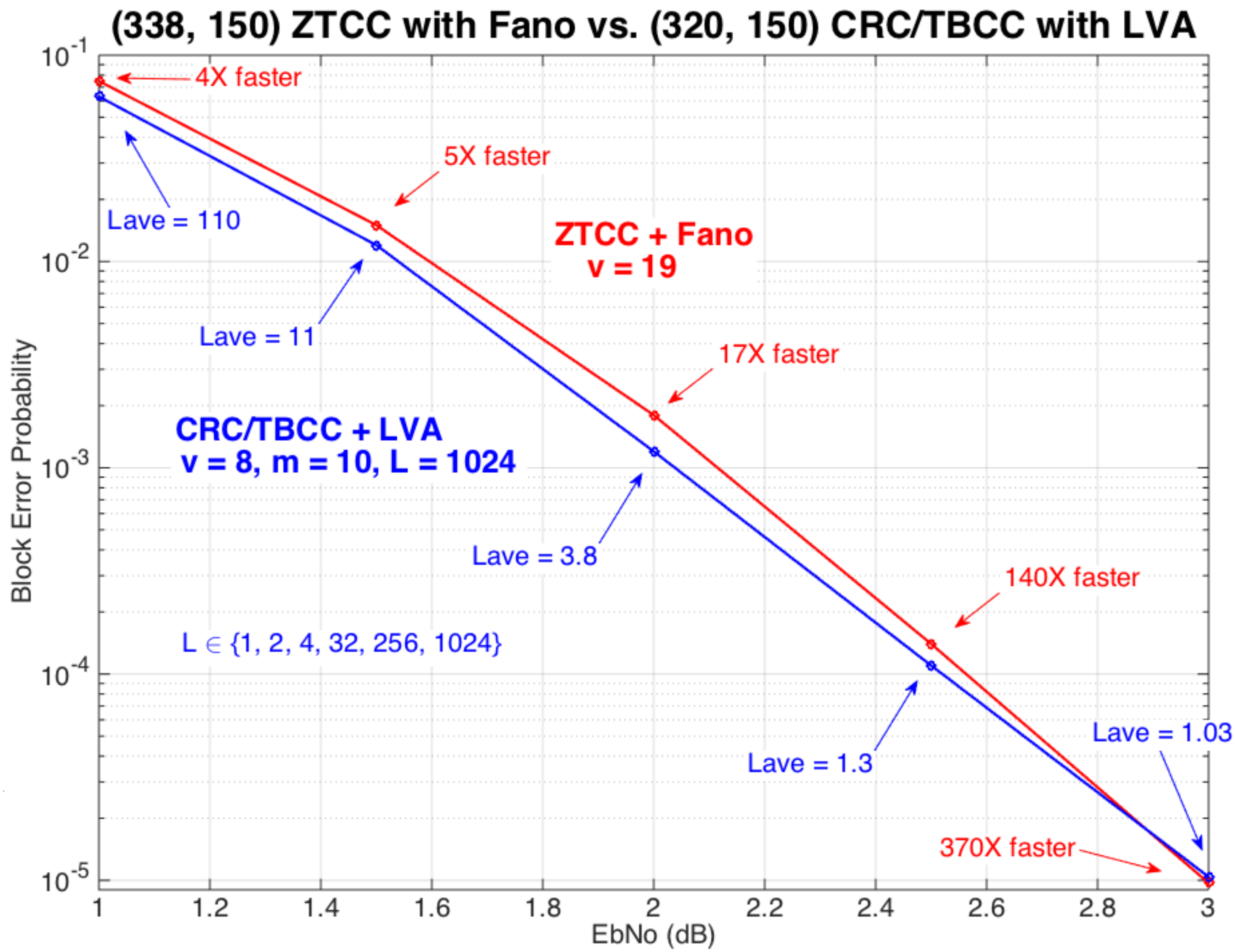
(320, 150) CRC/TBCC code vs. (338, 150) ZTCC

List Viterbi
vs.
Fano

Rates:
0.44 vs. 0.47



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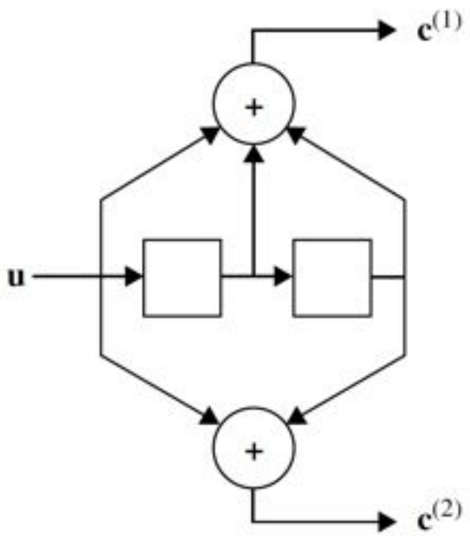
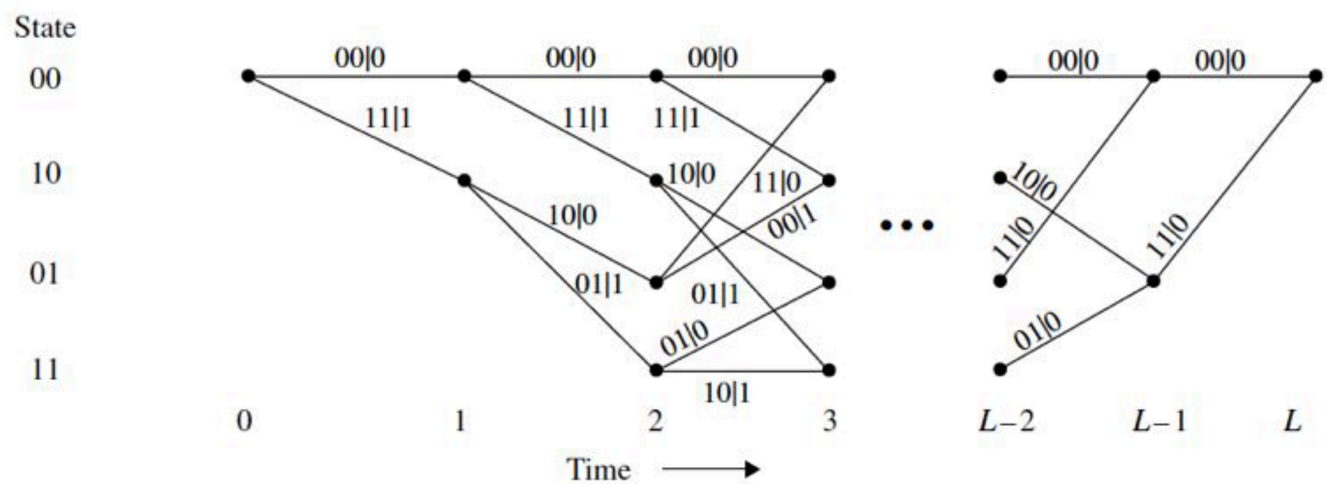
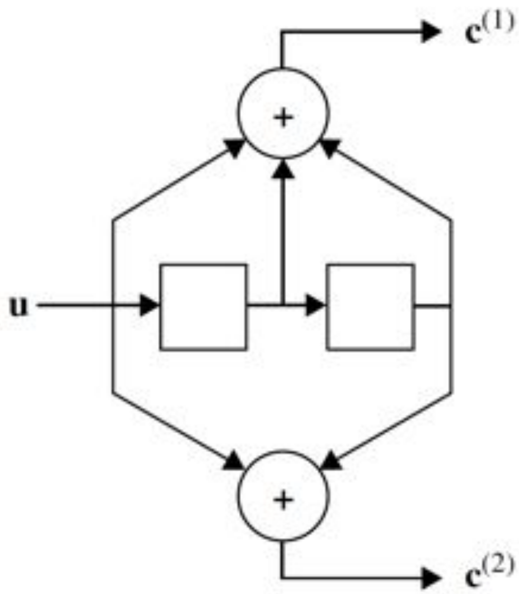
Fano Algorithm Tutorial

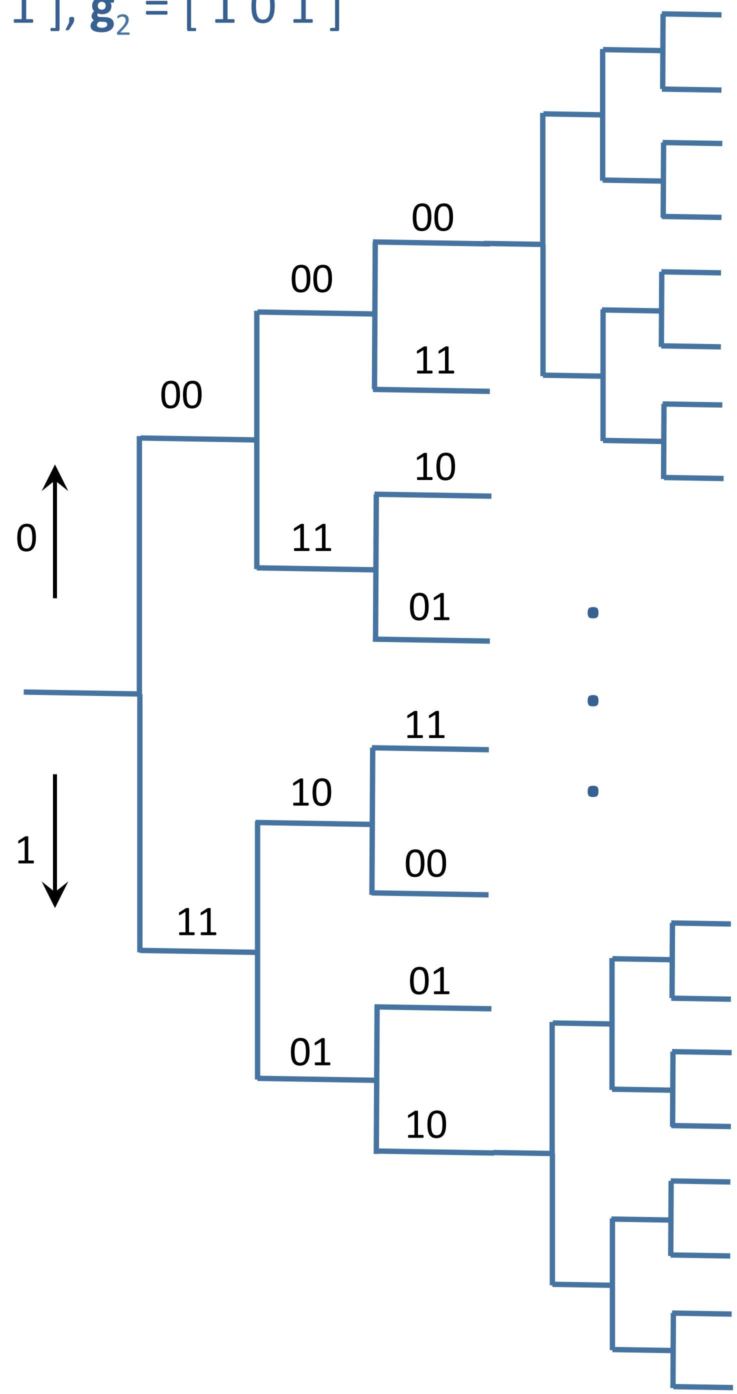
Decoding **trellis** for v = 2 code with $\mathbf{g}_1$ = [1 1 1], $\mathbf{g}_2$ = [1 0 1]





Decoding **tree** for v = 2 code with $\mathbf{g}_1$ = [1 1 1], $\mathbf{g}_2$ = [1 0 1]





Description of Fano Algorithm [based on Lin & Costello]

- Using the *Fano metric**, the decoder steps through the tree forward, and often backward, on the basis of cumulative metric comparison with a dynamically changing threshold.
- If the cumulative metric is greater than the threshold, the decoder steps forward to the next branch; otherwise, the decoder looks back to the previous metric and moves backward if that metric is also less than the threshold.
- If the look-back metric is greater than the threshold, the threshold is decremented and the look-forward sequence begins again, but this time the alternative path is taken.
- Each time a tree node is visited in the forward direction, the threshold is lower than it was on the previous visit. This eliminates the possibility of an infinite loop.
- The path that reaches the end of the tree is the decoded path.

Variable-Length Codes and the Fano Metric

JAMES L. MASSEY, FELLOW, IEEE

* $M_{bit}(k) = \log_2[\, p(R_k/C_k) \,/\, p(R_k)\,] - R = 1 - \log_2[\, 1 + \exp(-4R_kC_k/N_0)\,] - R$

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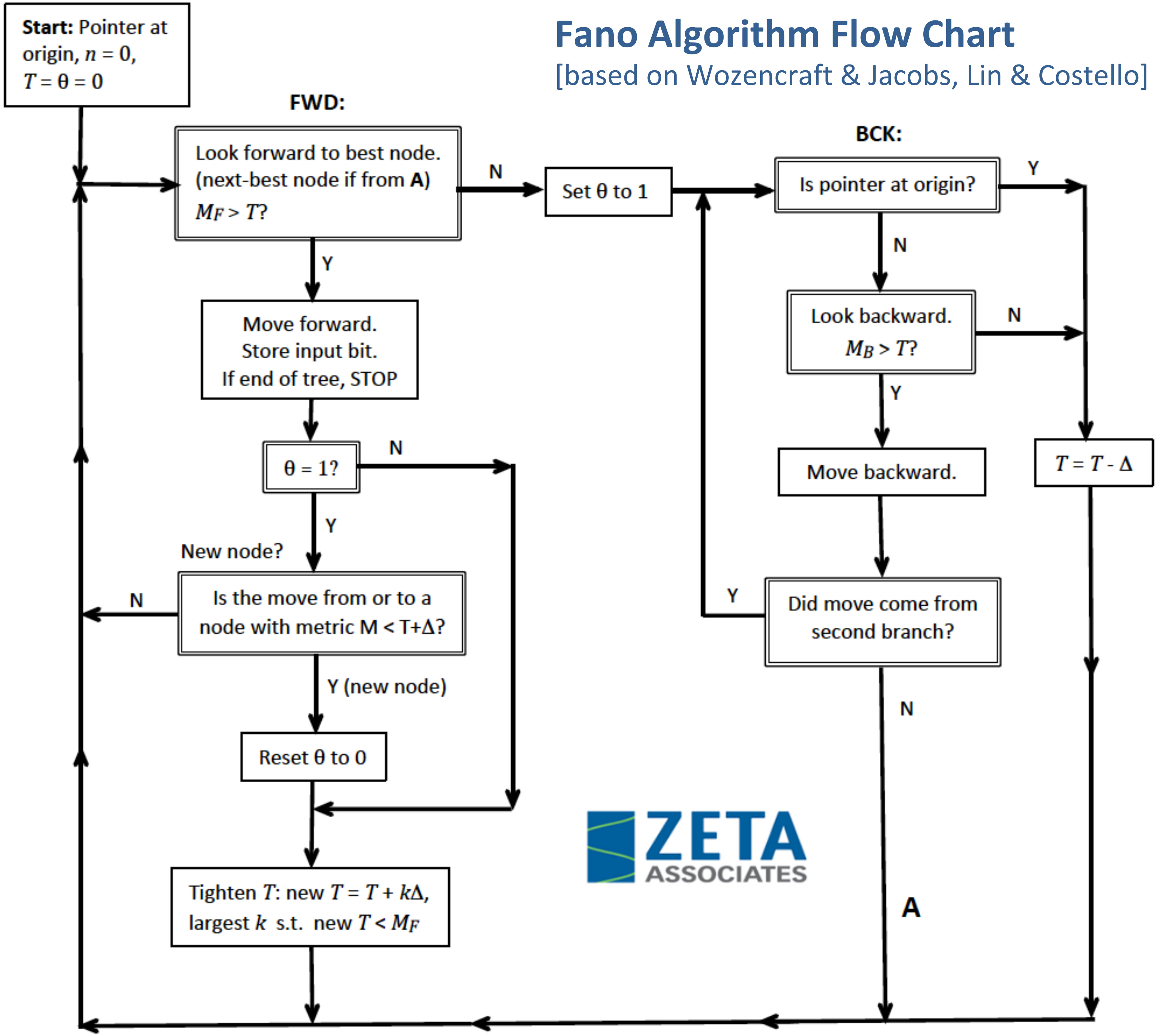
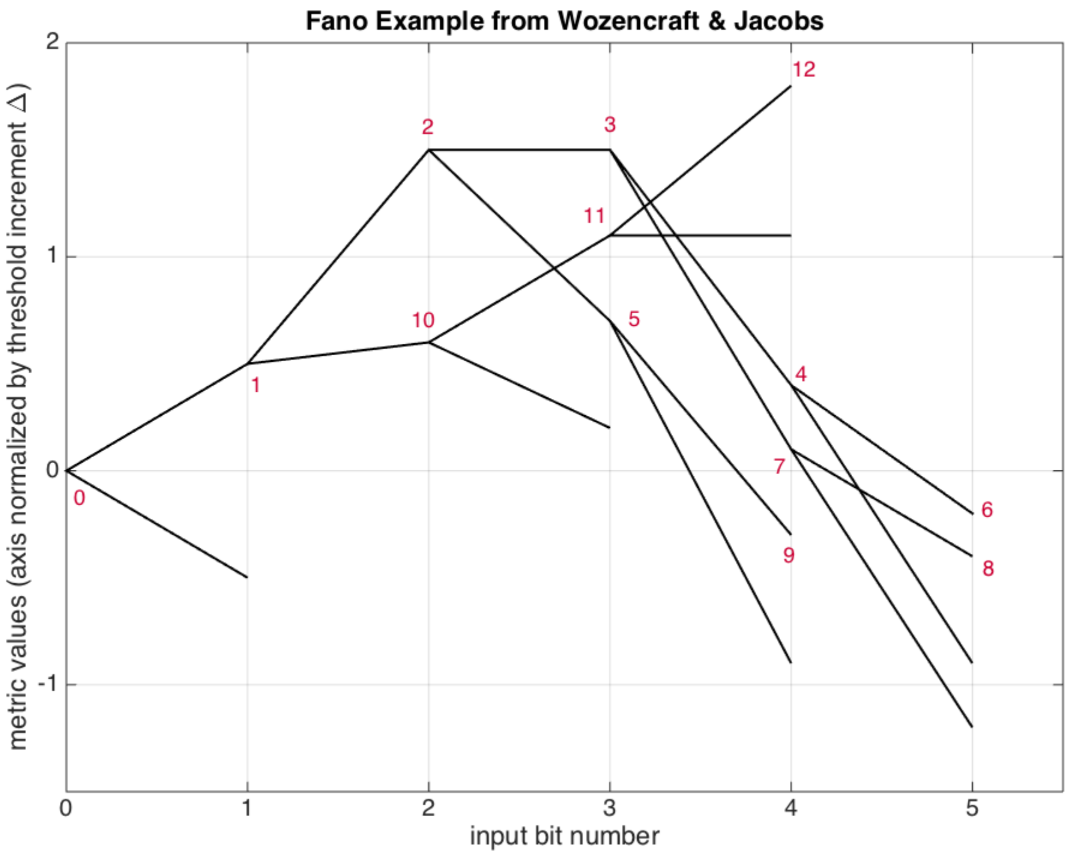
Fano Algorithm Flow Chart

[based on Wozencraft & Jacobs, Lin & Costello]



Fano Example (from Wozencraft & Jacobs)



threshold violation

Pointer at Node	Running Threshold	θ	Action ($\times$ indicates "set $\theta = 1$")		
0	0	0	look at 1	move to 1	
1	0	0	look at 2	move to 2	set $T = \Delta$
2	Δ	0	look at 3	move to 3	
3	Δ	0	look at 4	$\times$ look at 2	move to 2
2	Δ	1	look at 5	$\times$ look at 1	set $T = 0$
2	0	1	look at 3	move to 3	
3	0	1	look at 4	move to 4	set $\theta = 0$
4	0	0	look at 6	$\times$ look at 3	move to 3
3	0	1	look at 7	move to 7	set $\theta = 0$
7	0	0	look at 8	$\times$ look at 3	move to 3
3	0	1	look at 2	move to 2	
2	0	1	look at 5	move to 5	set $\theta = 0$
5	0	0	look at 9	$\times$ look at 2	move to 2
2	0	1	look at 1	move to 1	
1	0	1	look at 10	move to 10	set $\theta = 0$
10	0	0	look at 11	move to 11	set $T = \Delta$
11	Δ	0	look at 12	move to 12	

Fano vs. Viterbi

(300+2v, 150) ZTCC

Fano is much faster, speed ratio not shown.

FER ratio ↓

Memory v FER ~ 1e-4	Fano/Viterbi Ratio
v = 6	1.27
v = 8	1.03
v = 11	1.03
v = 12	1.04
v = 14	1.03

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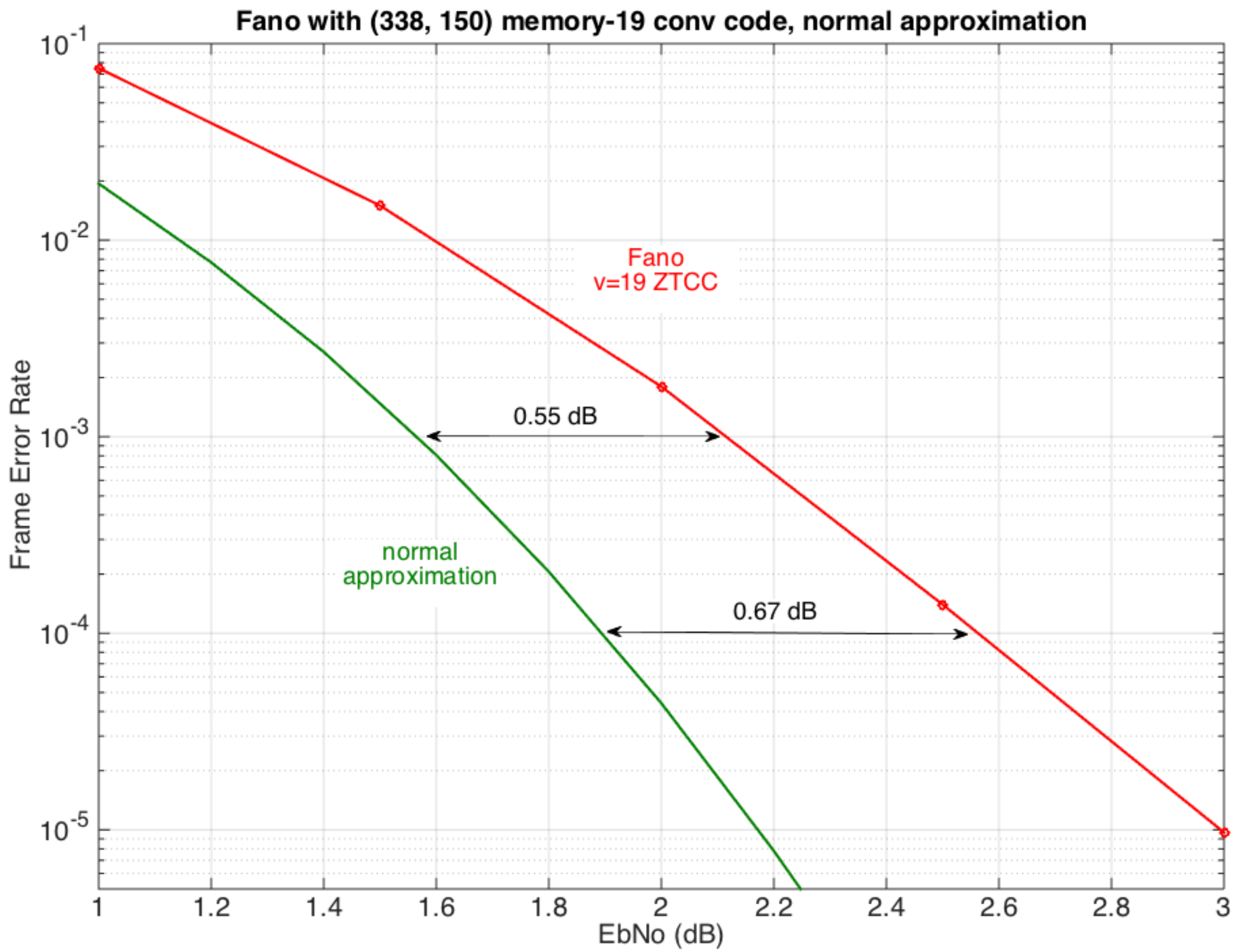
Memory-19 (338, 150) ZTCC

- A tail-biting conv code with ~ maximum-likelihood decoding would reduce this gap.
- Code-rate loss for using ZTCC: $10 \log_{10}(338/300) = 0.52$ dB



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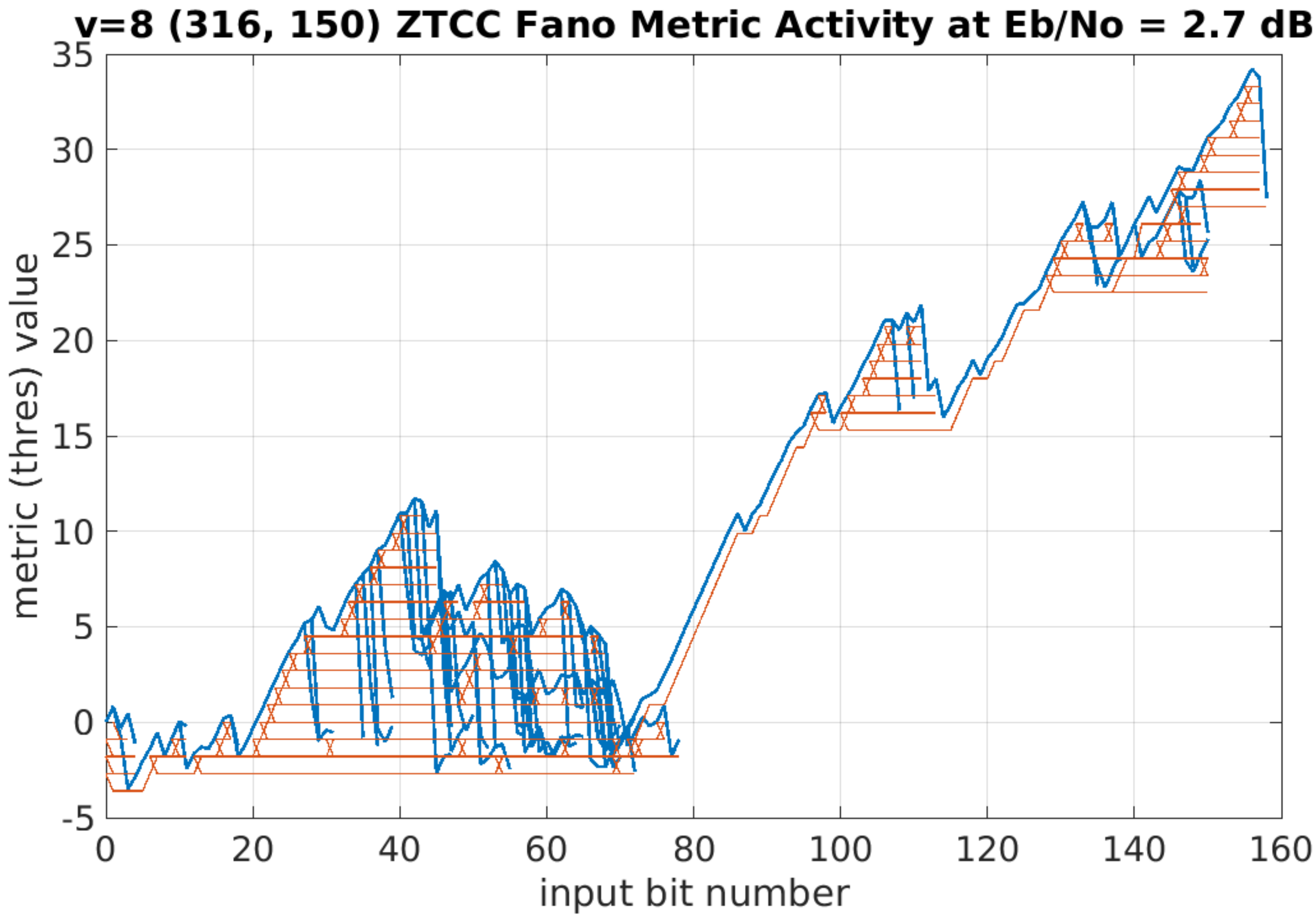
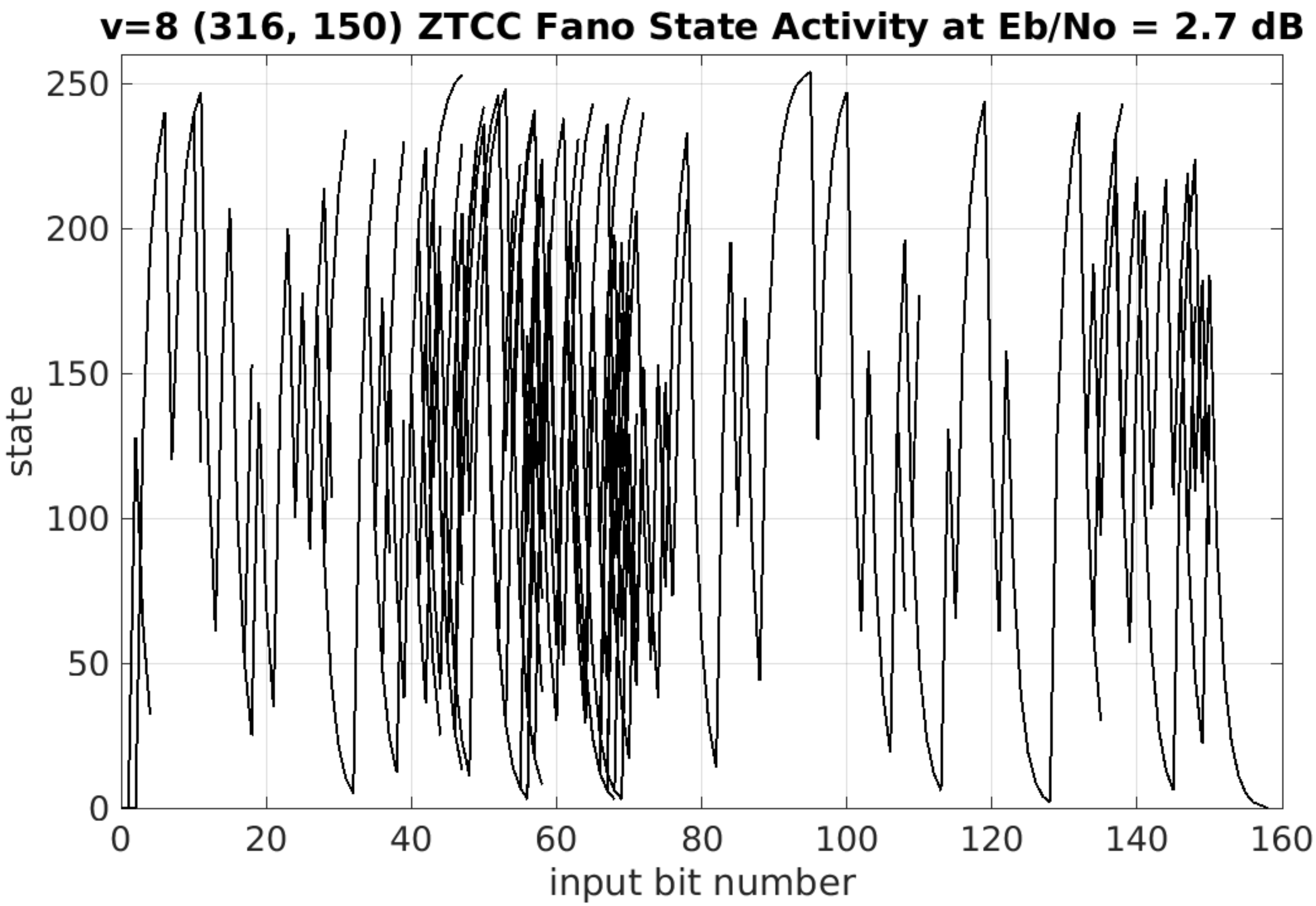
Memory-8 ZTCC

- Metric and dynamic threshold trajectories (Δ = 0.9)



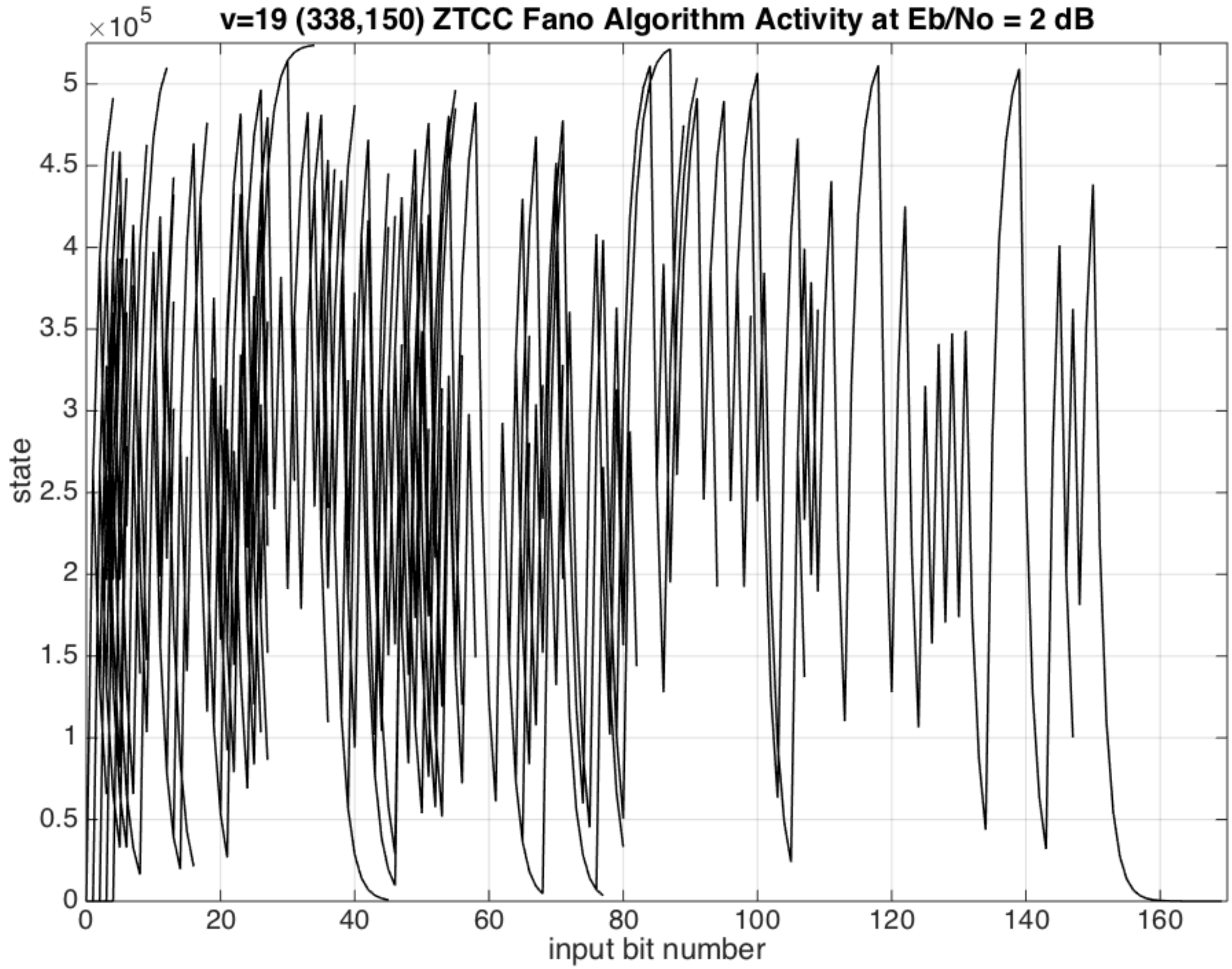
Memory-8 ZTCC

- Trellis paths explored at 2.7 dB



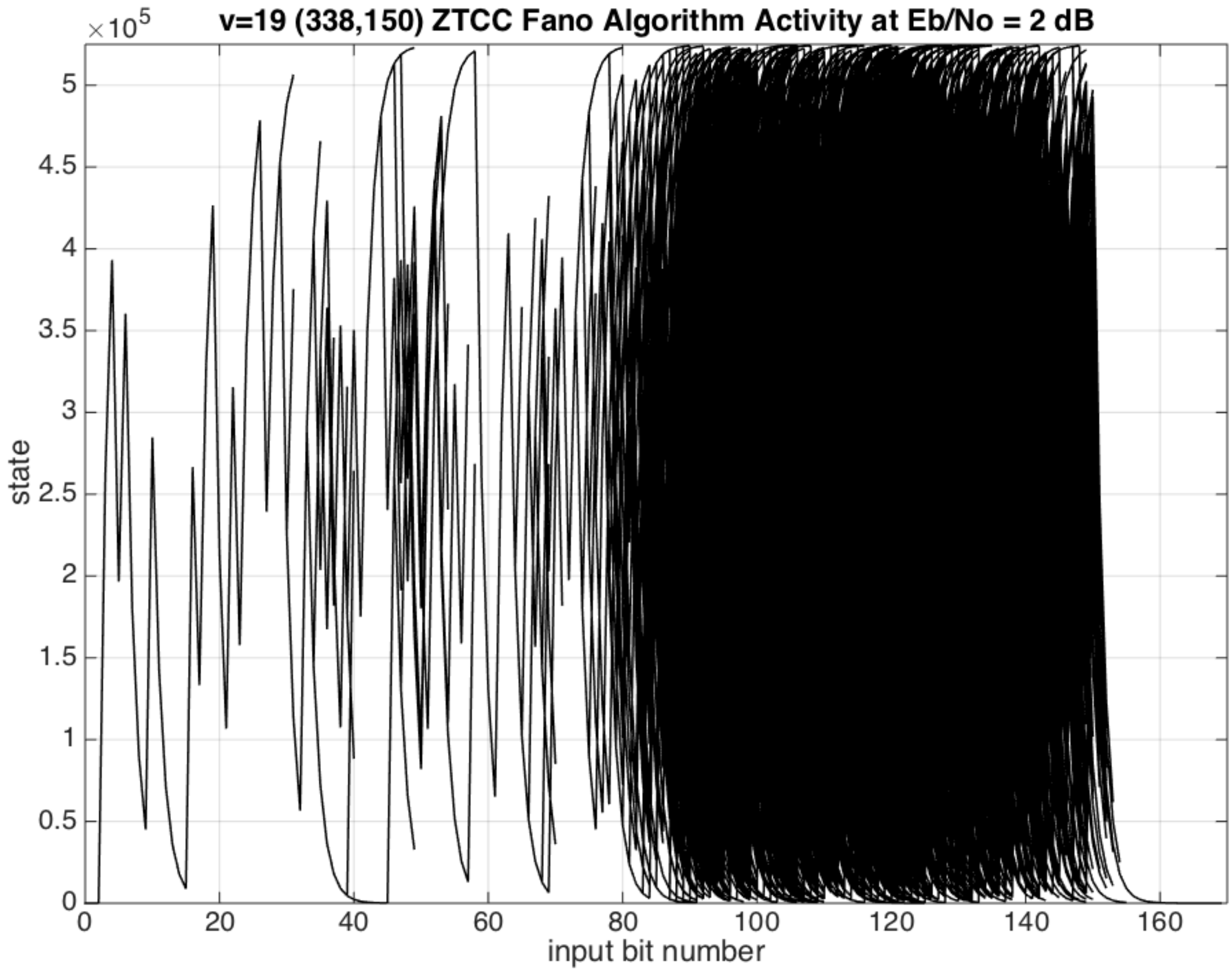
Memory-19 ZTCC

- Trellis paths explored at 2 dB



Memory-19 ZTCC

- Trellis paths explored at 2 dB, different codeword and noise vectors.
- Decoding complexity follows Pareto distribution (heavy tail)



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Comment on Catastrophic ZTCCs

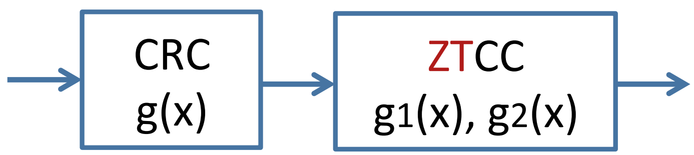
- Simulated *Fano* algorithm on ZTCC with polynomials

  $g_1'(x) = g(x)\, g_1(x)$
  $g_2'(x) = g(x)\, g_2(x)$

- Observed catastrophic error propagation.

- This is not the case with the *Viterbi* algorithm for such ZTCCs.

- Note this models the CRC/ZTCC combination,
  but *not* the CRC/TBCC case we work with.



Wrap-Around Fano Algorithm for TBCCs

Wrap-Around Viterbi Algorithm (WAVA)

Two Decoding Algorithms for Tailbiting Codes

Rose Y. Shao, *Member, IEEE*, Shu Lin, *Life Fellow, IEEE*, and Marc P. C. Fossorier, *Senior Member, IEEE*

Wrap-Around BCJR Algorithm (WABA)

Tailbiting MAP Decoders

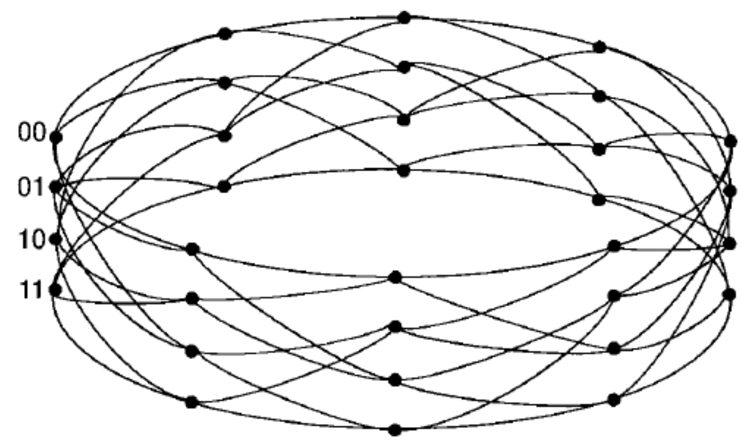
John B. Anderson, *Fellow, IEEE*, and Stephen M. Hladik, *Member, IEEE*

WAVA

1. Initialize state metrics to zero.

2. After each pass through (around) the trellis, continue with *new initial metrics set to final metrics*.

3. Choose as decision *the tail-biting path* with the maximum cumulative metric.



(from Anderson)

Wrap-Around Fano Algorithm (WAFA)

- Start at state 0
- Run through two periods of the tree
- Final state after two passes through tree becomes initial state for third and final pass through tree.

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WAFA vs. WAVA

(300, 150) TBCC

FER ratio

Memory v FER ~ 1e-4	Fano/Viterbi Ratio
v = 6	1.23
v = 8	1.06
v = 11	1.59
v = 12	2.00
v = 14	5.80

- Near-Viterbi performance for $v \leq 11$, but not for $v > 11$.
- WAFA has to find correct state, WAVA only has to get distribution favorable to correct state.
- For v=6 case, WAFA is ~10X slower than WAVA.

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Toronto Fano Algorithm for TBCCs

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Toronto Fano Algorithm

Sequential Decoding of Short Length Binary Codes: Performance Versus Complexity

Bo Lian and Frank R. Kschischang, *Fellow, IEEE*

From personal communication with Bo Lian:

1. Start with generator matrix $\mathbf{G}_{tb}$ for TBCC
2. create upper triangular echelon form of $\mathbf{G}_{tb}$, call it $\mathbf{G}_e$
3. obtain decoding tree for $\mathbf{G}_e$
4. use this tree in *stack-algorithm* sequential decoder

Below, the *Fano algorithm* is used.

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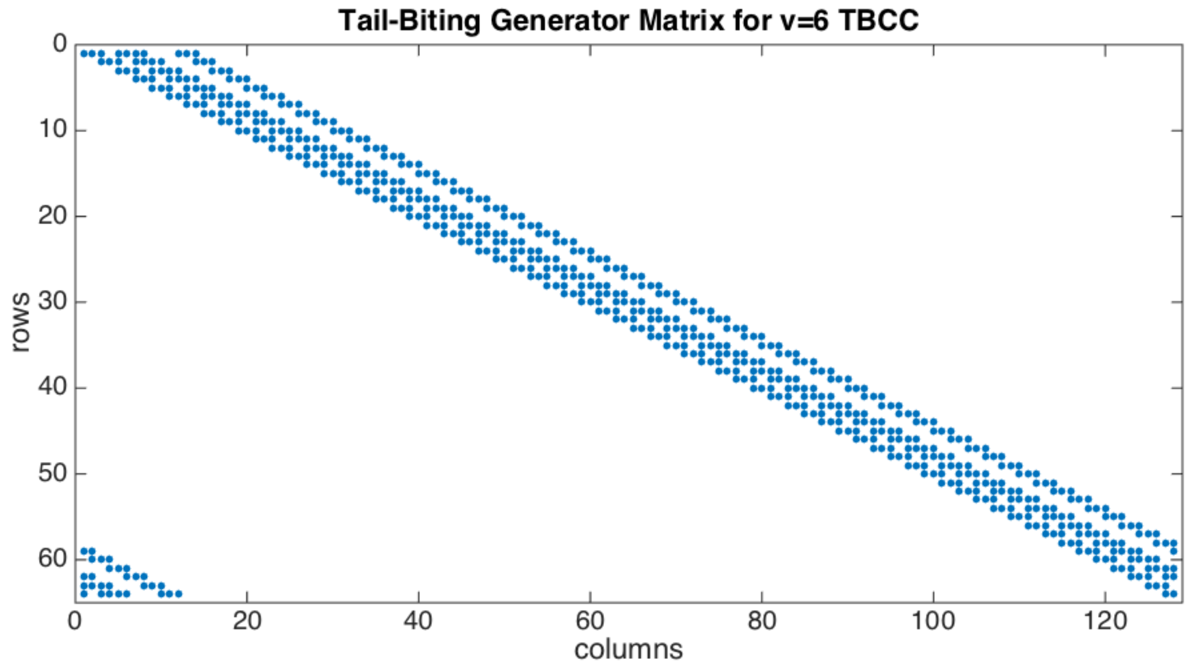
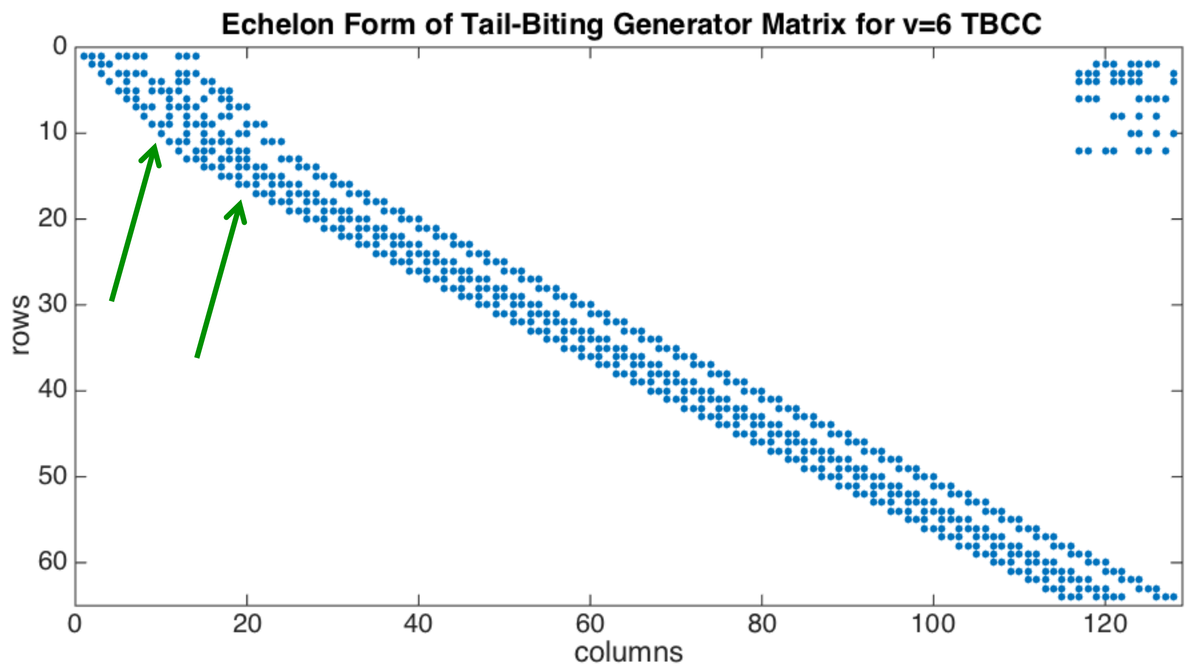
Memory-6 Generator Matrices for (128, 64) TBCC

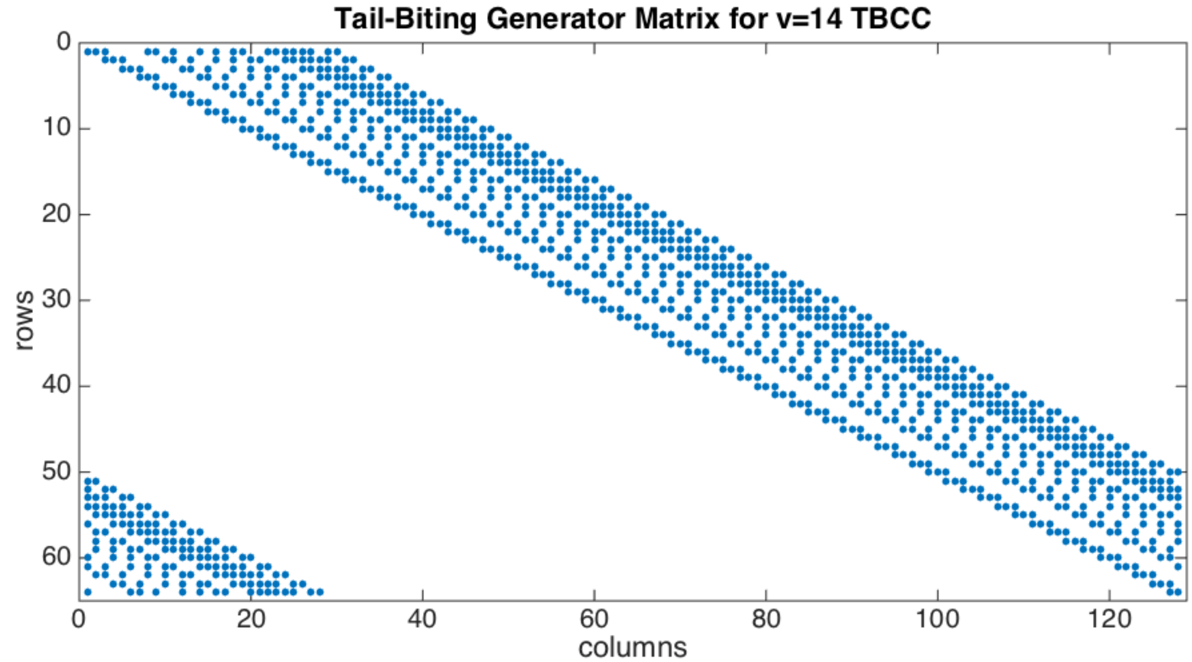
u_1
u_2
u_3
.
.
.
u_{64}

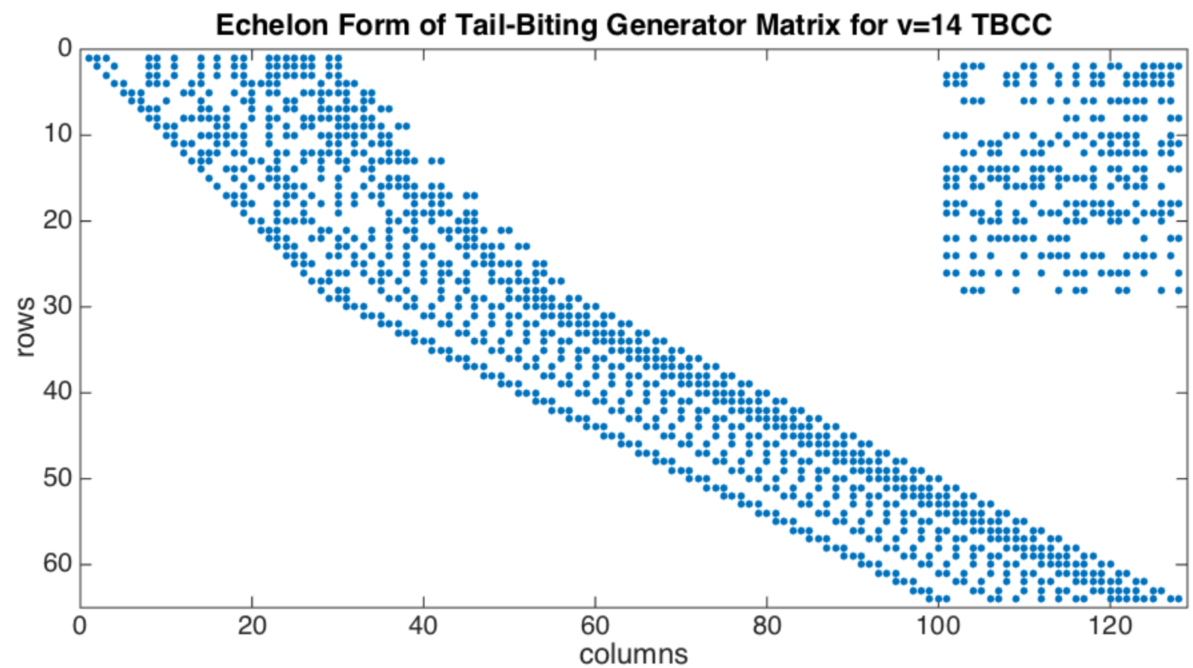


u_1
u_2
u_3
.
.
.
u_{64}



Memory-14 Generator Matrices for (128, 64) TBCC





Toronto Fano Performance for (128, 64) TBCC

Memory Size	Eb/No	TorFano/WAVA (FER)	TorFano/WAVA (speed)
v = 6	4.2 dB	1.3	0.2
v = 14	2.5 dB	10	0.1

- Lian/Kschischang paper proposes an alternative metric to speed up decoding:

  the improved variable bias-term (iVBT) metric

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Open Problems

PROBLEMS TO INVESTIGATE

1. Examine iVBT metric further.

2. Fano decoder for large-memory TBCC (stable, fast).

3. Fano decoder for catastrophic ZTCC (= CRC+ZTCC) and CRC+TBCC.

4. Use stack decoder in CRC/conv list decoding.

5. Fano decoder for cyclic and general linear block codes.

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MORE PROBLEMS TO INVESTIGATE

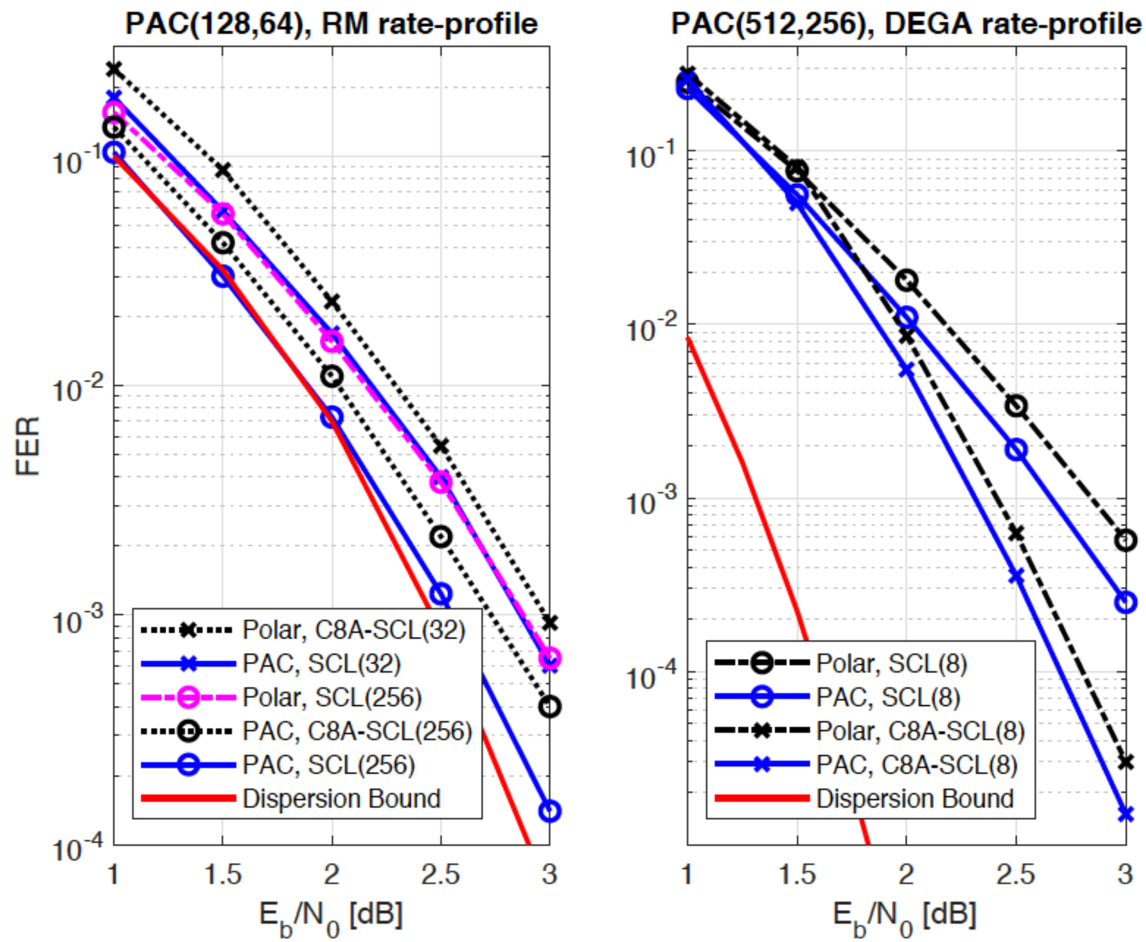

Figure 10: Performance of PAC codes under list decoding

Polarization-adjusted Convolutional (PAC) Codes: Sequential Decoding vs List Decoding

Mohammad Rowshan, *Student Member, IEEE*, Andreas Burg, *Member, IEEE*, Emanuele Viterbo, *Fellow, IEEE*