A Novel Technique for Extracting Parametric Models from MEM Resonator Test Data

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***Abstract*— This paper reports an innovative test and modeling procedure that is well-suited to high-Q resonators and overcomes several shortcomings of FFT-based approaches. The novel technique uses excitation-response data to generate parametric resonator models in the time-domain. The models possess the following advantages: immunity to parasitic coupling, a spectral resolution that is not limited by record length, and the ability to clearly distinguish nearly degenerate and degenerate modes by using multi-input, multi-output data sets. The parametrization readily yields transfer functions, natural frequencies, time constants and insight into mode shapes.**

I. Introduction

Modern signal and network analyzers use the fast Fourier transform (FFT) to produce non-parametric models of resonators. These data are subsequently analyzed for resonant frequencies, bandwidths and so forth. While this is a generally accepted technique for determining the resonator transfer function, it is ill-suited for resonators possessing long time constants (and possessing closely spaced modes). Adequate spectral resolution and minimizing data windowing errors requires records whose durations are longer than the dominant time constant. Furthermore, early in development, MEM resonators often do not have optimized electrode configurations and signal buffers because the focus is on improving the quality of the resonator. Thus, parasitic coupling between the excitation source and measurement point is often manifest, obscuring the salient features of the resonator dynamics. Although it is possible to remove some degree of coupling with calibration procedures, they are time consuming and success relies on the time-invariance of the coupling since commercial analyzers essentially subtract off the feed-through [1].

This paper reports a test and modeling procedure that overcomes the aforementioned shortcomings of FFT-based approaches. The procedure generates linear time-invariant (LTI) models from excitation-response data and is particularly well-suited for characterizing high-Q MEM resonators. The novel technique is summarized as follows: (1) a burst excitation with energy spectrum encompassing the modes to be characterized is applied to the resonator; (2) a segment of the input-free transient response is used to develop the input-free portions of a state-space model using a modified principal component analysis (based on the method introduced by Kung [2]); (3) the model is completed by using the parameterization's initialized free-response state along with the burst excitation record to identify the effect of inputs on the model dynamics. The complete model yields transfer functions, natural frequencies, time constants and eigenvectors, the latter of which can be related to mode shapes. This procedure rejects all parasitic coupling because the device response during the burst excitation is not used in the model development. By developing parametric models directly from time data, the approach provides two additional advantages: spectral resolution is not limited by record length and nearly degenerate and degenerate modes can be clearly distinguished by using multi-input, multi-output data sets.

This paper is organized as follows: In Sec. II, notation is established and details of the proposed testing and identification technique are for the first time presented. In Sec. III, the ringdown-based modeling approach is validated by application to MEM resonators: first, using a pair of UCLA-designed planar concentric ring resonators (URES) [3], model competency and immunity to input-output parasitics is established; next, using a quadruple mass gryo (QMG) [4], short-segment parametrizations for long ringdown devices are demonstrated. Section IV summarizes the work and comments on limitations of the LTI parametrization in application to MEM resonators.

II. Test and Modeling Procedure

A resonator whose response properties for some range of driving amplitudes and frequencies is assumed to have been chosen and instrumented with appropriate transducers. For the sake of generality it is assumed that there are m drivers, whose time signals will be denoted $\mathrm{u}^{(i)} : \mathbb{R} \mapsto \mathbb{R}$ for $i = 1, \dots, m$, and l pick-offs, respectively denoted as $\mathrm{y}^{(j)}$ for $j = 1, \dots, l$. Practically speaking, the sets of signals $\mathrm{u}^{(i)}$ and $\mathrm{y}^{(j)}$ are never known exactly because they are sampled at discrete intervals. Uniform sampling of these signals - with period T_s - starting at time t_{start} and through time t_{stop}, results in the representations

$$\begin{aligned} t_k &= t_{\text{start}} + kT_s, \quad k = 0, 1, \dots, \lfloor \frac{t_{\text{stop}} - t_{\text{start}}}{T_s} \rfloor, \\ \mathrm{u}_k^{(i)} &= \mathrm{u}^{(i)}(t_k), \quad i = 1, \dots, m, \\ \mathrm{y}_k^{(j)} &= \mathrm{y}^{(j)}(t_k), \quad j = 1, \dots, l, \end{aligned} \tag{1}$$

with column vector concatenations of the signals given as

$$\mathrm{y}_k = [\mathrm{y}_k^{(1)}, \dots, \mathrm{y}_k^{(l)}]^\intercal, \text{ and } \mathrm{u}_k = [\mathrm{u}_k^{(1)}, \dots, \mathrm{u}_k^{(m)}]^\intercal.$$

A. Empirical Procedure - Summary Step (1)

The goal of the experimental process is to obtain free-response data from a device under test (DUT) so that the dynamics of some structural mode(s) can be clearly assessed. A procedure for accomplishing this follows: Begin recording data at a time designated t_{start} and deliver a burst signal, denote it by $v^{(\cdot)}$ – designed to target energy into mode(s) of interest – to the DUT, initially at rest, via a single channel while any others remain quiescent, e.g. $\mathrm{u}^{(1)} \leftarrow v^{(1)}$, $\mathrm{u}^{(i)} \leftarrow 0$ for $i \neq 1$. Following the burst activity, transients associated with any parasitic coupling between the input and pick-off transducers are anticipated to very rapidly settle so that free ringdown of any excited DUT dynamics dominates the persisting observations. A certain time, to be exactly specified in Sec. II-B, satisfying this hypothesis will be denoted by t_{ring}. At some decided later time, t_{stop}, data collection ends. The sampled pick-off signals, y_k for $k = 0, 1, \ldots, \lfloor \frac{t_{\text{stop}} - t_{\text{start}}}{T_s} \rfloor$, due to the burst applied via the first input channel are designated by the column vector

$$\gamma_k^{(1)} = \left[\gamma_k^{(1,1)},\ \gamma_k^{(2,1)}, \ldots, \gamma_k^{(l,1)} \right]^{\mathsf{T}}. \tag{2}$$

Similar experiments are then carried out for any remaining drive channels, until all m are completed. The investigator is at liberty to tailor different burst inputs for each of the m tests.

B. Computational Procedure - Summary Steps (2) and (3)

With data collected, a parametrization of the Eqn. (16) model is undertaken. To accomplish this, a robust LTI technique known as principal component analysis (PCA) is used. PCA has the drawback of computational expense due to it entailing the singular value decomposition (SVD) [5] of a large matrix. The application of this technique to MEM resonator data (potentially featuring high oscillation frequencies and long ringdown times) is made tractable by one key realization. Namely, that the spectral signatures of any modeshape occupy only a narrow band of frequencies. Hence, processing may be performed with a less expensive baseband representation of the data that preserves longer-term amplitude and phase trends, after which the dynamics can be reassociated with the proper frequencies. This process represents the second step of the procedure as summarized in Sec. I.

One may go about constructing the low data rate baseband equivalent vector sequences in the following way: first, identify in the data the spectral range of interest, say $\omega \in [\omega_{\text{low}}, \omega_{\text{high}}]$. Next, for the frequency content in this range, make phase and energy preserving baseband equivalent representations of the data by spectrum shifting

$$\gamma_k^{(i)} e^{-\imath \omega_{\text{shift}} t_k}, \quad i = 1, 2, \ldots, m, \text{ where } \omega_{\text{shift}} = \frac{\omega_{\text{high}} + \omega_{\text{low}}}{2},$$

and then forward-backward filtering [6] these records with a unity gain low-pass filter with cutoff just outside $\pm\frac{\omega_{\text{high}} - \omega_{\text{low}}}{2}$ and linear phase within. The resulting complex baseband records – designate them by $\eta_k^{(i)}$ – may now be downsampled for analysis via PCA. By construction, the downsample factor

$$\mathrm{D} = \lfloor \frac{1}{T_s(\omega_{\text{high}} - \omega_{\text{low}})} \rfloor \tag{3}$$

results in maximal data reduction without risking aliasing. The baseband data pertinent to ringdown modeling are thus

$$\mathfrak{h}_k^{(i)} \triangleq \eta_{\mathrm{D}k}^{(i)}, \quad k = p, \ldots, p + 2M - 2 \leq \lfloor \frac{t_{\text{stop}} - t_{\text{start}}}{\mathrm{D}T_s} \rfloor, \tag{4}$$

where $p = \lceil \frac{t_{\text{ring}} - t_{\text{start}}}{\mathrm{D}T_s} \rceil$ and $M \in \mathbb{Z}$ are both elected by the user. Keeping M less than 10^3 for modest numbers of input and pick-off channels is advised as doing so will afford almost instant PCA calculations on a modern personal computer.

To proceed, PCA requires the construction of a certain block-Hankel structured matrix from the free-response data [2]. The authors recommend a two-step process to populate H with the downsampled baseband data: form m block-Hankel matrices of size $lM \times mM$ using the $2M - 1$ points

$$H^{(i)} = \left[\begin{array}{cccc} \mathfrak{h}_p^{(i)} & \mathfrak{h}_{p+1}^{(i)} & \cdots & \mathfrak{h}_{p+M-1}^{(i)} \\ \hline \mathfrak{h}_{p+1}^{(i)} & \mathfrak{h}_{p+2}^{(i)} & \cdots & \mathfrak{h}_{p+M}^{(i)} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathfrak{h}_{p+M-1}^{(i)} & \mathfrak{h}_{p+M}^{(i)} & \cdots & \mathfrak{h}_{p+2M-2}^{(i)} \end{array} \right]. \tag{5}$$

The columns of these matrices, $H^{(\cdot)}$, can then be stacked into the desired $lM \times mM$ block-Hankel matrix, H. Using $\mathrm{col}_i(\cdot)$ to denote an operator that picks off the i^{th} columns of the matrices upon which it acts, H is formed as

$$H = [\,\mathrm{col}_1(H^{(1)}, ..., H^{(m)}), \ldots, \mathrm{col}_M(H^{(1)}, ..., H^{(m)})\,]. \tag{6}$$

An SVD of the block Hankel matrix H is then computed

$$H = U \Sigma V^{\mathrm{H}}, \tag{7}$$

where Σ is an $lM \times mM$ matrix with the singular values of H arranged along its diagonal in decreasing order.

The data captured in H are presently assumed to convey dynamics realizable by an LTI system of order $n << M$ where $n \leq \min(m, l)$ is expected to be a more reliable model. The model order is chosen by an examination of the largest singular values of H and then truncating after some magnitude cutoff. The rationale is that the subspace of H corresponding to its largest singular values is descriptive of energetic LTI dynamics manifest in the data while that of the smaller singular values accounts for LTI features that are only weakly stimulated and phenomena not describable as such. Based upon the choice of n, observability and controllability matrices associated with a balanced n-state state-space realization can be estimated:

$$\mathcal{O} = \left[\mathrm{col}_{[1,\ldots,n]}(U)\right] \Sigma^{1/2}, \quad \mathcal{C} = \Sigma^{1/2} \left[\mathrm{col}_{[1,\ldots,n]}(V)\right]^{\mathrm{H}}, \tag{8}$$

where $\Sigma^{1/2}$ denotes the principal square root of the $n \times n$ leading principal submatrix of Σ.

A state matrix associated with a discrete LTI model for the baseband dynamics and timestep DT_s, Φ, is found by solving

$$\left[\mathrm{row}_{[1,\ldots,lM-l]}(\mathcal{O})\right] \Phi = \left[\mathrm{row}_{[l+1,\ldots,lM]}(\mathcal{O})\right], \tag{9}$$

where $\mathrm{row}_i(\cdot)$ picks off the i^{th} rows of the matrices upon which it operates. A least squares solution of Eqn. (9) is sufficient. The corresponding timestep T_s state matrix permitting dynamics in the band $\omega \in [\omega_{\text{low}}, \omega_{\text{high}}]$, A, can now be computed as

$$A = e^{\imath\omega_{\text{shift}} T_s}\, \Phi^{1/\mathrm{D}}. \tag{10}$$

The related pick-off model as well as a concatenation of the m initial ringdown states are respectively given by

$$C = \mathrm{row}_{[1,\ldots,l]}(\mathcal{C}), \tag{11}$$

$$\left[x^{(1)}_{\mathrm{D}p},\, \ldots,\, x^{(m)}_{\mathrm{D}p}\right] = \mathrm{col}_{[1,\ldots,m]}(\mathcal{O}) e^{\imath\omega_{\text{shift}} t_{\mathrm{D}p}}. \tag{12}$$

Thus, an input-free, timestep T_s state-space model has been parametrized, where m states – each associated with one of the burst experiments – at sample time $t_{\mathrm{D}p}$ are known:

$$x_{k+1} = Ax_k, \qquad y_k = Cx_k. \tag{13}$$

Equation (13) is complex-valued. The primary purpose of it's development, however, is to simulate strictly real-valued outputs, as would be measured in experiments. The model is therefore appended to only output the real-valued parts of y_k:

$$\begin{aligned} \underbrace{\begin{bmatrix} \mathrm{Re}\, x_{k+1} \\ \mathrm{Im}\, x_{k+1} \end{bmatrix}}_{\mathsf{x}_{k+1}} &= \underbrace{\begin{bmatrix} \mathrm{Re}\, A & -\mathrm{Im}\, A \\ \mathrm{Im}\, A & \mathrm{Re}\, A \end{bmatrix}}_{\mathsf{A}} \underbrace{\begin{bmatrix} \mathrm{Re}\, x_k \\ \mathrm{Im}\, x_k \end{bmatrix}}_{\mathsf{x}_k}, \\ \underbrace{\mathrm{Re}\, y_k}_{\mathsf{y}_k} &= \underbrace{\begin{bmatrix} \mathrm{Re}\, C & -\mathrm{Im}\, C \end{bmatrix}}_{\mathsf{C}} \mathsf{x}_k. \end{aligned} \tag{14}$$

The final step of the procedure parametrizes a matrix B relating DUT inputs to the state dynamics. This matrix is determined by simulating the system from $\mathsf{x}^{(\cdot)}_0 = 0$ up to the known state $\mathsf{x}^{(\cdot)}_{\mathrm{D}p}$ using the sampled input sequences, $v^{(\cdot)}_k$. The fact that only one channel is non-zero for any given experiment is exploited to set up m systems of equations that can be solved (in a least squares sense, generally) for the columns of B:

$$\mathsf{x}^{(i)}_{\mathrm{D}p} = \left[\sum_{k=0}^{\mathrm{D}p-1} \mathsf{A}^{\mathrm{D}p-1-k} v^{(i)}_k\right] \mathrm{col}_i(\mathsf{B}), \quad i = 1, \ldots, m. \tag{15}$$

The matrix B is put together by arranging the computed columns appropriately. The timestep T_s model is summarized

$$\begin{aligned} \mathsf{x}_{k+1} &= \mathsf{A}\mathsf{x}_k + \mathsf{B}\mathrm{u}_k, \quad \text{where} \quad \mathsf{A} \in \mathbb{R}^{2n\times 2n},\, \mathsf{B} \in \mathbb{R}^{2n\times m} \\ \mathsf{y}_k &= \mathsf{C}\mathsf{x}_k, \qquad\qquad \text{and} \quad \mathsf{C} \in \mathbb{R}^{l\times 2n}. \end{aligned} \tag{16}$$

Provided that the DUT dynamics are very nearly LTI, and that the user has been careful with both the forward-backward filtering and choice of state dimension, n, the eqn. (16) model is expected to provide a very good representation of the parasitic-free DUT dynamics within the frequency range $\omega \in [\omega_{\text{low}}, \omega_{\text{high}}]$. If, moreover, parasitic coupling effects are negligible for the DUT then one should find that $\mathsf{y}_k \approx \mathrm{y}_k$. Important DUT characteristics, including natural frequencies, time constants and mode shape features, may be found by examining the eigenvalues and vectors of the state matrix A.

III. Validation

For the purpose of validation and establishing utility, the procedure presented in Section II is applied to a pair of planar concentric ring resonators (URES) [3] as well as a quadruple mass gryoscope (QMG) [4]. The data obtained from each of the devices exhibit distinctive qualities, in turn permitting several aspects of the parametric modeling capability to be highlighted.

A. Planar Concentric Ring Resonators

Owing to their intended cyclical symmetry, URES devices ideally possess spectrally degenerate pairs of in-plane flexural mode shapes. Practically, however, processing non-idealities always conspire to detune the modes. Even same-wafer device-to-device variations in dynamics are sometimes significant. Modal identification is, moreover, potentially complicated because of parasitic coupling between drivers and pick-offs. Characterizing devices both expediently and accurately under these circumstances can be a challenge. The results shown here demonstrate that this challenge can be met by using the ringdown-based parametric modeling technique.

Two devices are instrumented for electrostatic actuation and sensing of the in-plane flexural mode shape pair featuring 2 nodal chords (n=2 modes). The first device is chosen because it exhibits little parasitic coupling and permits a compelling verification of the technique. The second device, on the other hand, is plagued by signaling parasitics and is showcased to illustrate the modeling technique's power in a very adverse characterization scenario.

Consistent with the procedure listed in Sec. II-A, ringdown experiments are conducted on both devices. The DUTs are stimulated using 2 volt amplitude, 1 second in duration burst chirps that target the pair of n=2 modes. The dynamics are parametrized using 2 seconds of ringdown data starting 100 milliseconds after the chirp has ceased. In line with the expectation of two modeshapes dominating these data, there is a dramatic drop-off observed between the 2nd and 3rd largest singular values of the block Hankel matrices, H, for both devices. Hence, n is chosen as 2, ultimately resulting in 4-state models in the form of Eqn. (16). The parametrized models are used to generate transfer functions to facilitate comparison with non-parametric spectrum-based models.

The benchmark case, detailed in Fig. 1 by single-input,

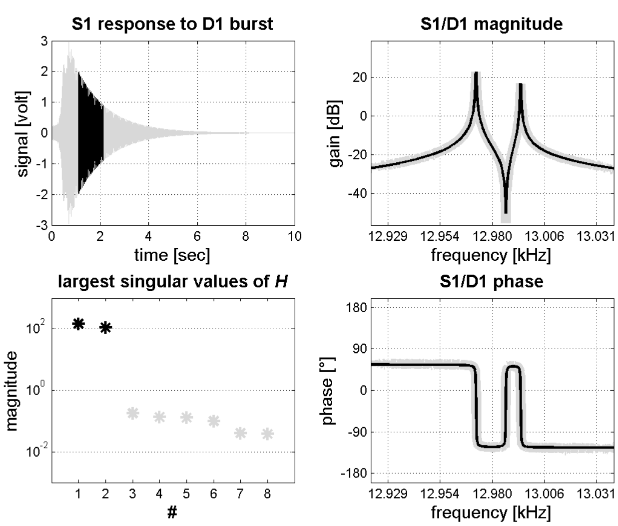

Fig. 1. Plots relating an identification experiment on the n=2 modes of a low-parasitic URES. The black traces are associated with parametrization choices (left side sub-plots) and the resulting model (right). The gray traces in the right side compare a non-parametric FFT-based frequency response function.

single-output experiment data, illustrates that for a URES with negligible parasitic coupling the frequency response of the LTI model is well-matched to the FFT-based frequency response (the same data have been used for both approaches). From the model, time constants of resonances near 12.972 and 12.994 kHz are respectively estimated to be 1.291 and 1.285 seconds.

Figure 2 depicts the magnitude portion of the two-input, two-output frequency response functions associated with the device having excessive parasitic coupling between drivers and pick-offs. The gray traces show the coupling almost completely obscures the n=2 mode shape resonances when using an uncompensated non-parametric FFT-based model. Nonetheless, the parametric technique is able to identify an electromechanical transfer function associated with the n=2 modes. Time constants on the identified 14.009 and 14.026 kHz resonances are 1.011 and 1.009 seconds, respectively.

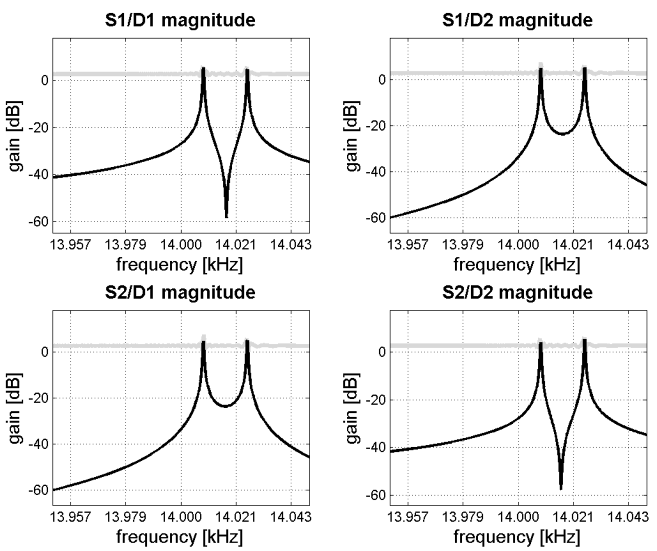

Fig. 2. Frequency response plots associated with the n=2 modes of a URES having excessive parasitic coupling between drivers and pick-offs. The nearly flat gray traces correspond to an FFT-based chirp analysis and offer almost no insight into the modes. The black traces, on the other hand, are based off the parametric model and unambiguously convey the modal characteristics.

B. Quadruple Mass Gyroscope

The function and utility of the ringdown-based parametric modeling technique have been established using the URES device data. Here, the modeling technique is applied to data from a QMG device, notably exhibiting a very long ringdown time. This provides an opportunity to showcase the precise identifications possible using only very short segments of data. The upper plot in Fig. 3 depicts the sensed motion associated with a burst excitation of one of the device's proof masses. Three small highlighted sections of various lengths correspond to ringdown segments used for parametric modeling (the singular values of H motivate the choice $n = 1$ in all cases). The lower plot superimposes the magnitude portion of the derived single-input, single-output transfer functions associated with the three different parametrizations. These traces are nearly indistinguishable, all featuring a 2.791 kHz resonant peak.

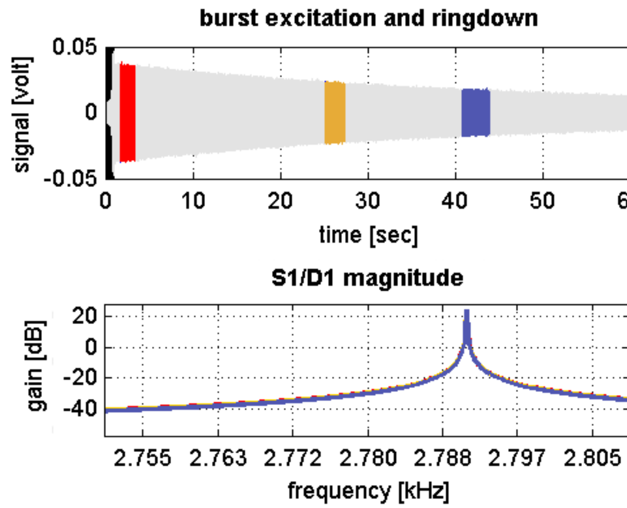

Fig. 3. Ringdown data and parametric-model based frequency response functions from QMG data. The top plot highlights three short segments of ringdown data used to construct parametric models. The lower plot shows the magnitudes of all three frequency response functions are nicely superimposed.

IV. Conclusion

A technique for generating models from ringdown experiments has been presented and verified using test data from several real MEM resonators. It was shown that the ringdown modeling technique is an effective replacement for a signal/network analyzer when studying the narrow band dynamics of high-Q MEM resonators and it offers a number of advantages. The use of parametric models offers facile and precise modeshape frequency and time-constant estimates. Another major advantage is the technique's immunity to parasitic coupling from drive to sense signals. The technique, though, has one notable drawback: the LTI parametrizations are inadequate for capturing the behavior of DUTs exhibiting nonlinear and non-stationary dynamics. Nonetheless, the technique may be used advantageously in such situations by parametrizing models from successive ringdown segments, and studying the model variations with time/ringdown amplitude. Future work will elaborate on this matter and analyze in greater detail the features that can be extracted from the parametrizations.

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