INTERACTION OF RELATIVISTIC ELECTRONS WITH RELATIVISTIC
SPACE-CHARGE WAVES DRIVEN BY LASER OR PARTICLE BEAMS

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ABSTRACT

In this paper novel aspects of the interaction of relativistic electrons with relativistic space charge waves, driven by laser or particle beams, are reviewed. A single particle interacting with such a wave can either be accelerated, in which case the wave is damped; or it can be decelerated, in which case it can emit radiation; or it can be focused toward or defocused away from the wave axis. These properties are usefully utilized in various plasma based concepts such as the Plasma Beat Wave Accelerator (PBWA), the Plasma Wake Field Accelerator (PWFA), the Plasma Wiggler, and the Plasma Lens.

INTRODUCTION

Whenever an electric field interacts with a charged particle, the particle can either gain or lose energy and it may also be deflected. It is well known that dense plasmas can support enormously large electric fields whenever ions and electrons are separated. When this charge separation is periodic, we have a space-charge density or a plasma wave. These waves obey the Bohm-Gross dispersion relation

\[ \omega_B^2 = \omega_p^2 + 3k_p^2v_e^2 \]  \hspace{1cm} (1)

where \( \omega_p \) is the plasma frequency, \( \omega_B \) and \( k_p \) are the frequency and wavenumber of the Bohm-Gross wave and \( v_e \) is the thermal velocity of the electrons.

A special class of plasma waves is the relativistic space-charge waves, so called because their phase velocity \( v_{ph} \) is so close to the speed of light that we can ascribe a Lorentz factor \( \gamma_{ph} \) to the wave. Defining the relativistic parameters \( \beta_{ph} = v_{ph}/c \) and \( \gamma_{ph} = (1-\beta_{ph}^2)^{-1/2} \), the Lorentz factor

- 1 -
is given by

\[ \gamma_{ph} = \left[ 1 - (1 - n/n_c) \right]^{-\frac{1}{2}} \equiv (n_c/n)^{\frac{1}{2}} = \omega_o/\omega_p \]  \hspace{2cm} (2)

where \( n_c \) is the plasma density and \( n_c \) is the critical density where \( \omega_o = \omega_p \). Using Poisson's equation it is easy to show that the longitudinal electric field of such a wave is given by

\[ E = \varepsilon \sqrt{n} \text{ V/cm} \]  \hspace{2cm} (3)

where \( \varepsilon = n_c/n \) is the fractional density modulation. This gives rather impressive values of between 1-10 GeV/m for a 10% modulated wave for plasma densities in the range \( 10^{16}-10^{18} \text{ cm}^{-3} \). We shall review in this paper how such large fields may be used to:

1) accelerate particles at ultrahigh accelerating gradients,
2) wiggle particles with extremely short effective wiggle periodicity, and
3) focus high energy electrons in an extremely short distance to a very small spot size.

ULTRAHIGH GRADIENT ACCELERATION OF PARTICLES

Laser Driven Plasma Accelerators

The basic mechanism for excitation of a longitudinal plasma wave by laser light is illustrated in Fig. 1. In this scheme proposed by Tajima and Dawson\(^1\), two parallel laser beams are co-propagated into a plasma. The difference frequency of the lasers is chosen to match the plasma frequency, so that the ponderomotive force of the beating pattern resonantly drives up the plasma wave. This accelerator is, therefore, named the Plasma Beat Wave Accelerator (PBWA)\(^2\).

\[ \omega_o \quad k_o \quad \omega_p \quad k_p \]

\[ \omega_1 \quad k_1 \quad \omega_p \quad k_p \]

BEATWAVE

\[ F_{NL} \]

PLASMA WAVE

2D DENSITY FLUCTUATIONS

Fig. 1. Schematic of the beat wave idea.
If $\alpha_i = eE/m\omega_0c$ is the normalized amplitude of the laser, it can be shown that the plasma wave exhibits secular growth with amplitude $\varepsilon(t) = n_1/n_0$, given by

$$\varepsilon(t) = \frac{\alpha_1\alpha_2}{4} \omega_p t.$$  \hspace{1cm} (4)

The maximum amplitude is reached when the plasma wave and the beat wave driver become out of phase with one another due to relativistic detuning\(^3\). The saturated amplitude is given by

$$\varepsilon_{\text{max}} = \left[ \frac{16}{3} \alpha_1\alpha_2 \right]^{1/3}.$$ \hspace{1cm} (5)

Expression (5) is valid in principle for $\alpha_1$, $\varepsilon \ll 1$, but simulations indicate that it is a reasonable approximation for values of $\alpha < 0.5$ and $\varepsilon < 0.8$\(^4\).

Causality demands that the phase velocity of the plasma wave has to be the same as the group velocity of the light waves. If the laser frequencies are much higher than $\omega_p$ then $v_{ph}$ is very close to $c$ and the injected relativistic particles may stay in phase with electric field of the plasma wave for a significant distance to be accelerated to high energy.

In practice the plasma wave will be driven by laser pulses that have a finite risetime $\alpha_i(t)$. Onset of plasma turbulence due to ion motion restricts the laser pulse widths that can be used to drive the plasma wave to a duration less than the ion plasma period. This turns out to be $\approx 43$ plasma wavelengths for a hydrogen plasma. The group velocity of the plasma wave behind the laser pulse is almost zero; thus the energy remains in the plasma oscillation until it is eventually dissipated by electrostatic wave-wave coupling. The trick in this plasma accelerator scheme is to efficiently beam load the plasma wave just behind the laser pulse to extract as much of the energy as possible, while maintaining the beam quality (energy spread and emittance).

**Plasma Wave Driven by Electron Bunches**

An alternative method for exciting a relativistic plasma wave was invented by J. Dawson who realized the energy density in existing electron or proton bunches can be comparable or exceed that in the most powerful of today's laser beams. In a scheme proposed by Chen, Dawson, Huff and Katsouleas\(^5\), a dense compact bunch is shot through a high density plasma. The space charge force of such a bunch displaces the plasma electrons and leaves behind a wake of plasma oscillations. The phase velocity of this wake, like that of a wake behind a boat, is tied to the velocity of the driving bunch $= c$. The acceleration mechanism in this scheme is exactly the same as in the PBWA. A trailing beam appropriately phased in this plasma wake can gain energy from it, much in the same way that a surfer gains energy from an ocean wave. This scheme is called the Plasma Wake Field Accelerator (PWFA)\(^6\).

If one uses a single symmetric bunch with a length that is much smaller than the wavelength of the plasma oscillation, the maximum energy gained by the trailing particles (assuming like particles) is limited to $< 2\gamma_b mc^2$ where $\gamma_b$ is the Lorentz factor associated with the driving particles. A solution to this dilemma was pointed out by Bane et al.\(^7\) who showed that by using a properly shaped bunch, the trailing particles could exceed this limit of $2\gamma_b mc^2$ imposed by the "wake field theorem." The precise shape of the driving bunch is not critical. It should have a
ramped density, with a risetime of ten or so plasma wavelengths, with a rather sharp cut-off. The cut-off time should be less than $\omega_p^{-1}$.

The basic mechanism behind obtaining large wake fields with shaped bunches is as follows$^8$. When a shaped electron bunch enters the plasma the plasma sees an excess of negative charge. Since the charge builds up slowly, the plasma electrons move out both transversely and longitudinally to neutralize the bunch field. The shielding continues until the tail of the bunch exists the region. Then suddenly the plasma, which was nearly neutral, is left with a non-neutral space charge of amplitude that is equal to the charge density at the peak of the driving beam. Clearly, the peak beam density should not be greater than the plasma density. Each plasma electron at this point acts like a spring pulled out to its maximum amplitude and released, setting up an oscillation at frequency $\omega_p$ and the phase velocity tied to the driving bunch velocity. This scheme is conceptually similar to other wake field transformer schemes$^9$ which utilize the wake fields of a low voltage, but high current driving beam to accelerate a low current bunch to high energies. In this scheme the plasma rather than a structure, acts as the transformer which has a high transformer ratio.

Computer Simulations of Plasma Wave Excitation

**Plasma Beat Wave Accelerator.** Figure 3 shows the results of a 1-D, relativistic, self-consistent particle simulation$^{10}$ in which the beating lasers are injected from the right. In 3(a) the beating pattern of the two lasers (which are rising in intensity from left to right) is clearly visible. In 3(b) we see the longitudinal electric field pattern of the plasma wave. The peak amplitude of the laser fields was 100 GeV/m, whereas the beat excited plasma wave saturated due to relativistic detuning at 55 GeV/m. The results of the numerical solution to the fluid equation$^3$ are also shown for comparison with the particle code prediction. The two are seen to be in excellent agreement.
Recently the computer models have been extended to 2D to examine the transverse stability of the lasers and the plasma wave. A parameter regime has been identified which essentially reproduces the 1D results. Basically, if the laser beams are less than $\omega_p^{-1}$ long, then the ion motion is not too worrisome and one can obtain a fairly planar plasma wave right up to saturation. There are 2D effects that influence particle acceleration such as the radial electric fields, but these are common to both the PBWA and the PWFA will be discussed later.

**Plasma Wake Field Accelerator.** Figure 4 shows the numerical solutions of 1D wake fields produced by various bunch shapes. The ideal bunch shape is one for which the retarding field within the bunch is uniform. Such an ideal situation can only be reproduced by placing a delta function precursor in front of a triangular bunch. Without such a precursor the wake field inside the triangular bunch $E_\omega$ is small, although much smaller than the wake field behind it $E_\omega$ is a displaced sinusoid. Defining the transformer ratio $R$ as $E_\omega/E_\omega$ we can see that it is now possible to obtain large values of $R$.

The linearly ramped and sharply cut-off bunches can only be approximated in an experiment. A more realistic driving bunch would have a long Gaussian rise time and a short Gaussian fall time. Such a bunch turns out to be more desirable. The retarding field $E_\omega$ inside the Gaussian bunch is smoother. The driving beam particles, therefore, will slow down roughly at the same
rate leading to less distortion of the bunch. The sharp cut-off of the driving bunch is not too critical as long as it is shorter than c/\omega_p.

The beam shaping and cut-off requirement may prove to be one of the toughest technological challenges in realizing the PWFA. For example in a plasma density of 10^{15} \text{ cm}^{-3}, the cut-off time must be on the order of one picosecond.

**Acceleration Mechanism and Limit to Energy Gain**

In the PBWA, the phase velocity of the plasma wave is \( c(1 - \frac{1}{2}\omega_p^2/\omega_0^2) \), where \( \omega_p \) is the plasma frequency and \( \omega_0 \) is the frequency of the laser. The ratio \( \omega_p^2/\omega_0^2 \) is equal to the ratio \( n_e/n_0 \), where \( n_e \) is the electron density and \( n_0 \) is the critical density at which \( \omega_0 = \omega_p \). The relativistic factor associated with this phase velocity \( \gamma_{ph} = \omega_0 / \omega_p \). In the wave frame a particle can only see an accelerating and focusing field for a maximum distance of quarter of a wavelength. In the lab frame, after Lorentz transformation, this turns out to be \( \frac{1}{4} \gamma_{ph}^2 c/\omega_p \). Since the longitudinal accelerating field is Lorentz invariant, the maximum energy gain limited by this dephasing is \( \int \mathbf{E} \cdot d\mathbf{l} = c/4\sqrt{n_e} \gamma_{ph}^2 c/\omega_p \), where \( \epsilon \) is the fraction of the density perturbation.

For optimum energy gain we want the plasma density to be as high as possible. At the same time the waves should have a high phase velocity. In other words, the ratio of \( \omega_0/\omega_p \) should be as large as possible. An example based on using a KrF laser, shown below, gives a maximum energy gain of 16 GeV limited by phase slip which is probably the best that can be done in a
single stage of the PBWA.

<table>
<thead>
<tr>
<th>Table 1: Single Stage Parameters for the PBWA</th>
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<tbody>
<tr>
<td>Laser wavelength</td>
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<tr>
<td>Plasma density</td>
</tr>
<tr>
<td>$E_z$</td>
</tr>
<tr>
<td>$\varepsilon$</td>
</tr>
<tr>
<td>Length</td>
</tr>
<tr>
<td>Energy gain</td>
</tr>
<tr>
<td>Plasma wavelength</td>
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</table>

What are the laser requirements for such a compact accelerator? There are considerable uncertainties at present in evaluating this. The main uncertainty is in estimating the laser to plasma wave coupling efficiency. At a first glance it would appear that the maximum efficiency is simply $\omega_p/\omega_0$ as demanded by the Manley-Rowe relation for a three wave process, or the conservation of wave action. This is fortunately not the case. In the beat excitation process, we get a series of frequency upshifted (anti-Stokes) and frequency downshifted (Stokes) sidebands, each shifted in frequency from its neighbor by $\omega_p$. If we could preferentially excite the frequency downshifted sidebands, while for each cascade transfer $\omega_p/\omega_0$ amount of energy to the plasma wave, we might improve the efficiency. In principle the two original electromagnetic waves will cascade until they downshift to frequencies close to $\omega_p$. Unfortunately, because of dispersion, the group velocities of the cascaded waves are not the same and the cascades might actually broaden the k spectrum of the plasma wave. Only experiments will determine this important issue. Nevertheless, we give some idea of what might be required from the laser. These are to be taken as non-optimized parameters.

<table>
<thead>
<tr>
<th>Table 2: Laser and Plasma Wave Parameters for Example in Table 1</th>
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<tr>
<td>Laser-plasma wave coupling efficiency</td>
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<tr>
<td>Build-up time of the plasma wave</td>
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<tr>
<td>Laser pulse-width (square pulse)</td>
</tr>
<tr>
<td>Laser beam diameter</td>
</tr>
<tr>
<td>$\alpha = \nu/\nu_0$</td>
</tr>
<tr>
<td>Laser energy</td>
</tr>
<tr>
<td>Plasma wave diameter (plane wave)</td>
</tr>
<tr>
<td>Plasma volume</td>
</tr>
<tr>
<td>Plasma wave energy</td>
</tr>
</tbody>
</table>
Clearly a proof-of-principle experiment to a few GeV level is not impossible with a modest extrapolation of todays KrF laser technology. In order to gain more energy one has to either contemplate staging or invent a scheme for phase-locking the particles\textsuperscript{12}.

In the PWFA, the situation is somewhat better. Here the requirement that the driving bunch be truncated in a time \( < \omega_p^{-1} \) means that plasma densities must be at most \( 10^{15} \text{cm}^{-3} \). This assumes that 1 ps is probably the sharpest cut-off time that can be achieved. The phase velocity of the wave; however, can be arbitrary depending on the beam energy. For beams with low energy, as shown in Fig. 5, dephasing still limits the maximum energy gain but for high energy beams pump depletion limits the energy gain.

![Graph](image)

Fig. 5. Maximum energy gain for a fixed beam current of 125 amps in PWFA. The spot size is assumed to be \( c/\omega_p \).

The useful design equations which describe the acceleration gradient and final energy which can be gained in the PWFA are:

\[
eE = \varepsilon \sqrt{n_e} \text{ eV/cm} \tag{6}
\]

\[
\Delta \gamma = N \pi/\gamma_b \frac{\varepsilon \gamma_b}{N + \varepsilon \gamma_b} \tag{7}
\]

where \( \varepsilon \) is defined as the ratio of the peak driving beam density to the plasma density (\( \varepsilon = n_b/n_e < 1 \)) and \( N \) is the length of the driving bunch normalized to the plasma wavelength. The two other design criteria are the spot size (\( 2n_b \)) and the cut off time (\( \tau \)) of the driving bunch.
\[
2n_b > c/\omega_p \approx 5 \times 10^5 \sqrt{n_e} \text{ cm} \\
\tau < \pi/\omega_p \approx 5 \times 10^{-5} \sqrt{n_e} \text{ sec}
\]

If the spot size is chosen to match $c/\omega_p$, the normalized bunch density $n_b/n_e$ can be calculated in terms of the current $I$ (Amps)

\[
\frac{n_b}{n_e} = 10^{-3} I
\]

Thus, $n_b/n_e = 1$ for $I = 1 \text{ kA}$. The corresponding wake field amplitude is simply

\[
eE = (n_b/n_e)^{\sqrt{n_e}} V/cm = 10^{-3} I(\text{amps})^{\sqrt{n_e}(\text{cm}^{-3})} V/cm.
\]

**Transverse Effects and Instabilities**

In addition to the longitudinal $E_z$ accelerating field the plasma wave in the PBWA has a significant radial field component $E_r$. This force is associated with the transverse variation of the plasma density due to the spatial intensity variation of the pump beams. The radial fields are both focusing and defocusing and the relative magnitude of $E_r$ compared to $E_z$ depends on the transverse dimension of the plasma wave. Figure 6 shows radial variation of $E_r$ and $E_z$ for a Gaussian laser beam of size $k_pR = 3$. One can see that the maximum value of $E_r$ is $0.5E_z$. Figure 7 shows $E_z$ and $E_r$ variation with $z$. It can be seen that the accelerating particles will see a focusing and an accelerating field over only a quarter of the wavelength.

In the beat wave accelerator it is of some importance to know not only the field distribution in the focal plane of the lasers but also their phase variation. This is particularly true if diffraction limited beams are not used.

In both the PBWA and the PWFA most of the usual laser or beam plasma instabilities are avoided by using driving beams that are only a few picoseconds long. An extensive discussion of the possible instabilities is to be found in Refs. (4) and (11). We note here that in the case of the PBWA, one instability that cannot be suppressed by simply going to very short laser pulses is the relativistic self-focusing. The threshold for this instability ($10^{12} \text{ W for CO}_2$) is rather high. It may be possible to actually use this instability to keep the laser beams focused over a distance greater than the Rayleigh length, thus utilizing the laser energy more efficiently.

**Beam Loading and Efficiency**

The term "beam loading" refers to the fraction of the energy in the accelerating mode that can be transferred to the accelerating beam. Clearly, we want to maximize this while keeping the energy spread and the emittance of the beam as small as possible. Because of the extremely small size of the accelerating buckets in the plasma accelerators, efficient beam loading consistent with emittance and energy spread requirements is going to be a challenge to achieve.
Fig. 6. Self-consistent longitudinal, $E_z$, and transverse, $E_r$, electric field profiles of the plasma wave.

Fig. 7. Schematic of the variation of $E_z$ and $E_r$ with $z$. 
Katsouleas et al. have calculated the beam loading achievable in both the beat wave and the wake field case\textsuperscript{13}. In the former case we assume that a very short laser pulse excites the plasma wave and the accelerating bunch or bunches are injected immediately behind the laser pulse before the plasma wave loses its coherence. In the wake field case, the second bunch should ideally be immediately behind the driving bunch.

In order to reduce the energy spread of the beam without lowering the beam loading efficiency, the use of an accelerating bunch that is ramped down in density has been suggested\textsuperscript{13}. In the 2D analysis beam loading efficiencies of up to 20\% appear feasible\textsuperscript{13}. Of course, beam loading efficiency is only one of the factors determining the overall efficiency of the plasma accelerator. The efficiency $\eta$ is given by

$$\eta = \eta_1 \eta_2 \eta_3$$

where

$\eta_1 =$ wall plug to laser (10\% maximum) or wall plug to driver beam from linac (30\%)

$\eta_2 =$ laser to plasma wave (10\% for $\omega_d/\omega_p = 30$) or beam to plasma wave (50\%)

$\eta_3 =$ plasma wave to accelerated beam (20\%)

Thus

$$\eta_{\text{beat wave}} \approx 2 \times 10^{-3} \quad \text{and} \quad \eta_{\text{wake-field}} \approx 3 \times 10^{-2}.$$ 

Experiments

Experiments are being carried out at UCLA\textsuperscript{15}, Rutherford Laboratory\textsuperscript{16} (U.K.), ILE (Japan)\textsuperscript{17}, INRS (Canada)\textsuperscript{18}, and elsewhere on the reproducible excitation of the relativistic plasma wave by beating laser beams and controlled acceleration of externally injected test particles by the plasma wave. In a recent UCLA experiment\textsuperscript{15}, the relativistic plasma wave was excited by beating the 9.6 $\mu$m and 10.6 $\mu$m lines of a CO$_2$ laser, with a modest intensity of $2 \times 10^{13}$ W/cm$^2$, in a $10^{17}$ cm$^{-3}$ density plasma. The plasma wave electric field was inferred from Thomson scattering of a probe laser beam to be $10^5$ MeV/m, a substantial improvement over the current benchmark gradient for accelerators. The results of the UCLA experiment that conclusively demonstrated the excitation of the relativistic plasma wave by beating lasers are shown in Fig. 8. A new mechanism which saturates the beat-excited plasma wave in this parameter regime was discovered\textsuperscript{19}. The relativistic plasma wave saturates, on the time scale of a few picoseconds, by coupling to other plasma modes which have a much lower phase velocity, via an ion ripple due to stimulated Brillouin scattering of the laser beams. A scaled up experiment which will demonstrate controlled acceleration of injected electrons is currently underway at UCLA. Experiments on the wake field concept are currently underway at the Argonne Wake Field Measurement Test Facility by the Argonne/University of Wisconsin collaboration.

Prospects of a Plasma Accelerator

So what are the prospects of building a future collider using either the PBWA or the PWFA scheme? Clearly, it is too early to tell. Even if the proposed proof-of-principle
Fig. 8. Infrared scatter spectrum (a) showing Stokes and anti-Stokes sidebands in the forward direction and ruby Thomson scattering (b) of the probe beam showing frequency shifted signal by $\omega_p = \omega_{\text{beat}}$. (UCLA experiment$^{15}$)

Experiments demonstrate high gradient, controlled acceleration, thereby confirming the promise of these schemes, several important technological issues still need to be addressed. The first, is the effect of plasma fluctuations on the reproducibility of the acceleration process, and on the ability to collide two beams that must be aligned to within a few angstroms of one another. The second, is the efficiency of the accelerator. It is clearly not enough to miniaturize the accelerator; it must also be cheaper. Staging is another important issue, particularly for the PBWA.
PLASMA WAVE WIGGLERS FOR SYNCHROTRONS AND FEL's

The geometry of the plasma wiggler is illustrated in Figure 9. In this simplified picture, the wiggler consists of a purely electric field oscillating perpendicular to the electron beam with a frequency $\omega_p$ but with no spatial dependence since $k$ of the radiation is transverse to $k_p$. The amplification of radiation by such a field in vacuum has been modelled recently by Y. T. Yan and J. M. Dawson\textsuperscript{20}. Using a plasma wave as a purely electric wiggler is attractive for two reasons: First, the effective wiggler wavelength ($\sim 2\pi c/\omega_p$, typically of order 100 $\mu$m) is shorter than that available with conventional magnet wigglers and second, the effective wiggler strength can be extremely large (equivalent to 1 megagauss wiggler fields at 100 $\mu$m wavelength).

Fig. 9. Geometry of the plasma wiggler FEL.

FEL Mechanism in a Plasma Wiggler

The equivalence between magnetic, electromagnetic and purely electric (plasma) wiggler schemes can most easily be seen from the point of view of the electrons. Figure 10 illustrates the fact that although the three cases appear quite different in the laboratory frame, in the electron frame all appear to be electromagnetic waves. Here, $\omega_o$ and $\omega_p$ represent the frequencies of the electromagnetic and electrostatic (plasma) waves respectively, and $k_o$, $k_w$ represent the wavenumbers of the electromagnetic and magnetic wigglers respectively in the laboratory frame. The purely electric wiggler has a zero $k$ whereas the purely magnetic wiggler has a zero $\omega$. In the electron frame the momentum four-vectors are
\[
\begin{bmatrix}
[k'_{\omega}] \\
[10] \\
\end{bmatrix} = \begin{bmatrix}
\gamma & i\beta \\
-i\beta & \gamma \\
\end{bmatrix} \begin{bmatrix}
k_o \\
[10] \\
\end{bmatrix} \text{electromagnetic}
\]

\[
\begin{bmatrix}
k_w \\
0 \\
\end{bmatrix} \text{purely magnetic}
\]

\[
\begin{bmatrix}
0 \\
10p \\
\end{bmatrix} \text{purely electric}
\]

**Fig. 10.** Equivalence among magnetic, electromagnetic, and purely electric (plasma) wigglers.

For \(\beta = 1\), the transformed quantities satisfy \(\omega' = k'c\) for all the cases except their wavelengths are different; \(\lambda' = \lambda_w / \gamma\) (magnetic), \(\lambda_o / 2\gamma\) (electromagnetic), and \(2\pi c / \omega_p \gamma\) (purely electric or plasma). Each electromagnetic wave has a wiggler strength \(a_w = eE' / m\omega'c = eA / mc^2\) equal to its corresponding value, \(eB\lambda_w / 2\pi mc^2\) (magnetic), \(eE_o / m\omega_o c\) (electromagnetic) and \(eE_p / m\omega_p c\) (purely electric or plasma) in the lab frame. In the plasma wiggler, as in the other two cases, the noise that seeds the lasing process comes from Compton scattering, with the radiated frequency in the exact forward direction \(\omega_r \equiv 2\gamma^2 \omega_p\). Once, there is spontaneous emission at \(\omega_r\), there is a resonant ponderomotive force on the radiating electrons. This force is \(F = -e v_w \times B_r / c\), where \(v_w\) is the wiggle velocity and \(B_r\) is the magnetic field of the spontaneous...
or stimulated radiation. For a plasma wave (wiggler) of the form \( E = E_p \sin(\omega_p t) a_x \) and radiation field

\[
B_r = \frac{c k_r}{\omega_r} E_r \sin (k_z z - \omega_r t + \phi) a_y ,
\]

the ponderomotive force is

\[
F = \frac{e}{2} \frac{e E_p}{m \omega_c c} \left( \frac{e E_r}{m \omega_c c} \right) mc^2 k_r \times \left[ \sin [k_z z - (\omega_r - \omega_p) t + \phi] + \sin [k_z z - (\omega_r + \omega_p) t + \phi] \right] \tag{12}
\]

In order to obtain a net energy exchange from the electron beam to the radiation, the electrons must have velocities in a range that is slightly greater than the phase velocity of the ponderomotive bucket. The second term in Eq. (12) has a phase velocity slightly greater than \( c \) and thus time averages to zero. The phase velocity of the resonant (first) term is Eq. (12) is

\[
v_\phi = \frac{\omega_r - \omega_p}{k_r} = \frac{\omega_r}{k_r} \left[ 1 - \frac{\omega_p}{\omega_r} \right] \equiv c \left[ 1 - \frac{\omega_p}{\omega_r} + \frac{\omega_p^2}{2\omega_r^2} \right] \tag{13}
\]

The last term \( \omega_p^2/2\omega_r^2 \) in Eq. (13) arises because \( \omega_p/k_r \) is not exactly \( c \) in a plasma. However, for relativistic beams \( \gamma = 1 - \frac{1}{v^2/c^2} \), so that this term can be neglected. Thus, the resonance condition, \( v = v_\phi \) and \( \gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \) becomes \( \omega_r \leq 2\gamma^2 \omega_p \), in complete analogy to conventional FEL's.

The effective maximum undulator strength parameter \( a_w \) of the plasma wiggler can be easily estimated from 1-D Poisson's equation: \( \nabla \cdot E = -4\pi n_1 \) where \( n_1 \) is the perturbed electron density associated with the plasma wave. The maximum density rarefaction occurs when \( n_1 = n_0 \) known as the cold plasma wavebreaking limit. In this limit, \( |k_p E_{p,\text{max}}| \approx 4\pi e n_0 \) or \( \omega_p/k_p c = 1 \). Thus the maximum value of \( E_p \) attainable in the plasma corresponds to \( a_w = 1 \), independent of the plasma density. The effective maximum magnet strength of a plasma wave is

\[
B_{\text{eff}} \text{ (gauiss)} = \frac{2\pi mc^2}{e\lambda_p} a_w \leq \frac{2\pi mc^2}{e\lambda_p} \equiv \frac{10^4}{\lambda_p \text{(cm)}} \equiv 3 \times 10^{-3} \sqrt{n_0 \text{(cm}^{-3})} \tag{14}
\]

It can be seen that effective magnetic field strengths of order 1 megagauss are possible in a plasma of density \( 10^{17} \text{ cm}^{-3} \).

The undulating mechanism of the plasma wiggler is shown schematically in Fig. 11. The electron beam is assumed to be thin; diameter of the beam is small compared to \( \lambda_p \approx 2\pi c/\omega_p \). In Fig. 11(a), the electric field \( E_p \) of the plasma wave deflects the electrons toward the left of the figure where there is an excess of ions. Since both the electrons and the plasma wave are moving at the speed of light, by the time the electrons move up by a distance \( \pi c/\omega_p \) the plasma wave moves to the right by the same amount. The electric field \( E_p \) now drives the electron to the right. This is shown in Fig. 11(b). At a time \( \pi/\omega_p \) later still, Fig. 11(c) the electrons move up another \( \pi c/\omega_p \) while the plasma wave moves to the right by the same amount. The electric field \( E_p \) once
Fig. 11. Undulating mechanism of the plasma wiggler. Solid contours are increasing in density and dotted contours are decreasing in density compared to the background plasma density.
again drives the electrons to the left. Thus the apparent wiggler wavelength is $\lambda_{w} = 2\pi c/\omega_{p}$, which is the same as the plasma wave wavelength

$$\lambda_{w}(\text{cm}) \equiv 3 \times 10^{6} / \sqrt{n_{e}}(\text{cm}^{-3})$$

Thus, for plasma densities in the range $10^{13}$ to $10^{18} \text{ cm}^{-3}$, $\lambda_{w}$ ranges from 1 cm to 30 $\mu$m.

**Plasma Wiggler Excitation**

As discussed previously the relativistic plasma waves, used for wigglers, can be excited by two schemes known as laser beat wave excitation$^{2}$ and wakefield excitation$^{5}$. Both are actively being investigated for accelerator applications in which the electrons are injected in the same direction as the phase velocity of the wave. If instead, the electrons are injected perpendicular to the plasma wave (parallel to the wavefronts), they are wiggled transversely causing them to radiate.

Table 3 below shows the possible combination of laser wavelengths that could be used to excite plasma wigglers of various wavelengths. The intensity required, in Table 3, is for $a_{w} = 0.5$ assuming that both lasers have square wave shapes and equal intensities. The intensity required changes somewhat when one takes into consideration the risetime of the laser pulse$^{11}$.

We have carried out two dimensional computer simulations to examine the transverse (to $k_{p}$) coherence of the plasma wave$^{4,11}$. In order to act as a wiggler the potential contours of the plasma wave must be planar. The results of a typical 2D simulation, using a fully relativistic, electromagnetic particle code, are depicted in Fig. 12. The simulation parameters were as follows: two laser beams $\omega_{e} = 5\omega_{p}$ and $\omega = 4\omega_{p}$ each with an r.m.s. $a_{w}^{0.1} = 0.4$, a transverse intensity profile of $\cos^{2}y$ and a beam width of 30 $c/\omega_{p}$ were injected into a homogeneous plasma 60 $c/\omega_{p}$ long and 60 $c/\omega_{p}$ wide with a temperature of 2.5 keV. The laser beams had a cubic rise from zero to maximum intensity in 300$/\omega_{p}$, after which the laser intensity remained constant. The ion-to-electron mass ratio was 1836 and temperature ratio was 1. Thus for an IR laser with wavelengths 9.6 $\mu$m and 12 $\mu$m, the parameters were: a hydrogen plasma with $n_{e} \sim 4.9 \times 10^{17} \text{ cm}^{-3}$, $\lambda_{w} \sim 47 \mu$m, laser risetime $\sim 7.6 \text{ ps}$, laser intensity $\sim 2 \times 10^{15} \text{ W/cm}^{2}$, beam width $\sim 25$ wavelengths and system length 50 wavelengths. The simulation parameters were chosen to study both the growth and saturation of the plasma wave and its transverse coherence and stability.

In Figure 12(a) the laser beat wave contours at $150/\omega_{p}$ are shown as the beams propagate from left to right. Figure 12(b) shows the contour plot of the potential of the plasma density wave at the same time. According to the fluid theory, the plasma wave should reach peak amplitude at this time. The plasma wave wavefronts in the contour plot are seen to be planar. A section through the center of the transverse axis of the longitudinal electric plot is shown in Figure 12(c). An extremely coherent plasma wave builds up rapidly in time/Space and saturates at $a_{w} \approx 0.5$ in excellent agreement with the value expected from the fluid theory$^{4}$. We find that the plasma wave typically remains coherent for a few ion plasma periods which translates to between a few to few tens of picoseconds for parameters in Table 3 before disrupting due to ion dynamics.
Fig. 12. Laser beat excitation of plasma wiggler. (a) Laser beat contours as lasers propagate from left to right, (b) wavefronts of the relativistic plasma wave, and (c) longitudinal electric field or, equivalently, $a_w$ on axis. The dotted line is the predicted value of the $a_w$ on axis from fluid theory.
Table 3: Parameters for Exciting Plasma Wigglers of Different Wavelengths

<table>
<thead>
<tr>
<th>Laser Type</th>
<th>Laser Wavelengths (μm)</th>
<th>Intensity (W/cm²)</th>
<th>Wiggler Wavelength (μm)</th>
<th>Approximate Plasma Density (cm⁻³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CO₂</td>
<td>9.6 and 10.6</td>
<td>2×10¹⁴</td>
<td>100</td>
<td>10¹⁷</td>
</tr>
<tr>
<td>CO₂</td>
<td>9.6 and 10.3</td>
<td>2×10¹⁴</td>
<td>140</td>
<td>5.8×10¹⁶</td>
</tr>
<tr>
<td>CO₂</td>
<td>10.6 and 10.3</td>
<td>2×10¹⁴</td>
<td>364</td>
<td>10¹⁶</td>
</tr>
<tr>
<td>Nd:glass</td>
<td>1.05 and 1.06</td>
<td>2×10¹⁶</td>
<td>110</td>
<td>10¹⁷</td>
</tr>
</tbody>
</table>

In this scheme we need a wide plasma wave which has $E_z$ nearly constant over 100 or so wiggle periods. The actual length of the wave (in the direction of the driving lasers) need only be 2 or 3 wavelengths with the central potential trough used for wiggling the electron beam. The laser beam pump depletion length is given by²¹:

$$L_d \geq \frac{\omega_p^2}{\omega_c^2} \frac{c}{\omega_p} \frac{4}{a_w^{\text{plasma}}}$$

This assumes that the maximum effective length of the laser pulse is determined by relativistic detuning³. For parameters in Table 3, $\omega_c/\omega_p \geq 10$ and $a_w^{\text{plasma}} \approx 0.5$, $L_d$ can be hundreds of plasma wavelengths long. Thus by using small F number cylindrical focus optics which produce a line focus that is greater than 100λₘ wide but only has a Rayleigh length of about 3λₘ, it may be possible to reuse the laser light.

FEL PARAMETERS FOR GAIN

As mentioned earlier, the plasma wigglers in principle are exactly the same as a static magnetic wiggler of comparable wavelength and strength. To the zeroth order, the requirements for obtaining a significant gain are exactly the same for the plasma wigglers as for a static magnetic wiggler as far as the radiating electron beam is concerned. Here we work out some typical parameters assuming that we have a wiggler with 100 wiggle periods with λₘ of 100 μm and an $a_w$ of 0.5. We work out beam parameters for obtaining a single pass gain of 20.

We start with the maximum small-signal gain formula for an FEL driven by a perfectly monoenergetic beam:\⁸

$$G = 2 \times 10^{-3} \frac{a_w^2}{1 + a_w^2} N^2 \frac{1}{1/\gamma}$$  \hspace{1cm} (8)
where the optimum beam cross section \( \sigma = N_{\gamma \lambda} \lambda \sqrt{3} \) was assumed. The FEL beam parameters consistent with the energy spread requirement \( \Delta \gamma / \gamma = 1 / (2N) \) and divergence angle \( \Delta \theta < 1/(\gamma \sqrt{N}) \) are listed below in Table 4.

The gain formula provides valuable insight, but is not quantitatively reliable. By not taking into account the change in the amplitude of the radiation field it underestimates the actual gain; by neglecting the energy spread it overestimates the gain.

The angular contribution to the energy spread is \( \Delta \gamma / \gamma = \gamma^2 \Delta \theta^2 \). The emittance consistent with this is \( \pi \alpha \Delta \theta \) which for the parameters listed in Table 4 is between 0.76-0.07 mm mrad. There is another constraint on the electron beam emittance. An electron entering the wiggler at an angle \( \theta \) to the beam axis has a displacement \( N \lambda \Delta \theta \) after it has traversed the length of the wiggler. Unless this displacement is much smaller than the plasma wave wavelength a significant fraction of the electrons will absorb power from the electromagnetic radiation. Thus letting \( N \lambda \Delta \theta < \lambda / 10 \), we have for \( N = 100 \) that \( \theta = 1 \) milliradian and \( \epsilon = 0.1-0.035 \) mm mrad for the parameters listed in Table 4. Such high quality beams will be required to obtain the FEL action in the short wavelength regime no matter what kind of wiggler is used.

### Table 4: Parameters for Obtaining Gain

<table>
<thead>
<tr>
<th>Beam Energy (( \gamma ))</th>
<th>Radiated Wavelength (( \lambda_r(\mu m) ))</th>
<th>Beam Current (amps)</th>
<th>Beam Diameter (( \mu m ))</th>
<th>Beam Density (( cm^{-3} ))</th>
<th>Divergence Angle ( \Delta \theta (rad) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>0.22</td>
<td>75</td>
<td>36</td>
<td>( 1.5 \times 10^{15} )</td>
<td>6.7 \times 10^{-3}</td>
</tr>
<tr>
<td>20</td>
<td>0.125</td>
<td>100</td>
<td>27</td>
<td>( 3.6 \times 10^{15} )</td>
<td>5 \times 10^{-3}</td>
</tr>
<tr>
<td>50</td>
<td>0.02</td>
<td>250</td>
<td>11</td>
<td>( 5.4 \times 10^{16} )</td>
<td>2 \times 10^{-3}</td>
</tr>
</tbody>
</table>

Detailed calculations and simulations of the plasma wiggler FEL which take into account the variation in \( a_w \) across the beam cross section, edge effects, beam energy spread, emittance and plasma wave harmonics are being carried out presently. It turns out that the plasma medium imposes some additional constraints on the electron beam. These are discussed in detail in Ref. (22).

THE PLASMA LENS

At the interaction point of a TeV class \( e^-e^+ \) collider, the colliding beams will be submicron size in order to achieve the necessary luminosity. The luminosity is defined as

\[
L = \frac{N^2 f H}{4 \pi r_0^2}
\]  

(30)
where $N$ is the number of particles in the colliding bunch, $f$ is the repetition rate, $H$ is the so-called pinch-enhancement factor which is on the order 1-5 and $r_0$ is the bunch radius. It is clear that besides increasing the number of particles per bunch, the luminosity of an accelerator can be increased by reducing the spot size at the collision point. If the bunch had no transverse energy spread or had zero emittance then a perfect lens would focus the bunch to a point. However, for a given emittance beam one obtains a smaller spot size as the focusing strength of the lens is increased. This is analogous to geometric optics where, for a given divergence of the optical beam, we obtain the smallest spot size with the shortest focal length lens, assuming no other aberrations are present.

A plasma has the potential of providing focusing forces that are up to three orders of magnitude larger than conventional magnetic lenses used currently in colliders. The physical mechanism for this effect can be visualized as follows. The self-force on a relativistic electron beam in vacuum is nearly zero, because the space-charge repulsive force is nearly balanced by the inward Lorentz force. When such a beam enters a plasma, however, it expels some or all of the plasma electrons so that there is charge neutrality within the beam.

Since the beam's current is not neutralized the $\frac{\nabla \times B}{c}$ pinching force still exists and the net result is a focusing of the beam. This mechanism is analogous to that which occurs at the final focus of an $e^+e^-$ collider. This is called disruption in accelerator terminology and gives the pinch enhancement factor $H$ in the expression for luminosity.

In order to act as a lens, the focusing force of plasma lens must increase linearly with the transverse spatial dimension, $y$. The $\nabla \times B$ pinching force can be straightforwardly shown to be $en_\parallel4\pi y/c$ which satisfies this requirement. The focusing gradient in useful units is

$$\sim 5 \left[ \frac{n_\parallel}{10^{15} \text{ cm}^{-2}} \right] \text{ MGauss/cm}.$$ 

There are two plasma lens concepts at present. They are called the thin and the thick plasma lens. Thin and thick refer to the plasma's thickness compared to the lens's focal length. For the thin case the beam is hardly pinched before it exits the plasma, while for the thick case the beam is completely pinched while in the plasma. When the pinching occurs in the plasma, the beam dynamics unavoidably become both nonlinear and convolutional. The problem then is no longer tractable analytically, therefore particle simulations have to be used. Both the thin and the thick plasma lenses are analyzed in detail in Ref. (23).

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