Tunable Radiation Generation
Using Underdense
Ionization Fronts

R.L. Savage Jr., W.B. Mori,
C. Joshi, T.W. Johnston, G. Shvets

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University of California at Los Angeles
Department of Electrical Engineering
56-125B Engineering IV Building
Los Angeles, CA 90024

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TUNABLE RADIATION GENERATION USING UNDERDENSE IONIZATION FRONTS

R. L. Savage Jr., W. B. Mori, and C. Joshi
Electrical Engineering Dept., U C L A, Los Angeles, CA 90024

T. W. Johnston
INRS-Energie, Varennes, Quebec J3X-1S2, Canada

G. Shvets
Plasma Fusion Center, M I T, Cambridge, MA 02139

ABSTRACT

Recently, underdense, relativistically-propagating, laser-produced ionization fronts have been used to upshift the frequency of impinging electromagnetic radiation from 35 GHz to more than 138 GHz in a continuously tunable fashion [1]. These experiments utilized a resonant microwave cavity operating near its cutoff frequency through which an ionizing laser pulse propagated. We present a derivation of the expected upshifted frequencies, carried out in the laboratory frame using a space-time picture of the upshifting process, and a detailed derivation of the reflection and transmission coefficients within the waveguide.

INTRODUCTION

Almost 25 years ago [2], it was proposed that the frequency of electromagnetic radiation directed toward, and reflected from, a rapidly moving boundary between a region of plasma and one of neutral gas would be altered by the relativistic Doppler effect [3, 4]. Thus, if such a boundary, or ionization front, were to propagate at a velocity close to that of light, frequency upshifts of many orders of magnitude would be obtained. However, in order for the incident radiation to reflect from the front, the density of the plasma would have to be extremely high. Advances in laser technology have enabled the creation of relativistically propagating ionization fronts via photo-ionization [5]. These have led to a renewed interest in reflection front ionization fronts [6, 7, 8]. As an ionizing laser pulse passes through a region of neutral gas, a front which propagates at the velocity of the laser light is created. While such a front may be highly
relativistic, in most cases it is not possible to make it dense enough to reflect an impinging electromagnetic wave. In order to reflect the impinging radiation, the front must be overdense; that is, the density of the plasma in the front must be above the critical density for electromagnetic radiation when viewed from the rest frame of the front. The critical density is defined as that density for which the characteristic plasma frequency, $\omega_p$, equals the radiation frequency. The plasma frequency is defined by $\omega_p^2 = 4\pi n_0 e^2/m$ where $n_0$ is the plasma density, $m$ is the electron's mass, and $e$ is its charge. Because the frequency of the incident radiation is much higher when viewed from this frame and the plasma frequency is Lorentz invariant, creation of an overdense front requires that the plasma be more dense than for a stationary boundary.

The early theoretical work in this area concentrated on reflection from overdense ionization fronts, but more recently it has been shown that significant upshifts are possible even for underdense fronts, where the incident wave is transmitted into the plasma [6]. Furthermore, under certain conditions the upshifted radiation may propagate in the plasma in the same direction as the ionization front. This type of “reflection” from an underdense front yields frequency shifts that are much different from those given by the usual Doppler effect.

We have experimentally confirmed most of the predictions of the underdense ionization front theory in a recent set of experiments. These experiments utilize a resonant microwave cavity that is filled with neutral gas (see Fig. 1). As an ionizing laser pulse passes through the cavity, the frequency of the microwave radiation is upshifted. The front encounters
a standing microwave field. It is composed of radiation which propagates in the same direction as the ionizing laser pulse, which we label the forward wave, and radiation which counterpropagates with respect to the laser pulse, which we call the backward wave. The plasma density in the ionization front is controlled by adjusting the neutral gas density in the cavity. Source radiation at 35 GHz has been upshifted to more than 138 GHz in a continuously tunable fashion.

**SPACE-TIME ANALYSIS**

The derivation of the upshifted frequencies can be carried out either in the laboratory frame or in the rest frame of the ionization front. By making a Lorentz transformation to the front's rest frame, the derivation is simplified to that of a stationary boundary with neutral gas and a higher frequency electromagnetic wave impinging upon it from one side and plasma and the upshifted radiation streaming away on the other [1]. However, in certain geometries the front may propagate faster than the speed of light. Because one cannot transform to a frame moving faster than c, the analysis for superluminal fronts must be undertaken in the laboratory frame. Space-time plots are particularly useful because they help one to visualize the upshifting mechanism over the full range of front velocities, from stationary to infinitely fast.

Fig. 2 is a space-time plot for an ionization front interacting with radiation confined within a microwave cavity operating near cutoff. The ionizing laser pulse, which passes through the cavity from right to left, is propagating at approximately c, and is represented by the heavy black line at 45 degrees to the space and time axes. The parallel vertical lines in the figure represent the stationary cavity windows from which the circulating microwave radiation reflects before the arrival of the laser pulse. The group velocity of the circulating radiation inside the cavity, \( v_g \), is \( 0.5c \). The parallel red lines in Fig. 2 show the propagation along the cavity axis of points of equal phase in the source radiation. Because the product of the group velocity and the phase velocity, \( v_p \), is \( c^2 \), the phase velocity of the source radiation, given by the inverse of the slope of the red lines, is greater than c.

As the source radiation is transmitted into the underdense front, its frequency is upshifted. The degree of upshift can easily be derived by taking a close look at the interaction of the radiation with the moving boundary. Fig. 3 shows a magnified detail of Fig. 2 along the front's space-time line. The parallel red lines track points of constant phase for the
Figure 2: Space-time diagram for an ionizing laser pulse passing through a resonant microwave cavity.
Figure 3: Detail of Fig. 2 along the ionization front's space-time line for the *forward* radiation.
radiation inside the cavity that is propagating in the same direction as, and being overtaken by, the ionization front, the forward wave. The parallel blue lines track points of constant phase for the upshifted, transmitted radiation. The derivation proceeds from three simple requirements: 1) the source radiation must obey the dispersion relation (Lorentz-invariant in our geometry) for an electromagnetic wave in a conducting waveguide, \( \omega_0^2 = \omega_c^2 + c^2 k^2 \), where \( \omega_c \) is the cutoff frequency for the particular operating mode in the waveguide; 2) the number of cycles of the source radiation impinging on the front must equal the number of cycles of upshifted radiation leaving the front; and 3) the upshifted radiation must obey the dispersion relation for an electromagnetic wave in a plasma filled waveguide (also Lorentz-invariant), \( \omega_{up}^2 = \omega_c^2 + \omega_p^2 + c^2 k^2 \), where \( \omega_p \) is the plasma frequency. The first and third requirements thus dictate the slope of the red and blue lines in Fig. 2, and the second requirement is simply that the spacing between the lines, measured along the front's space-time line, must be equal on both sides.

The full range of front velocities can be considered by simply changing the slope of the front's space-time line. The two limiting cases are a stationary boundary and one that propagates at an infinitely high velocity. For the stationary boundary, the front's space-time line is vertical and requirement 2 is simply that the frequency is fixed. As is the usual case for stationary interfaces, the wavenumber must adjust itself in order to obey requirement 3. For the infinitely fast front, also known as “flash ionization” [9], the boundary is horizontal in the space-time plot. Thus, the wavenumber is fixed by requirement 2 and the frequency adjusts in order to satisfy requirement 3.

It can easily be seen in Fig. 3 that the temporal periods of the radiation on the left and right sides of the front are given by:

\[
T_L = \frac{2\pi}{\omega_L} = X \left( \frac{1}{v_{\text{front}}} - \frac{1}{v_{\text{phase}_L}} \right) \quad \text{and} \quad T_R = \frac{2\pi}{\omega_L} = X \left( \frac{1}{v_{\text{front}}} - \frac{1}{v_{\text{phase}_R}} \right).
\]

Using these expressions, we can write the upshifted frequency as a function of the source frequency as:

\[
\omega_R = \omega_L \frac{1/v_{\text{front}} - 1/v_{\text{phase}_L}}{1/v_{\text{front}} - 1/v_{\text{phase}_R}}.
\]

Viewed another way, this relationship follows from the requirement that the phase be continuous across the boundary, which is a consequence of the requirement that \( E \) and \( B \) be continuous at the front. The dispersion
relations to the left of (before) and to the right of (after) the front are
\[ \omega_L^2 = \omega_c^2 + c^2 k_L^2 \] and \[ \omega_R^2 = \omega_c^2 + \omega_p^2 + c^2 k_R^2. \]

By eliminating \( k_R \) and \( k_L \) from the above expressions, we obtain a quadratic equation in \( \omega_R \) which can be solved to give
\[
\omega_R = \omega_L \gamma^2 (1 - \beta v_g/c) \left\{ 1 - \beta \left[ 1 - \frac{\omega_c^2 + \omega_p^2}{\omega_L^2 \gamma^2 (1 - \beta v_g/c)^2} \right]^{1/2} \right\}. \tag{1}
\]

Here \( \beta = v_{\text{front}}/c \) and \( \gamma = (1 - \beta^2)^{-1/2} \) is the relativistic Lorentz factor associated with the front. \( v_g \) is the group velocity of the source radiation in the cavity, and is related to \( v_{\text{phase}} \) by \( v_g v_{\text{phase}} = c^2 \). A similar derivation for the radiation inside the cavity which is propagating in the opposite direction to the ionization front (left to right in Fig. 2), the backward wave, yields an upshifted frequency given by
\[
\omega_R = \omega_L \gamma^2 (1 + \beta v_g/c) \left\{ 1 - \beta \left[ 1 - \frac{\omega_c^2 + \omega_p^2}{\omega_L^2 \gamma^2 (1 + \beta v_g/c)^2} \right]^{1/2} \right\}. \tag{2}
\]

The upshifted forward wave is shown by the blue lines in Fig. 2, and the backward wave is shown by the green lines. These parallel lines represent the propagation of points of equal phase, and are bounded by black lines which represent the group velocity of the radiation. Eq. 1 and Eq. 2 both predict upshifted frequencies that are proportional to the front density, with the forward wave always upshifted to a higher frequency than that of the backward wave for a given plasma density. We note that \( \beta \) is not limited to being less than unity, in fact, we recover the flash-ionization results for \( \beta \to \infty \) [9].

The group velocity of the upshifted radiation inside the waveguide is plotted as a function of the front density in Fig. 4. The group velocity of the forward wave, which is always greater than that of the source radiation, increases as the plasma density in the front increases. On the other hand, the group velocity of the backward wave initially decreases as the front density increases. It goes to zero when the front density is such that \( \omega_p^2 = \omega_L^2 (1 + \beta v_g/c)^2 - \omega_c^2 \). At this point the group velocity of the backward wave changes sign, then increases in the forward direction as the plasma density continues to increase. For our experimental parameters, this occurs at a plasma density of \( 2 \times 10^{13} \) cm\(^{-3} \).

Fig. 5 is a space-time plot for a front density of \( 1 \times 10^{13} \) cm\(^{-3} \), where the group velocity of the backward wave is small, but it is still propagating.
Figure 4: Group velocity of the *forward* and *backward* upshifted radiation vs. the plasma density in the front.
Figure 5: Space-time diagram for a front density of $1 \times 10^{13}$ cm$^{-3}$. 
in the backward direction. The red stippled area indicates source radiation that leaks out of the cavity before the arrival of the laser pulse. For a front density of $4 \times 10^{13} \text{ cm}^{-3}$, shown in Fig. 4, the backward wave has turned around and is propagating in the same direction as the ionizing laser. At this density the forward wave has already been significantly compressed in duration. Also, because the backward wave is now following along behind the front, its duration decreases significantly as its group velocity approaches that of the front at higher plasma densities.

**REFLECTION AND TRANSMISSION COEFFICIENTS**

The reflection and transmission coefficients for a TE wave polarized perpendicular to the plane of incidence (s-polarization) can easily be calculated in the front’s frame (recall that in this frame the problem is reduced to that of a stationary boundary). We first derive the modes that can exist in the streaming plasma. The momentum equation for the electrons in the front’s frame is

$$\frac{\partial \vec{v}}{\partial t} + \vec{U} \frac{\partial \vec{v}}{\partial z} = -\frac{e}{m_0 \gamma} (\vec{E} + \frac{1}{c} \vec{U} \times \vec{B}).$$  \hspace{1cm} (3)

$\vec{U}$ is the velocity of the neutral gas, which in this frame is equal to the velocity of the front, and $m_0$ is the rest mass of the electrons. We use two of Maxwell’s equations,

$$\nabla \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$  \hspace{1cm} (4)

and

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} - \frac{4\pi}{c} \vec{J},$$  \hspace{1cm} (5)

where $\vec{J}$, the free current, is $n e \vec{v}$. By taking the curl of Eq. 4, considering only TE waves ($\nabla \cdot \vec{E} = 0$), and substituting the right side of Eq. 5 for $\nabla \times \vec{B}$, we can write

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi n e}{c^2} \frac{\partial \vec{v}}{\partial t}.$$  \hspace{1cm} (6)

We fourier analyze by assuming solutions of the form $e^{i(kx - \omega t)}$. Differentiating Eq. 6 gives

$$\left(\frac{\omega^2}{c^2} - k^2 \right) \vec{E} = \frac{-i4\pi n e \omega}{c^2} \vec{v}.$$  \hspace{1cm} (7)
Figure 6: Space-time diagram for a front density of $4 \times 10^{13}$ cm$^{-3}$. At this density the upshifted backward wave is "reflected" into the forward direction.
We can now differentiate Eq. 3 and eliminate $\vec{E}$ and $\vec{B}$ using Eqs. 4 and 7 to write, after a bit of algebra,

$$(\omega - Uk_\parallel) \left( 1 - \frac{\omega_p^2}{\omega^2 - c^2 k^2} \right) = 0,$$

(8)

where the subscript $\parallel$ denotes along the direction of $U$, which is opposite to the direction of propagation of the front. This equation has three roots,

$$k_{1\parallel} = \frac{\omega}{U},$$

(9)

$$k_2 = \frac{\omega}{c} \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{\frac{1}{2}},$$

(10)

and

$$k_3 = -\frac{\omega}{c} \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{\frac{1}{2}}.$$

(11)

We neglect the third root because we assume a positive value for the wavenumber of the source wave.

Now that we know that there are two possible modes in the streaming plasma, we can apply the boundary conditions to solve for the reflection and transmission coefficients. We assume a discontinuous boundary between the neutral gas and the plasma, which lies in the x-y plane. $U$ is along the z axis and an electromagnetic wave with its electric field along the x axis is incident upon the boundary with its wave vector in the y-z plane (s-polarization). There are three conditions that must be satisfied at the plasma-neutral boundary: 1) continuity of the tangential component of $\vec{E}$, 2) continuity of the normal component of $\vec{B}$, and 3) the total current must be zero because the electrons and ions have identical initial velocities when they are created at the boundary. Considering an incident mode (subscript $i$), a reflected mode (subscript $r$), and the two transmitted modes derived earlier (subscripts 1 and 2) these boundary conditions can be written as:

$$\left( \vec{E}_i + \vec{E}_r \right)_\perp = \left( \vec{E}_1 + \vec{E}_2 \right)_\perp$$

(12)

$$\left( \vec{B}_i + \vec{B}_r \right)_\parallel = \left( \vec{B}_1 + \vec{B}_2 \right)_\parallel$$

(13)

and

$$\nabla \times (\vec{B}_1 + \vec{B}_2) = \frac{1}{c} \frac{\partial}{\partial t} (\vec{E}_1 + \vec{E}_2).$$

(14)
In our geometry, Eqs. 12 and 13 become

\[ E_i + E_r = E_1 + E_2 \]  \hspace{1cm} (15)

and

\[ B_i \frac{k_{i||}}{k_i} - B_r \frac{k_{r||}}{k_r} = B_1 \frac{k_{1||}}{k_1} - B_2 \frac{k_{2||}}{k_2}. \]  \hspace{1cm} (16)

Realizing that \( B = (ck/\omega)E \) for each mode and that \( k_{i||} = k_{r||} \), we can eliminate the reflected mode from the two equations above and write

\[ E_1 = \frac{2E_i - E_2 \left(1 + \frac{k_{2||}}{k_{2||}}\right)}{\left(1 + \frac{k_{1||}}{k_{1||}}\right)} \]  \hspace{1cm} (17)

Eq. 14 can be differentiated and rearranged to give

\[ E_1 = E_2 \left(\frac{\omega^2 - c^2k_2^2}{c^2k_2^2 - \omega^2}\right). \]  \hspace{1cm} (18)

We can now equate Eqs. 17 and 18 and reduce them to

\[ \frac{E_2}{E_1} = \frac{2}{\left(1 + \frac{k_{2||}}{k_{1||}}\right) + \left(1 + \frac{k_{2||}}{k_{2||}}\right)\left(\frac{\omega^2 - c^2k_2^2}{c^2k_2^2 - \omega^2}\right)}. \]  \hspace{1cm} (19)

Using this and Eq. 18, we obtain

\[ \frac{E_1}{E_1} = \frac{2}{\left(1 + \frac{k_{1||}}{k_{1||}}\right) + \left(1 + \frac{k_{1||}}{k_{2||}}\right)\left(\frac{\omega^2 - c^2k_2^2}{c^2k_2^2 - \omega^2}\right)}. \]  \hspace{1cm} (20)

And finally, Eq. 15 simply gives us

\[ \frac{E_r}{E_i} = \frac{E_1}{E_i} + \frac{E_2}{E_i} - 1. \]  \hspace{1cm} (21)

Thus Eqs. 19, 20, and 21 give us the reflection and transmission coefficients in the front's frame. In order to obtain the laboratory frame coefficients, we must transform back to the laboratory frame.

We make the Lorentz transformation using the relations

\[ \bar{E}_{\text{lab}} = \gamma(\bar{E} + \bar{\beta} \times \bar{B}) - \frac{\gamma^2}{\gamma + 1} \bar{\beta}(\bar{\beta} \cdot \bar{E}) \]  \hspace{1cm} (22)

and

\[ \bar{B}_{\text{lab}} = \gamma(\bar{B} + \bar{\beta} \times \bar{E}) - \frac{\gamma^2}{\gamma + 1} \bar{\beta}(\bar{\beta} \cdot \bar{B}). \]  \hspace{1cm} (23)
Using these relations, we can write the electric fields in the laboratory frame as

\[ \vec{E}_{d_{lab}} = \vec{E}_1 \gamma \left( 1 - \beta \frac{k_{i\parallel}c}{\omega} \right), \]  
\[ \vec{E}_{r_{lab}} = \vec{E}_r \gamma \left( 1 + \beta \frac{k_{i\parallel}c}{\omega} \right), \]  
\[ \vec{E}_{1_{lab}} = 0, \]  
and

\[ \vec{E}_{2_{lab}} = \vec{E}_2 \gamma \left( 1 - \beta \frac{k_{o\parallel}c}{\omega} \right). \]

Because \( \vec{E}_1 \) transforms to zero in the laboratory frame, we see that this mode is simply a static magnetic field. We can use the relation \( B_1 = (ck_1/\omega)E_1 \) to obtain \( B_1 \), then transform back to the laboratory frame to give

\[ \vec{B}_{1_{lab}} = E_1 \gamma \left\{ \left( \frac{ck_{1\parallel}}{\omega} - \beta \right) \vec{i} + \frac{ck_{1\perp}}{\gamma \omega} \vec{k} \right\}, \]  

or

\[ B_{1_{lab}} = E_1 \gamma \left\{ \left( \frac{ck_{1\parallel}}{\omega} - \beta \right)^2 + \left( \frac{ck_{1\perp}}{\gamma \omega} \right)^2 \right\}^{1/2}. \]

We can now write, after a considerable amount of algebraic manipulation, the laboratory frame reflection and transmission coefficients as

\[ r = \frac{E_{r_{lab}}}{E_{i_{lab}}} = \frac{\sqrt{\epsilon_i} - \sqrt{\epsilon_t}}{\sqrt{\epsilon_i} + \sqrt{\epsilon_t}}, \]  
\[ t_2 = \frac{E_{2_{lab}}}{E_{i_{lab}}} = \frac{2\sqrt{\epsilon_i}}{\sqrt{\epsilon_i} + \sqrt{\epsilon_t}}, \]  
and

\[ t_1 = \frac{B_{1_{lab}}}{E_{i_{lab}}} = \left\{ \frac{1 - \beta^2}{1 - \beta^2 \epsilon_i} \right\}^{1/2} 2\beta \sqrt{\epsilon_i} \frac{\left( \sqrt{\epsilon_i} - \sqrt{\epsilon_t} \right)}{1 - \beta \sqrt{\epsilon_t}}. \]

Here \( \epsilon_i' = 1 - \omega_e^2/\omega^2 \) and \( \epsilon_t' = 1 - (\omega_e^2 + \omega_R^2)/\omega^2 \) with primed quantities specified in the front's frame.
For the range of plasma densities and $\sqrt{\epsilon_i}$ values used in our experiments, $\tau \approx 0$ and $t \approx 1$. Note that Eqs. 30, 31, and 32 reduce to the vacuum expressions in the limit $\epsilon_i \to 1$ ($\omega_c = 0$). Although the ratio of the transmitted to incident powers at the front is always expected to be nearly unity, for large upshifts most of the incident energy is left in the free-streaming mode's static $B$ field as the transmitted wave's duration shortens. The above model assumes that the ionization front is discontinuous. Similar results are obtained for finite length underdense fronts, except that the energy in the free-streaming mode is instead converted to thermal energy in the plasma, as is the case for finite length overdense fronts [3].

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References


[4] The degree of upshift is equal to what would be expected from a moving mirror, but because the moving boundary, or ionization front, carries no energy, the reflected electromagnetic energy is reduced for reflection from an ionization front.


