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Relaxation of Thermal Stress Singularities in Bonded Viscoelastic Quarter Planes

Singular thermal stress fields in bonded viscoelastic quarter planes are studied with the use of the viscoelastic analogy. The order of the singularity is shown to depend on the material properties, indicating that it will vary with time in viscoelastic materials. This is studied in detail for Maxwell materials, and it is shown that the order of the singularity generally increases with time. This evolution of the singularity can, for certain combinations of material properties, lead to initial increases in the stress levels near the edge of the interface before relaxation occurs.

1 Introduction

Engineering components often feature disssimilar materials bonded along planar surfaces. Elastic analyses of such configurations generally exhibit stress singularities of various types depending on the loadings, geometry, and materials. Bogy (1968, 1970), for instance, studied stresses in bonded elastic quarter planes under applied surface tractions and found, under certain conditions, stress singularities of the order $r^{-\lambda}$, where $0 \le \lambda \le 0.41$. Depending on the loadings and the material combinations, Bogy (1970) also identified situations in which the edge singularities either were logarithmic or nonexistent. Two studies (Hein and Erdogan (1971) and Dempsey and Sinclair (1981)) consider bimaterial wedges and show that geometries other than quarter planes can lead to oscillatory stresses near the edge of the interface. In all cases, the singularity depends on the elastic properties of the two materials.

In many applications, including energy conversion and space technology, high temperature operation of bonded structures is necessary, so time-dependent deformation often occurs in at least one of the materials. In other applications, a viscoelastic adhesive is used to bond two elastic materials, thus forming a pair of interfaces in which one material is elastic and the other is viscoelastic. The latter problem has been studied by Delale and Erdogan (1981), who consider a lap joint composed of two plates bonded by a thin viscoelastic adhesive. The results exhibit a redistribution of the peak shear stresses near the edge of the interface, but no singularities are encountered because of the assumptions associated with plate theory.

Because elastic stress singularities in crack-free bonded structures are a function of the material properties, the order of the singularity in viscoelastic materials is time-dependent and the relaxation of the near-edge stress fields depends on its evolution. In this paper, thermal stress singularities in bonded quarter planes are analyzed using the viscoelastic analogy. A rather simple thermal field, consisting of a semicircular disk in which the temperature changes uniformly, is considered in order to allow the study to focus on the effects of viscoelastic behavior on the singular stress field. The elastic thermal stresses are derived, following Bogy, with the use of the Mellin transform and the asymptotic solutions for the near-edge stresses are obtained by application of the residue theorem to the inversion integral. This result is then used as the Laplace transform of the time-dependent viscoelastic solution (with appropriately transformed material properties), whereupon numerical inversion of this transform leads to the desired result for the relaxation of singularities in viscoelastic bonded quarter planes.

This procedure is outlined in Sections 2 and 3, while some examples of viscoelastic relaxation of the singular fields near the edge of the interface are given in Section 4. Conclusions follow in Section 5.

2 Elastic Solution

The model to be studied herein features two quarter planes bonded along one surface, as shown in Fig. 1. The layers are assumed to be in perfect contact, with no defects or cracks anywhere in the structure. Slipping or debonding at the interface is not allowed. The elastic and viscoelastic properties in the lower quarter plane are referred to with a single prime (e.g., μ', ν', and α'), while those of the upper quarter plane are denoted by a double prime (μ'', ν'', and α''). The materials are assumed to be homogeneous, isotropic, and linear viscoelastic. The free surfaces of the model (at $x = 0$) are assumed to be traction-free and the thermal field provides the only loading.

The well-known equations of small strain elasticity are solved by way of the Airy stress function Φ, which is governed by the following equation:

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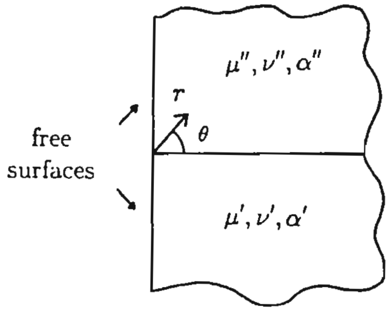

Fig. 1 Model for determination of thermal stresses in bonded quarter planes

$$\nabla^4\Phi + q\alpha E\nabla^2 T = 0, \tag{1}$$

where T is the temperature change from the stress-free reference temperature, α is the thermal expansion coefficient, and

$$q = \begin{cases} 1 & \text{for plane stress} \\ 1/(1-\nu) & \text{for plane strain} \end{cases}. \tag{2}$$

By definition, the stresses are found from the stress function Φ in the following manner:

$$\sigma_{rr} = \frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta}$$
$$\sigma_{\theta\theta} = \Phi_{,rr}$$
$$\sigma_{r\theta} = \frac{1}{r^2}\Phi_{,\theta} - \frac{1}{r}\Phi_{,r\theta} \tag{3}$$

and the displacements can be shown to be given by

$$u_{r,r} = \frac{1}{2\mu}\left[\frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta} - \left(1 - \frac{m}{4}\right)\nabla^2\Phi\right] + n\alpha T$$
$$u_{\theta,r} - \frac{u_\theta}{r} + \frac{1}{r}u_{r,\theta} = \frac{1}{\mu}\left(\frac{1}{r^2}\Phi_{,\theta} - \frac{1}{r}\Phi_{,r\theta}\right), \tag{4}$$

where

$$m = \begin{cases} 4/(1+\nu) & \text{for plane stress} \\ 4(1-\nu) & \text{for plane strain} \end{cases} \tag{5}$$

and

$$n = \begin{cases} 1 & \text{for plane stress} \\ (1+\nu) & \text{for plane strain} \end{cases}. \tag{6}$$

Combining these equations with the traction-free boundary conditions (at $\theta = \pm\pi/2$)

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0, \tag{7}$$

and the interface conditions (at $\theta = 0$)

$$\sigma'_{\theta\theta} = \sigma''_{\theta\theta}$$
$$\sigma'_{r\theta} = \sigma''_{r\theta}$$
$$u'_r = u''_r$$
$$u'_\theta = u''_\theta, \tag{8}$$

one can solve the problem. Solving for the stress function Φ, subject to the appropriate traction or displacement boundary conditions, provides a means for computing the steady-state thermal stresses, strains, and displacements in a planar medium.

The solution of this problem is facilitated by the Mellin transform, defined as

$$\hat{\Phi}(s,\theta) = \int_0^\infty \Phi(r,\theta) r^{s-1} dr, \tag{9}$$

where $\hat{\Phi}$ denotes the Mellin transform of Φ and s is the transform parameter. Similar transforms are defined for the stresses, displacements, and the temperature field, according to:

$$\hat{\sigma}_{ij}(s,\theta) = \int_0^\infty \sigma_{ij}(r,\theta) r^{s+1} dr,$$
$$\hat{u}_i(s,\theta) = \int_0^\infty u_i(r,\theta) r^s dr,$$
$$\hat{T}(s,\theta) = qE\alpha\int_0^\infty T(r,\theta) r^{s+1} dr. \tag{10}$$

For this study, the thermal field is assumed to consist of a uniform temperature change T_0 over a semicircle of radius R_0. This field can be thought of as an initial temperature distribution that has not yet had a chance to diffuse to a steady field. By assuming this temperature distribution to be steady, one can isolate the effects of the viscoelastic relaxation from the effects of the transient temperature. Also, this simple thermal field avoids the need to derive complicated solutions for steady temperature distributions in bonded structures. Inserting this thermal field into equation (10) leads to a transformed field of the form:

$$\hat{T} = \frac{qE\alpha T_0}{s+2} R_0^{s+2}. \tag{11}$$

Under the Mellin transformation the equation for the stress function (equation (1)) becomes

$$\left(\frac{d^2}{d\theta^2} + s^2\right)\left(\frac{d^2}{d\theta^2} + (s+2)^2\right)\hat{\Phi} + \left(\frac{d^2}{d\theta^2} + (s+2)^2\right)\hat{T} = 0 \tag{12}$$

and the stresses and displacements are conveniently written as:

$$\sigma_{r\theta} + i\sigma_{\theta\theta} + (s+1)\left(\frac{d}{d\theta} + is\right)\hat{\Phi} \tag{13}$$

and

$$2\mu(u_r + iu_\theta) = -i\left(\frac{d}{d\theta} + is\right)$$
$$\left\{1 + \frac{m\left(\frac{d}{d\theta} - is\right)\left[\frac{d}{d\theta} - i(s+2)\right]}{4(s+1)(s+2)}\right\}\hat{\Phi} - \frac{m\hat{T}}{4(s+1)}. \tag{14}$$

The solution to equation (12), for $\hat{T}$ independent of θ, is:

$$\hat{\Phi} = Ae^{is\theta} + \bar{A}e^{-is\theta} + Be^{i(s+2)\theta} + \bar{B}e^{-i(s+2)\theta} - \frac{\hat{T}}{s^2} \tag{15}$$

where A and B are unknown complex constants and $\bar{A}$ and $\bar{B}$ denote their complex conjugates. The complex notation is used in order to reduce the algebra involved in the problem, as demonstrated by Hein and Erdogan (1971).

Given a solution for the stress function in each quarter plane of the model, the full solution is obtained by using the boundary and interface conditions to determine A', B', A'', and B''. The traction-free conditions on the free surfaces can be used to determine the A's in terms of the B's, giving

$$A's = B'(s+1) - \bar{B}'e^{is\pi} + \frac{\hat{T}'}{2s}e^{is\frac{\pi}{2}} \tag{16}$$

and

$$A''s = B''(s+1) - \bar{B}''e^{-is\pi} + \frac{\hat{T}''}{2s}e^{-is\frac{\pi}{2}}. \tag{17}$$

This allows determination of the stresses in terms of the complex constant B

$$\hat{\sigma}'_{r\theta} + i\hat{\sigma}'_{\theta\theta} = 2i(s+1)\Big\{(s+1)(e^{is\theta} + e^{i(s+2)\theta})B' -$$
$$(e^{is(\theta+\pi)} + e^{-i(s+2)\theta})\bar{B}' + (e^{is(\theta+\pi/2)} - 1)\frac{\hat{T}'}{2s}\Big\}$$

$$\hat{\sigma}''_{r\theta} + i\hat{\sigma}''_{\theta\theta} = 2i(s+1)\Big\{(s+1)(e^{is\theta} + e^{i(s+2)\theta})B'' -$$

$$(e^{is(\theta-\pi)} + e^{-i(s+2)\theta})\bar{B}'' + (e^{is(\theta-\pi/2)} - 1)\,\frac{\hat{T}''}{2s}\Big\}. \qquad (18)$$

Using the interface conditions, which require continuous displacements and shear and normal stresses across the interface, and breaking B' and B'' into real and imaginary parts according to:

$$B' = 2E + 2iF \qquad (19)$$

and

$$B'' = 2G + 2iH, \qquad (20)$$

the following linear system of four equations is obtained

$$\begin{bmatrix} \gamma & \eta & -\gamma & \eta \\ \eta & \lambda & \eta & -\lambda \\ \gamma+m' & \eta & -k(\gamma+m'') & k\eta \\ \eta & \lambda-m' & k\eta & -k(\lambda-m'') \end{bmatrix} \begin{Bmatrix} E \\ F \\ G \\ H \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} \qquad (21)$$

where

$$k = \frac{\mu'}{\mu''}$$
$$\gamma = 2[(s+1) - \cos^2\xi],$$
$$\eta = -2\sin\xi\cos\xi,$$
$$\lambda = 2[(s+1) + \cos^2\xi],$$
$$\xi = \frac{s\pi}{2},$$
$$R_1 = (1-\cos\xi)\left(\frac{\hat{T}' - \hat{T}''}{4s}\right),$$
$$R_2 = -\sin\xi\left(\frac{\hat{T}' + \hat{T}''}{4s}\right),$$
$$R_3 = (1-\cos\xi)\left(\frac{\hat{T}' - k\hat{T}''}{4s}\right),$$
$$R_4 = -\sin\xi\left(\frac{\hat{T}' + k\hat{T}''}{4s}\right). \qquad (22)$$

The solution of the steady-state problem now reduces to the inversion of the matrix in equation (21).

The system represented by equation (21) is similar to previous solutions for different loadings. Hein and Erdogan (1971), for instance, considered the same geometric model with a dislocation at the interface, and came up with the same matrix on the left side and a different loading vector on the right side. Bogy (1968), on the other hand, used a different notation and developed an 8×8 system, but the determinant of the coefficient matrix was identical to that of equation (21).

After extensive algebra, the transformed stresses can be found by solving equation (21) for the constants E, F, G, and H and substituting them into equation (18). The resulting interface stresses are given by

$$\hat{\sigma}_{r\theta}(\theta=0) = \frac{-(s+1)}{s\|X\|}\,[Q_1 m'\hat{T}' + Q_2 k m''\hat{T}''] \qquad (23)$$

and

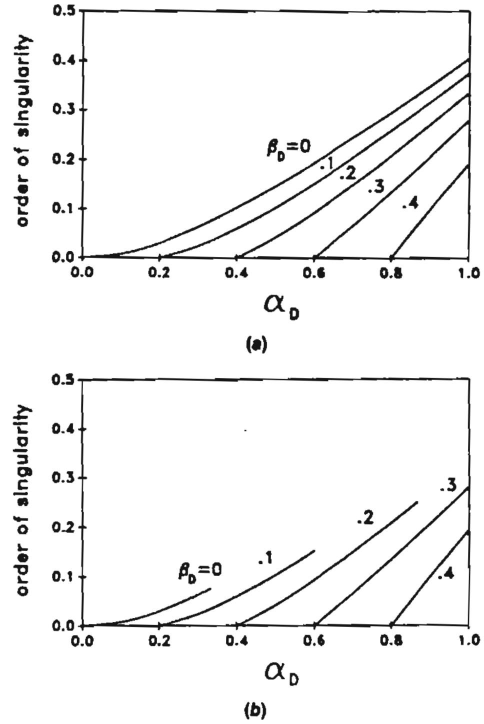

Fig. 2 Order of the singularity for various values of Dundurs' parameters under (a) plane-strain and (b) plane-stress conditions

$$\hat{\sigma}_{\theta\theta}(\theta=0) = \frac{-(s+1)}{s\|X\|}\,[Q_3 m'\hat{T}' + Q_4 k m''\hat{T}''], \qquad (24)$$

where the four functions Q_i are given by

$$Q_1 = \eta(1-\cos\xi)[(1/2)k_1(\gamma\lambda-\eta^2) + 2k\lambda m'']$$
$$+ \sin\xi[(1/2)k_1\gamma(\gamma\lambda-\eta^2) + k_2\gamma\lambda + k_3\eta^2],$$
$$Q_2 = \eta(1-\cos\xi)[(1/2)k_1(\gamma\lambda-\eta^2) - 2\lambda m']$$
$$+ \sin\xi[(1/2)k_1\gamma(\gamma\lambda-\eta^2) + k_2\gamma\lambda - k_3\eta^2],$$
$$Q_3 = (1-\cos\xi)[-(1/2)k_1\lambda(\gamma\lambda-\eta^2) + k_2\gamma\lambda + k_3\eta^2]$$
$$+ \eta\sin\xi[-(1/2)k_1(\gamma\lambda-\eta^2) + 2km''\gamma],$$
$$Q_4 = (1-\cos\xi)[(1/2)k_1\lambda(\gamma\lambda-\eta^2) - k_2\gamma\lambda + k_3\eta^2]$$
$$+ \eta\sin\xi[(1/2)k_1(\gamma\lambda-\eta^2) + 2m'\gamma], \qquad (25)$$

where

$$k_1 = 2(k-1)$$
$$k_2 = km' - m'$$
$$k_3 = km'' + m'. \qquad (26)$$

Also, $\|X\|$, the determinant of the matrix in equation (21), is given by

$$\|X\| = k_3^2\eta^2 + (k-1)^2(\gamma\lambda-\eta^2)^2 - k_2^2\gamma\lambda$$
$$+ k_2(k-1)(\lambda-\gamma)(\gamma\lambda-\eta^2). \qquad (27)$$

This determinant has a profound effect on the solution of the problem because its zeroes locate the poles of the transformed stresses in the Mellin domain, so it warrants further investigation.

Both the number and location of the zeroes of the determinant in equation (27) depend on a parameter P, defined by

$$P = k_2[2k_1 - k_2]. \qquad (28)$$

In order for the transforms of the stresses to exist, the real part of the transform parameter s must be less than -1, so the Mellin inversion depends only on the zeroes of the determinant for $\mathrm{Re}(s) < -1$. For all k_1, k_2, and k_3 (i.e., for any combination of materials), $s=-1$ and $s=-2$ are always zeroes. Hence, we concern ourselves with roots of the determinant that lie between −2 and −1. For $P>0$, there is exactly one zero on the interval $-2 < s < -1$, and it is a simple zero. For all admissible values of the material constants (i.e., $0 < \nu < ½$), this zero occurs between −1.59 and −2.0. As P approaches zero, the zero of the determinant moves closer to $s=-2$ until, when $P=0$, there are no zeroes on $-2 < s < -1$ and the zero at $s=-2$ becomes a double root. Finally, for $P<0$, there are no zeroes on $-2 < s < -1$ and the zero at $s=-2$ is simple; there is also a simple zero between −2.4 and −2.0. In this paper, only those material combinations which lead to algebraic singularities (i.e., only those for which P is positive) will be considered.

As shown by Dundurs (1969), calculation of the roots of the determinant actually can be reduced to two material parameters, one possible set of which is:

$$\alpha_D = \frac{k_2}{k_3} \text{ and } \beta_D = \frac{k_2 - k_1}{k_3}. \tag{29}$$

By confining Poisson's ratios to $0 < \nu < 0.5$, one limits Dundurs' parameters to $-1 < \alpha_D < 1$ and $(\alpha_D - 1)/4 < \beta_D < (\alpha_D + 1)/4$ for plane strain and $(3\alpha_D - 1)/8 < \beta_D < (3\alpha_D + 1)/8$ for plane stress. Typical plots of the order of singularity for various values of these parameters are shown in Fig. 2. (See Bogy (1970) for a more detailed discussion of the roots of this determinant.)

The recovery of the stresses in real space from the transformed stresses involves inversion of the Mellin transform, using the following complex integral:

$$\sigma_{ij}(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\sigma}_{ij}(s,\theta) r^{-(s+2)} ds, \tag{30}$$

where c lies in a strip of regularity of the integrand. As mentioned previously, we are concerned only with $s < -1$, so c lies between $\mathrm{Re}(s) = -1$ and the first pole of the integrand associated with the inversion of the transformed stresses. In order to obtain the leading terms in the asymptotic expansions for the stresses, one can perform the inversion using contour integration and the residue theorem. It can be shown that for an appropriately chosen contour, the inversion integral of equation (30) is asymptotic to the residue as r approaches zero. Once s_1, the first pole of the integrand is found, the asymptotic solution follows directly. The form of the stress thus depends on the nature of the pole and is quite different for the three types of determinant mentioned earlier in this section.

For the assumed thermal field and for cases where there is an algebraic singularity, the shear and normal stresses in the interface are given by

$$\sigma_{r\theta}(\theta=0) = F(r;s)[Q_1 m'q'E'\alpha' + Q_2 k m''q''E''\alpha'']$$
$$\sigma_{\theta\theta}(\theta=0) = F(r;s)[Q_3 m'q'E'\alpha' + Q_4 k m''q''E''\alpha''], \tag{31}$$

where

$$F(r;s) = \frac{-(s_1+1)}{s_1(s_1+2)} \frac{T_0}{\frac{d}{ds}(|\mathbf{X}|)|_{s=s_1}} \left(\frac{R_0}{r}\right)^{(s_1+2)}. \tag{32}$$

This elastic solution provides a tool for determining the response of viscoelastic materials via the viscoelastic analogy.

3 Viscoelastic Solution

In order to determine the stresses and strains in a viscoelastic body, the constitutive relations must be modified to account for the time-dependent deformations. In general, these relations are written as

$$P_1(D)s_{ij} = P_2(D)e_{ij} \tag{33}$$

and

$$P_3(D)\sigma_{ii} = P_4(D)(\epsilon_{ij} - 3\alpha T), \tag{34}$$

where

$$P_k(D) = \sum_{n=1}^{N} C_{kn} \frac{\partial^n}{\partial t^n}. \tag{35}$$

The operator $P_k(D)$ is a general, linear, differential operator of order N, and C_{kn} are appropriate coefficients representing the material behavior of a viscoelastic solid. A common solution method for these types of problems is to take the Laplace transform of the viscoelastic equations and compare the resulting set of equations to the steady-state formulation (see, for instance, Lee (1955)). Because the bulk behavior is often different from the shear behavior in viscoelastic materials, the stress-strain relations are usually written in terms of the stress and strain deviators, s_{ij} and e_{ij}, defined as

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{ll}}{3} \tag{36}$$

and

$$e_{ij} = \epsilon_{ij} - \frac{\epsilon_{ll}}{3}. \tag{37}$$

In terms of these quantities, the elastic stress-strain relations are

$$s_{ij} = 2\mu e_{ij} \tag{38}$$

and

$$\sigma_{ll} = 3\kappa(\epsilon_{ll} - 3\alpha T), \tag{39}$$

where κ is the bulk modulus, i.e.,

$$\kappa = \frac{E}{3(1-2\nu)} = \frac{2\mu(1+\nu)}{3(1-2\nu)}. \tag{40}$$

Transforming equations (33) and (34) gives

$$\bar{s}_{ij} = \frac{P_2(p)}{P_1(p)} \bar{e}_{ij} \tag{41}$$

and

$$\bar{\sigma}_{ii} = \frac{P_4(p)}{P_3(p)} (\bar{\epsilon}_{ii} - 3\alpha\bar{T}), \tag{42}$$

where the Laplace transform of a function f is denoted by $\bar{f}$, and p is the independent variable in the Laplace domain. By comparing the elastic and viscoelastic constitutive equations, it is apparent that the solution of the viscoelastic problem in the Laplace domain is equivalent to the solution of the steady-state problem, with the elastic properties 2μ and 3κ replaced by $P_2(p)/P_1(p)$ and $P_4(p)/P_3(p)$, respectively. The time-dependent behavior of the viscoelastic problem is thus recovered by substituting the equivalent transformed properties into the steady-state elastic solution and inverting the Laplace transform.

The analysis of bonded structures composed of viscoelastic materials is difficult, even with the use of the analogy described in the previous section. The substitution of functions of the Laplace parameter p for the material properties in elastic solutions such as those represented by equation (31) leads to very complicated expressions for the stresses in the Laplace domain and analytical inversion is not possible. The primary difficulty is the fact that the eigenvalues s_k, which are functions of the Laplace parameter p, are not known explicitly because these are determined by solving a transcendental equation. Hence, numerical inversion is necessary. Unfortunately, numerical inversion of the Laplace transform is difficult because the operator's inherent unboundedness prevents explicit error control. Because a small change in the transformed function can lead to an arbitrarily large change in the real function, high precision is needed to obtain accurate results.

The method adopted for this study solves the Laplace transform integral definition as an integral equation, with $f(t)$ as the unknown, using Gaussian quadrature (Bellman (1966, p. 32)). Other methods, such as those suggested by Miller and Guy (1966), Papoulis (1957), and Piessens (1972) may be used if the accuracy of the quadrature method is insufficient. Each of these methods are useful because they do not require evaluation of the transformed function for complex values of the Laplace parameter p.

The first step is to transform the integral into one with finite limits, then approximate the integral as a finite sum, using Nth-order Gaussian quadrature. This can be evaluated for N arbitrary values of the Laplace parameter p, giving a linear system represented by

$$\bar{f}(p_k)=\frac{1}{2}\sum_{i=0}^{N} w_i\left(\frac{1+\tau_i}{2}\right)^{p_k-1} g(\tau_i). \tag{43}$$

This gives a system of N equations for the N unknowns $g(\tau_i)$, which represent values of the unknown function $g(\tau)$ at discrete locations τ_i. The solution from this point is trivial. For this study, order 15 quadrature ($N=15$) was used because it was found to be the most reliable for problems associated with exponential decay. This is useful for the present study because the relaxation of stresses in Maxwell materials tends to be exponential.

As an example of one type of viscoelastic material, the rest of this paper will consider Maxwell materials. For these materials, the stress-strain equations are

$$\frac{\partial s_{ij}}{\partial t}+\frac{s_{ij}}{\tau_0}=2\mu\frac{\partial \epsilon_{ij}}{\partial t} \tag{44}$$

and

$$\sigma_{ii}=3\kappa(\epsilon_{ii}-3\alpha T). \tag{45}$$

Hence, by comparison with equations (33) and (34),

$$P_1(D)=\frac{\partial}{\partial t}+\frac{1}{\tau_0}$$

$$P_2(D)=2\mu\frac{\partial}{\partial t}$$

$$P_3(D)=1$$

$$P_4(D)=3\kappa. \tag{46}$$

Taking the Laplace transform of these operators and substituting into equation (41) yields the following equivalencies between the elastic materials properties and the viscoelastic "properties" in the Laplace domain:

$$2\mu\rightarrow\frac{2\mu p}{p+\frac{1}{\tau_0}} \tag{47}$$

and

$$\kappa\rightarrow\kappa. \tag{48}$$

This represents a material for which the bulk behavior is elastic, and the stress decays exponentially (with a decay constant of τ_0) for a uniaxial fixed-grip test. Substituting equations (47) and (48) into the elastic stress solutions represented by equations (31) yields expressions for the stresses in the Laplace domain. This is then inverted numerically to obtain the time-dependent behavior of duplex structures.

Using these equivalent properties in the elastic solution for the interface stresses near the edge of the interface of bonded quarter planes, one finds transformed viscoelastic solutions of the form:

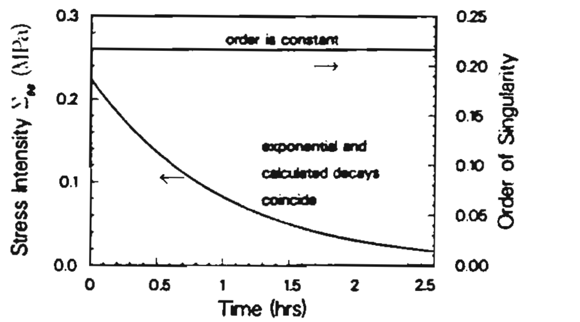

Fig. 3 Relaxation of the boundary layer stress intensity for constant order of the singularity

$$\sigma_{r\theta}(\theta=0)=\frac{1}{p}F(r;s_1)[Q_1 m' q' E'\alpha'+Q_2\bar{k}m'' q'' E''\alpha'']$$

$$\sigma_{\theta\theta}(\theta=0)=\frac{1}{p}F(r;s_1)[Q_3 m' q' E'\alpha'+Q_4\bar{k}m'' q'' E''\alpha'']. \tag{49}$$

If the decay constants are equal (i.e., $\tau_0'=\tau_0''$), and for plane stress conditions these equations simplify to:

$$\sigma_{r\theta}(\theta=0)=8F(r;s_1)[Q_1\alpha'+Q_2\alpha'']\frac{\mu'}{p+1/\tau}$$

$$\sigma_{r\theta}(\theta=0)=8F(r;s_1)[Q_3\alpha'+Q_4\alpha'']\frac{\mu'}{p+1/\tau} \tag{50}$$

If the order of the singularity for this problem were independent of time, then the transformed stresses could be inverted analytically, leading to a time dependence of the form:

$$\sigma_{ij}=C_{ij}e^{-t/\tau_0}, \tag{51}$$

where C_{ij} is the initial value of the stress σ_{ij}. In reality, though, the order of the singularity must be determined from the roots of the determinant given in equation (27), which becomes a function of the Laplace transform parameter p when the effective material properties are used. Hence, the order of singularity is time-dependent and numerical inversion of the transformed stresses is required. As an example, the transform of one of the material parameters, k, can be written:

$$\bar{k}=\frac{\mu'}{\mu''}\left(\frac{p+\frac{1}{\tau_0''}}{p+\frac{1}{\tau_0'}}\right). \tag{52}$$

As the Laplace parameter p ranges from 0 to ∞, $\bar{k}$ can take on virtually any value, depending on the creep constants of the two materials. This can be shown with the use of the limit theorems of the Laplace transform, giving the final value of $\bar{k}$ (at $t\rightarrow\infty$) as:

$$\bar{k}_{\infty}=\frac{\mu'\tau_0'}{\mu''\tau_0''}. \tag{53}$$

Therefore, depending on the ratios of the creep constants of the two materials, $\bar{k}$ can take on any value during the life of the component, thus leading to numerous possibilities for the final order of singularity. For Maxwell materials, the final value for the effective Poisson's ratio is 0.5 for any finite decay rate τ. Hence, the material parameter m as defined in equation (5) has the following limiting values:

$$\bar{m}_{\infty}=\begin{cases}8/3 & \text{for plane stress}\\ 2 & \text{for plane strain}\end{cases} \tag{54}$$

Given these limiting values for the material properties $\bar{k}$, $\bar{m}'$, and $\bar{m}''$, one can determine the limiting value for $\bar{P}$ =

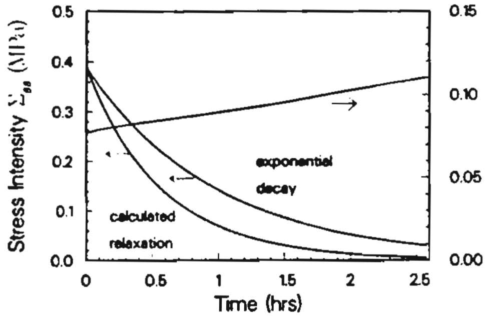

Fig. 4 Relaxation of the boundary layer stress intensity for a time-dependent order of the singularity

$\tilde{k}_2(2\tilde{k}_1 - \tilde{k}_2)$, which determines whether or not there will be a singularity for a given set of material properties. This limiting value is

$$\tilde{P}_\infty = \tilde{m}_\infty(4 - \tilde{m}_\infty)(\tilde{k}_\infty - 1)^2 \quad (55)$$

which is non-negative. Therefore, regardless of whether there is an initial singularity, there will always be a singularity at $t = \infty$, unless the limiting value of the shear modulus ratio $\tilde{k}_\infty$ is 1.

4 Results

The singular stress fields in this study can be characterized by two parameters: the order of the singularity and the associated "boundary layer stress intensity factor." This latter quantity, as defined by Wang and Choi (1982), is given by

$$K_{ij} = \lim_{r \to 0} r^{s_1+2} \sigma_{ij}(\theta = 0). \quad (56)$$

For this paper, this stress intensity is normalized by the quantity $R_0^{-(s_1+2)}$, giving it stress units. Hence, results are presented for a new parameter Σ_{ij}, given by:

$$\Sigma_{ij} = \lim_{r \to 0} \left(\frac{r}{R_0}\right)^{s_1+2} \sigma_{ij}(\theta = 0). \quad (57)$$

In addition, only those results for the normal stress at the interface $\sigma_{\theta\theta}$ ($\theta = 0$) will be presented, because it is generally the largest of the four interface stresses (σ'_{rr}, σ''_{rr}, $\sigma_{r\theta}$, and $\sigma_{\theta\theta}$).

As is evident from the previous section, choosing $\tau' = \tau''$ and $\nu' = \nu''$ provides a situation in which the order of the singularity is independent of time. Figure 3 shows the decay of the stress intensity factor for such a case, where

$$\begin{aligned} \tau' &= \tau'' = 1 \text{ hr} \\ \nu' &= \nu'' = 0.5 \\ \mu' &= 10 \text{ GPa} \\ \mu'' &= 100 \text{ GPa} \\ \alpha' &= 1 \times 10^{-6}\,{}^\circ\text{C}^{-1} \\ \alpha'' &= 2 \times 10^{-6}\,{}^\circ\text{C}^{-1} \\ T_0 &= 100^\circ\text{C} \end{aligned} \quad (58)$$

and plane stress conditions were assumed. As discussed previously, the relaxation of the stresses for a constant order should be exponential. This is verified in this figure where the comparison between the calculated relaxation, from numerical Laplace transform inversion and a purely exponential decay, is shown to be excellent. This result provides some verification of the numerical transform inversion.

Figure 4 shows the results for the relaxation of the stresses in a case where the order is varying. The properties used in this case were identical to those given above, except $\nu' = 0.1$ and $\nu'' = 0.3$. This figure shows that the order of singularity more than doubles from its initial value of 0.065. The relaxation of the stress intensity, in this case, is faster than exponential decay. It is unclear how these competing effects (decreasing stress intensity and increasing order) will effect failure or delamination.

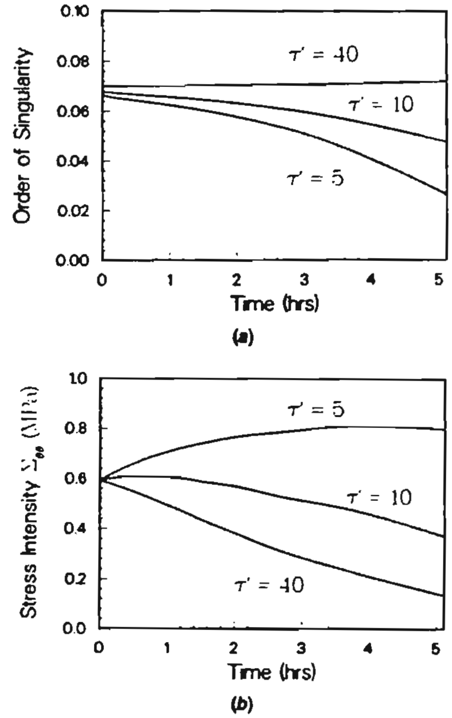

Fig. 5 Order of the (a) singularity and (b) boundary layer stress intensity factor for different ratios of the relaxation constant

In some cases the order of singularity can decrease before increasing. This behavior is accompanied by a stress intensity which increases before relaxing. This was explored using the following properties:

$$\begin{aligned} \tau'' &= 1 \text{ hr} \\ \nu' &= 0.3 \\ \nu'' &= 0.5 \\ \mu' &= 10 \text{ GPa} \\ \mu'' &= 100 \text{ GPa} \\ \alpha' &= 1 \times 10^{-6}\,{}^\circ\text{C}^{-1} \\ \alpha'' &= 2 \times 10^{-6}\,{}^\circ\text{C}^{-1} \\ T_0 &= 100^\circ\text{C} \end{aligned} \quad (59)$$

with a varying relaxation constant τ' in the lower quarter plane and assuming plane-strain conditions. These properties indicate that the shear modulus in the upper plane is greater than that in the lower plane, giving $k = 0.1$ initially, but if the softer material has a higher relaxation constant ($\tau' > \tau''$), the final value for the shear moduli ratio will be greater than one. Hence, for some time this ratio will be near one and the singularity will disappear. As shown in Fig. 5(*a*), the order of the singularity decreases faster for increasing τ' (i.e., for slower relaxation in the lower quarter plane). The subsequent increase in the order is not shown here. The companion Fig., 5(*b*), shows the corresponding relaxation of the boundary layer stress intensities. As the relaxation constant in the softer material is increased (and the order of the singularity drops at an increasing rate), the stress intensity tends to relax more slowly. In fact, it increases in some cases. Again, there is competition between the relaxation of the stress intensity and the increase (or decrease) of the order of the singularity.

5 Conclusions

Thermal stresses in perfectly-bonded dissimilar quarter planes are often singular, depending on the loading and the material properties. Because the order of this singularity is material-dependent, it tends to change with time in viscoelastic materials. It is apparent from the results in the previous section that as the order increases with time, the relaxation of the boundary layer stress intensity factor tends to be faster than exponential decay, while the relaxation is relatively slow when the order decreases. Experimental evidence is required to assess the impact that this phenomenon has on the onset of failure in crack-free structures.

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7 References

Bellman, R., Kalaba, R., and Lockett, J., 1966, *Numerical Inversion of the Laplace Transform*, Elsevier, New York.

Bogy, D. B., 1968, "Edge-Bonded Dissimilar Orthogonal Elastic Wedges Under Normal and Shear Loading," ASME JOURNAL OF APPLIED MECHANICS, Vol. 35, pp. 460–466.

Bogy, D. B., 1970, "On the Problem of Edge-Bonded Elastic Quarter-Planes Loaded at the Boundary," *International Journal of Solids and Structures*, Vol. 6, pp. 1287–1313.

Delale, F., and Erdogan, F., 1981, "Viscoelastic Analysis of Adhesively Bonded Structures," ASME JOURNAL OF APPLIED MECHANICS, Vol. 48, pp. 331–338.

Dempsey, J. P., and Sinclair, G. B., 1981, "On the Singular Behavior at the Vertex of a Bi-Material Wedge," *Journal of Elasticity*, Vol. 11, pp. 317–327.

Dundurs, J., 1969, discussion, ASME JOURNAL OF APPLIED MECHANICS, Vol. 36, pp. 650.

Hein, V. L., and Erdogan, F., 1971, "Stress Singularities in a Two-Material Wedge," *International Journal of Fracture Mechanics*, Vol. 7, pp. 317–330.

Lee, E. H., 1955, "Stress Analysis in Viscoelastic Bodies," *Quarterly of Applied Mathematics*, Vol. 13, pp. 183–190.

Miller, M. K., and Guy, W. T., 1966, "Numerical Inversion of the Laplace Transform by Use of the Jacobi Polynomials," *SIAM Journal of Numerical Analysis*, Vol. 3, pp. 624–635.

Papoulis, A., 1957, "A New Method of Inversion of the Laplace Transform," *Quarterly of Applied Mathematics*, Vol. 14, pp. 405–414.

Piessens, R., 1972, "A New Numerical Method for the Inversion of the Laplace Transform," *Journal of the Institute of Mathematics and its Applications*, Vol. 10, pp. 185–192.

Wang, S. S., and Choi, I., 1982, "Boundary-Layer Effects in Composite Laminates: Part 2 – Free-Edge Stress Solutions and Basic Characteristics," ASME JOURNAL OF APPLIED MECHANICS, Vol. 49, pp. 549.