Spectral Sparsification (User guide)

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This note provides a user guide to spectral sparsification procedure. Spectral sparsification produces spectral sparsifiers of (undirected) weighted graphs developed by Dan Spielman's group at Yale University (Spielman and Srivastava, 2011). We have applied this procedure in the field of fluid mechanics to sparsify vortical interactions and develop a sparsifed-dynamics model (Nair and Taira, 2015).

We introduce the notion of a network/graph in §1 and describe briefly the spectral sparsification procedure in §2. The MATLAB function called `spectral_sparsity.m` performs this sparsification procedure. The approach is demonstrated with the help of an example problem described in §3. `example.m` script executes the example problem. To refer to this code, please cite the references mentioned in this tutorial.

1 Network/Graph

A network is defined by a collection of vertices joined by edges. Any (undirected) graph $\mathcal{G}$ can be described by a set of vertices $V = \{v_1, v_2, ...v_N\}$, a set of edges E, and a set of weights w associated with the edges, i.e., $\mathcal{G} = \{V, E, w\}$. Network connections can be summarized by its adjacency matrix $\boldsymbol{A}_{\mathcal{G}} \in \mathbb{R}^{N\times N}$, which is given by

$$[\boldsymbol{A}_{\mathcal{G}}]_{ij} = \begin{cases} w_{ij} & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

2 Spectral sparsification

Sparsification involves the creation of a sparse graph $\mathcal{G}_S$ from the original graph $\mathcal{G}$ based on an approximation order of ϵ. The approximation with $\epsilon = 0$ indicates that $\mathcal{G}_S$ is same as original graph $\mathcal{G}$ and none of the edges are sparsified, while $\epsilon = 1$ yields heavily sparsified graphs.

- Leads to reduction in the number of edges from N^2 to $\mathcal{O}(N \log(N)/\epsilon^2)$ for large N.
- Maintains spectral similarity between original ($\mathbf{A}_{\mathcal{G}}$) and sparse ($\mathbf{A}_{\mathcal{G}_\mathbf{S}}$) configurations.
- Leads to redistribution of edge weights to preserve node strengths.

$\mathbf{A}_{\mathcal{G}_\mathbf{S}}$ =`spectral_sparsity`($\mathbf{A}_{\mathcal{G}}, \epsilon$) takes adjacency matrix $\mathbf{A}_{\mathcal{G}}$ and approximation order ϵ as input and provides sparsified adjacency matrix $\mathbf{A}_{\mathcal{G}_\mathbf{S}}$ as output.

Remarks:

- The adjacency matrix should be symmetric as obtained from an undirected graph. If the matrix provided is not symmetric, we symmetrize the matrix in the code by performing $(\boldsymbol{A}_{\mathcal{G}} + \boldsymbol{A}_{\mathcal{G}}^{\mathrm{T}})/2$, where $^{\mathrm{T}}$ indicates transpose. Also, the edge weights should be strictly positive.
- The graph cannot have any self-loops, i.e., the diagonal elements of the adjacency matrix should be zero, $[\boldsymbol{A}_{\mathcal{G}}]_{ii} = 0$.

3 Example

For the example, we consider $N = 40$ nodes equally spaced along the circumference of a unit circle. Let the position of the nodes be $r_i = (x_i, y_i)$. The adjacency matrix weights are chosen as follows,

$$[\boldsymbol{A}_{\mathcal{G}}]_{ij} = \begin{cases} 1/|r_i - r_j| & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Using inverse of the distance between the nodes as edge weights for computing the adjacency matrix $\mathbf{A}_{\mathcal{G}}$, we construct an original graph as shown in Figure 1 (top left). The sparsity plot (top right) indicates that as the distance between the nodes decreases, the edge weights increases. Spectral sparsification is performed on the original graph with approximation order $\epsilon = 1$ resulting in sparse graph $\mathbf{A}_{\mathcal{G}_\mathbf{S}}$ with reduced number of edges shown in Figure 1 (bottom left). The white spaces in the sparsity plot indicates absence of edge connectivity. In spite of the reduction in edge connectivity due to spectral sparsification as observed in the sparsity plot shown in Figure 1 (bottom right), we maintain spectral similarity between the original and sparse graphs. We show the comparison in eigenvalue spectra of the adjacency matrix (λ) for original graph and sparse graph with $\epsilon = 1$ in Figure 2.

References

SPIELMAN, D. A. & SRIVASTAVA, N. 2011 Graph sparsification by effective resistances. SIAM J. Sci. Comput. 40 (6), 1913-1926.

NAIR, A. G. & TAIRA, K. 2015 Network-theoretic approach to sparsified discrete vortex dynamics. J. Fluid Mech. 768, 549-571.

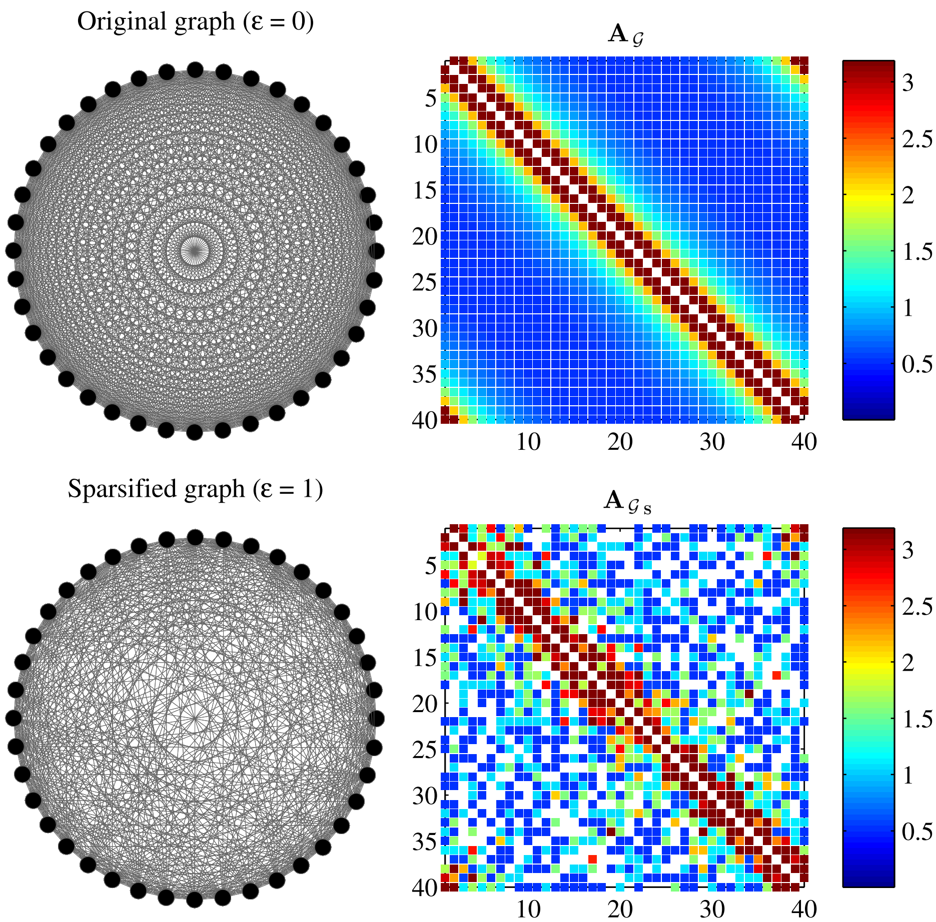

Figure 1: (Top row) Original graph and sparsity plot of adjacency matrix $\mathbf{A}_{\mathcal{G}}$. (Bottom row) Sparse graph and sparsity plot of sparsified adjacency matrix $\mathbf{A}_{\mathcal{G}_\mathrm{S}}$. In sparsity plot, white color indicates sparse edges while color indicates edge weights.

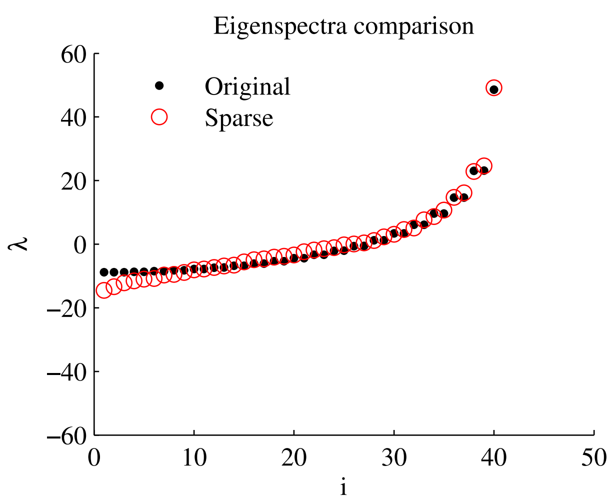

Figure 2: Original graph $\mathbf{A}_{\mathcal{G}}$ and sparse graph $\mathbf{A}_{\mathcal{G}_\mathrm{S}}$ spectra.