THE SHEATH CRITERION

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That ions arriving at a sheath on a negative electrode or wall must have energy greater than or equal to $\frac{1}{2}kT_e$ was shown by Langmuir[1] in 1929. In 1949 Bohm[2] gave an unconvincing proof of this theorem, which has since been known as Bohm's Criterion for Stability of a Sheath. Recently several authors[3,4] have made use of this criterion, although there is some doubt as to its ~~validity~~ applicability[5]. It is the purpose of this note to ~~show that~~ point out the conditions under which the sheath criterion is valid ~~for a wide range of conditions~~. That the ion velocity near a boundary should be given by the electron, rather than the ion, temperature is physically reasonable when $kT_e >> kT_i$, since the force driving the ions into the wall is provided by the plasma pressure $n_o kT_e$.

Consider a one-dimensional semi-infinite plasma in which $kT_e >> kT_i$ and $\lambda >> h$, where h is the Debye length $(kT_e/4\pi ne^2)^{\frac{1}{2}}$ and λ is the collision mean free path between ions and neutrals or electrons. We seek a time-independent solution of Poisson's equation in the region within a distance λ of the boundary, where the particles are assumed to suffer no collisions. The wall is assumed to be negative relative to the plasma, and the potential V will look somewhat as depicted in Fig. 1. Let us choose a point approximately $\frac{1}{2}\lambda$ away from the wall to be the origin $x = 0$, and define the value of V there to be 0.

The exact location of this point is immaterial; what matters is that the equations of motion for free-fall are obeyed for many lengths h on either side of this point.

The electron density is assumed to follow the Boltzmann relation:

$$n_e = n_{eo} e^{eV/kT_e} \qquad (1)$$

The subscript o will denote quantities at $x = o$. If the ions are assumed to arrive at $x = o$ with a uniform energy eV_o directed toward the wall, their density will be given by

$$n_i = n_{io} \left(1 + VV_o^{-1}\right)^{-\frac{1}{2}} \qquad (2)$$

In terms of the natural variables $\eta = -eV/kT_e$ and $\xi = x/h$, Poisson's equation becomes, for a perfectly absorbing wall,

$$\eta'' = a\left(1 + \eta\eta_i^{-1}\right)^{-\frac{1}{2}} - e^{-\eta} \qquad (3)$$

where the prime indicates $d/d\xi$, $\eta_i = -eV_o/kT_e$ is the initial ion energy, and $a = n_{io}/n_{eo}$, so that $a - 1$ is the value of η'', or the charge imbalance, at $\xi = 0$. Eq. (3) can be integrated to obtain

$$\tfrac{1}{2}\eta'^2 = 2a\eta_i\left[\left(1 + \eta\eta_i^{-1}\right)^{\frac{1}{2}} - 1\right] + e^{-\eta} - 1 + \tfrac{1}{2}b^2, \qquad (4)$$

where b is the value of η' at $\xi = 0$.

The behavior of the density terms as a function of η is shown in Fig. 2a, for various values of η_i, for the case $a = 1$. Since an electron excess produces a concave-downwards curvature of the η - ξ curve, it is clear that if $n_e > n_i$ for small positive η, the η - ξ curve cannot start with zero slope, since it would then have to be concave upwards and downwards at the same time. This difficulty does not arise if $\eta_i > n_e$ for positive η. The critical condition is that the slopes of the n_i and n_e curves be equal at $\eta = 0$. By differentiating (1) and (2), one sees that this is just the sheath criterion, $\eta_i = \frac{1}{2}$. For a=1 and b=0, Bohm[2] showed that the expression (4) for η'^2 is negative unless $\eta_i \geq \frac{1}{2}$.

However, a = 1 and b = 0 cannot be strictly true, since if $\eta' = \eta'' = 0$, all derivatives would vanish, and only the trivial solution $\eta = 0$ is possible. We now investigate the effect of the boundary conditions on the minimum value of η_i. Consider first the case b = 0, a > 1. As one varies η_i the curves of η vs ξ behave qualitatively as shown in Fig. 3. It is now possible to decrease η_i until the n_i curve dips slightly below the n_e curve, as shown in Fig. 2b. If η_i is too small, however, the solution becomes oscillatory. The critical η_i is reached when $\eta' = \eta'' = 0$ for some value of η. By setting Eqs. (3) and (4) equal to zero, one has two equations for the critical η_i and η for any given a. These can be solved by substituting from (3) into (4) to obtain

$$1 - e^{-\eta} = 2a\eta_i\,(ae^{\eta} - 1) \tag{5}$$

Denoting the left hand side by $f(\eta)$ and the right hand side by $g(\eta)$, we see that f and g behave qualitatively as in Fig. 4 for various η_i. For large η_i there is no solution, since η' is never zero except at the origin. For small η_i, there are two solutions, corresponding to a negative value for η'^2 and an oscillatory behavior for η. The critical value of η_i is that which makes the f and g curves tangent. Setting f = g and f' = g', we obtain

$$\eta_i = \tfrac{1}{2} \left[1 - \left(\frac{a-1}{a} \right)^{\frac{1}{2}} \right]^2 \tag{6}$$

This gives the reduction in the sheath criterion when $a > 1$.

It is characteristic of the sheath equation (3) that when η becomes of the order of 1, the curve develops catastrophically in a distance of the order of h. Hence if $\lambda >> h$, the value of a must be very close to unity to enable η to stay close to zero for such a long distance. Beyond a mean free path from the wall, collisions can alter the equation of motion so that (3) no longer has to hold. An upper limit to a can be obtained by assuming that $\eta'' = a-1$ throughout. With $b = 0$, the potential will then be $\eta(\xi) = \frac{1}{2}(a-1)\xi^2$. This is an underestimate of η since in practice η'' is not constant but increases with η. If we require that η be less than 1 when $\xi = \frac{1}{2}\lambda$, then $a-1 < 8/\lambda^2$. For almost all laboratory plasmas of interest, h lies between 10^{-2} and 10^{-3} cm. If $\lambda = 1$ cm, we see that $a-1 < 8 \times 10^{-4}$. Putting this into (6), we see that the correction to $\eta_i \geq \frac{1}{2}$ is less than 6%, usually much less. Thus for neutral densities below about 10 microns, the sheath criterion is valid. When the ion-neutral mean free path becomes comparable to h, however, the free-fall equations do not hold over a large distance, and the sheath criterion may be significantly changed by the boundary condition at the "sheath edge". Ion-electron mean free paths are usually greater than 1 cm.

It is apparent from Eqs. (3) and (4) that the $\eta - \xi$ curve must be symmetric about some ξ. If $\eta > 0$, the curve has a minimum there. If $a = 1$ but $b \neq 0$, we can shift the origin to this point, where $\eta' = 0$ and $\eta'' > 0$, and recover the previous case, since by our original choice of the origin the free-fall equations must hold for a long distance on either side of $x = 0$. In particular, if $\underline{b} > 0$, it is apparent from Fig. 2a that if one went to sufficiently negative η, the ion excess must eventually become large for any η_i, and hence the $\eta - \xi$ curve must reach a minimum. If $\underline{b}$ is so small that this minimum is not reached for a distance greater than $\frac{1}{2}\lambda$, and $|\eta| < 1$ there, it is easy to see that the formal solution must be the one we have considered as the actual solution, with $a-1 << 1$, and therefore $\eta_i \gtrsim \frac{1}{2}$. A similar argument for the case $a \neq 1$, $b \neq 0$ will show that we have already covered the most general boundary condition. Thus by the time ions reach the region of free-fall, they must have somehow or other acquired energy greater than or equal to $\frac{1}{2}$ kTe. The acceleration mechanism is not considered here. The solution for the collision-dominated region far from the sheath, whatever it may be, must join smoothly onto this free-fall solution at some point where η, η', and η'' are not zero.

If $\eta_i > \frac{1}{2}$, the n_i and n_e curves are no longer almost parallel at $\eta = 0$, and the boundary conditions $a = 1$ and $b = 0$ must be very closely satisfied in order for η to stay close to zero for many lengths $\underline{h}$. Although it is possible in principle for η_i to be greater than $\frac{1}{2}$, it is unlikely for energetic reasons.

So far we have only considered cold ions. If the ions have a velocity spread, those having more than the mean velocity will contribute more than average to the density in the sheath and vice versa. There is, however, a second order effect in the velocity spread which decreases the ion density. Thus if the ions are not cold, the critical value of η_i is greater than $\frac{1}{2}$, if η_i is computed from the mean velocity. In particular, if the ions are Maxwellian with a streaming velocity v_o, Eq. (2) will have to be replaced by an integral of the form

$$n_i = A \int_a^{\infty} e^{-\lambda^2 [(y^2 - a^2)^{\frac{1}{2}} - v_o]^2} dy \qquad (7)$$

If the wall potential η_w is so low that a large number of plasma electrons escape to the wall or electrode, Eq. (1) can be corrected to read

$$n_e = \frac{1}{2} n_{eo} e^{-\eta} [1 + \text{erf}(\eta_w - \eta)] \qquad (8)$$

This will decrease the critical value of η_i. If electrons are emitted from the wall, the density n_p of emitted electrons will behave like the dotted curve in Fig. 2b. This addition to n_e will increase η_i, but the effect is usually very small, even for space-charge limited emission.

Provided that the effects of the last two paragraphs are small, the sheath criterion should be valid whenever $\lambda >> h$.

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FIGURE CAPTIONS

1) The assumed geometry

2) Behavior of the ion and electron densities as functions of the potential, for various initial ion energies. (a) a = 1. (b) a > 1.

3) The potential distribution as the initial ion energy is varied.

4) Behavior of the functions f and g as η_i is varied.

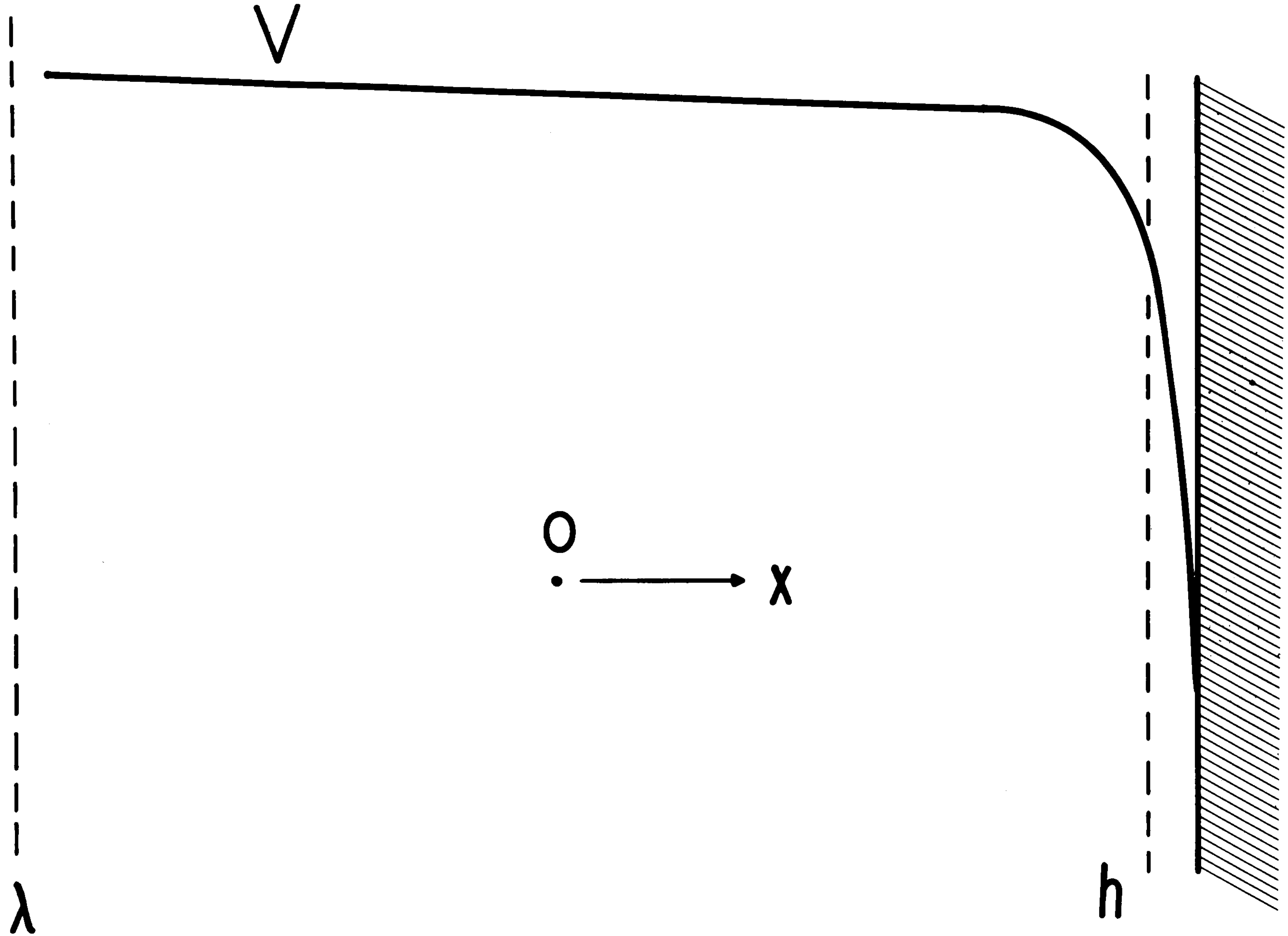

FIG. 1

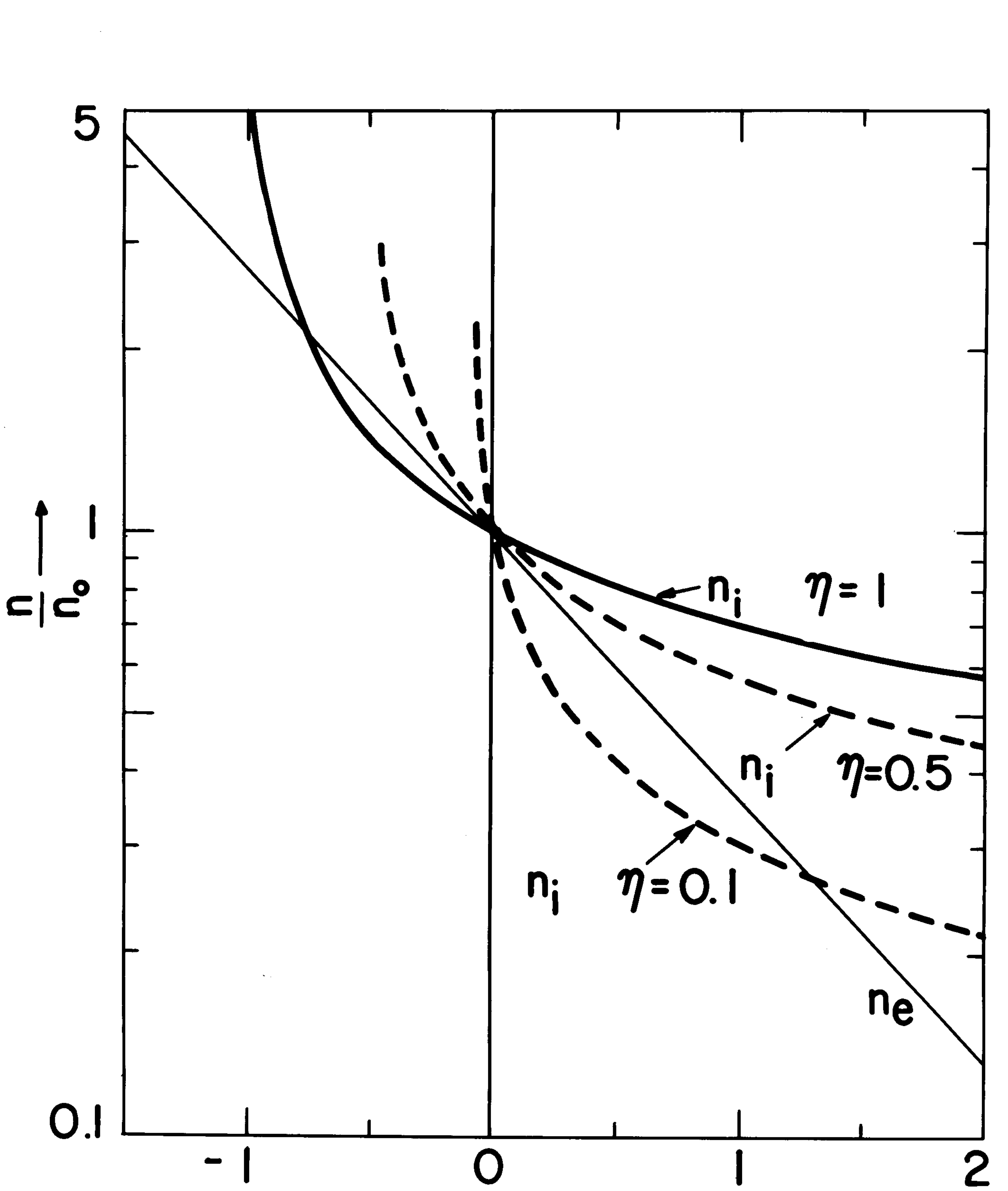

FIG. 2a

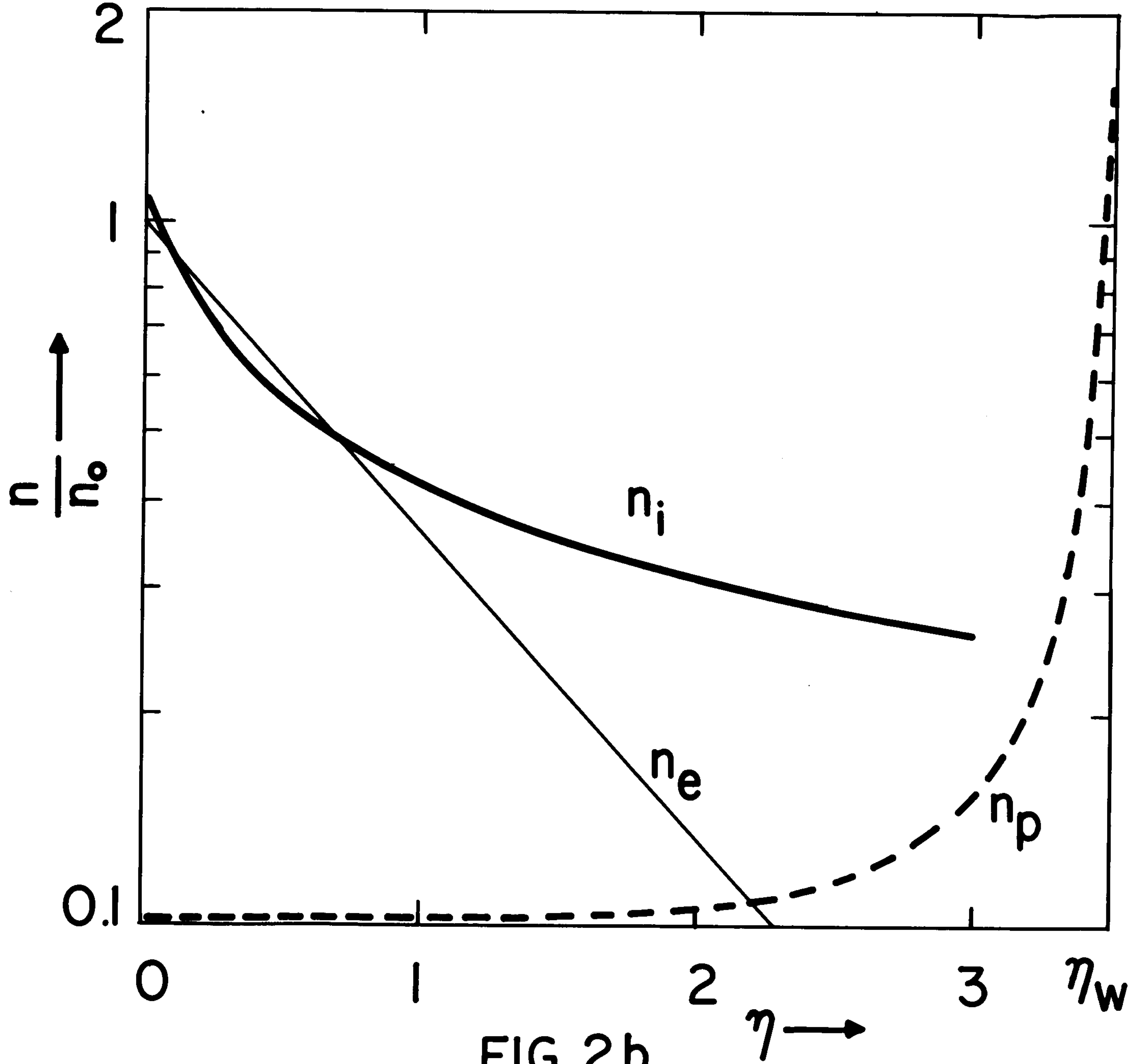

FIG. 2b

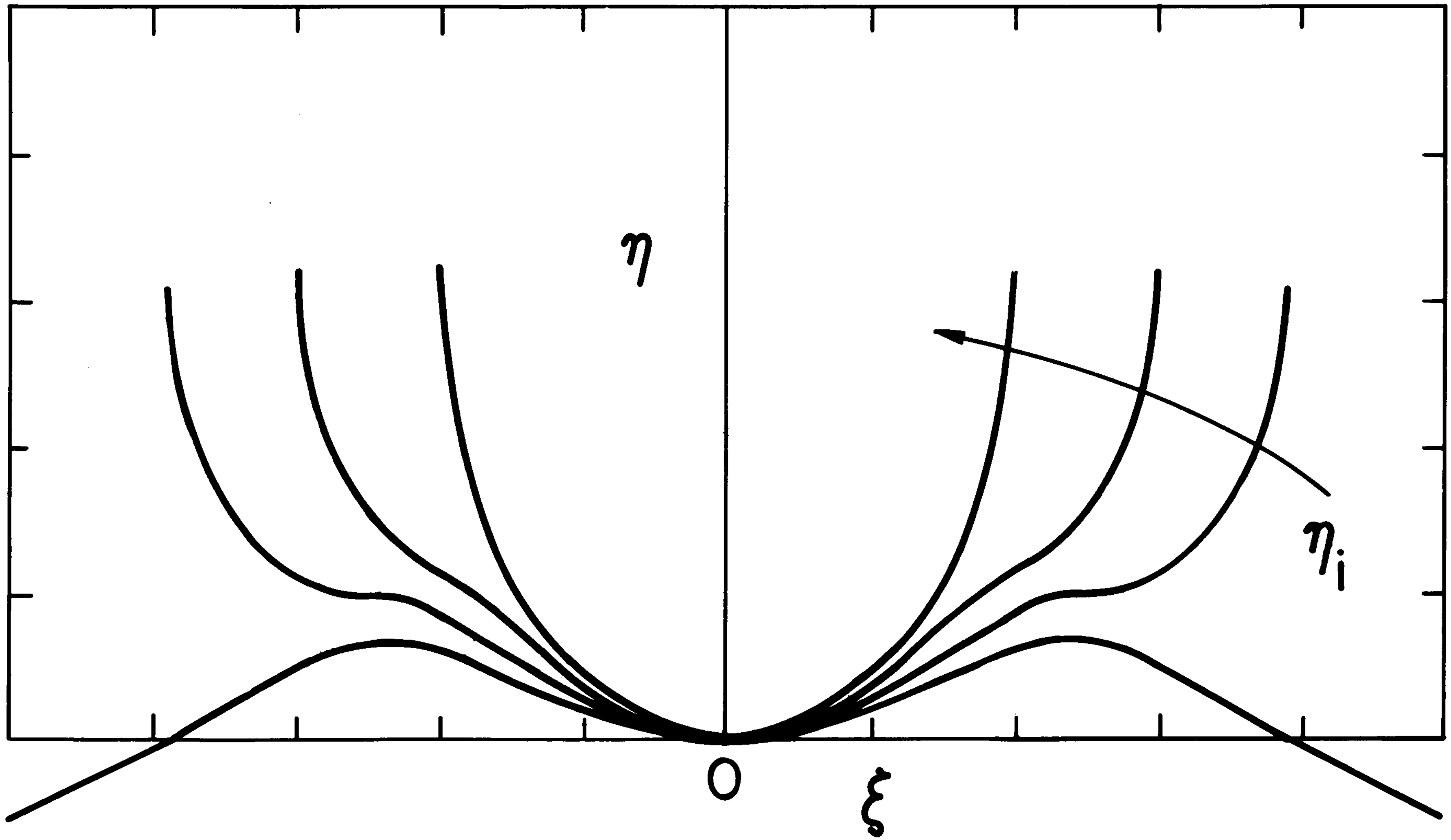

FIG. 3

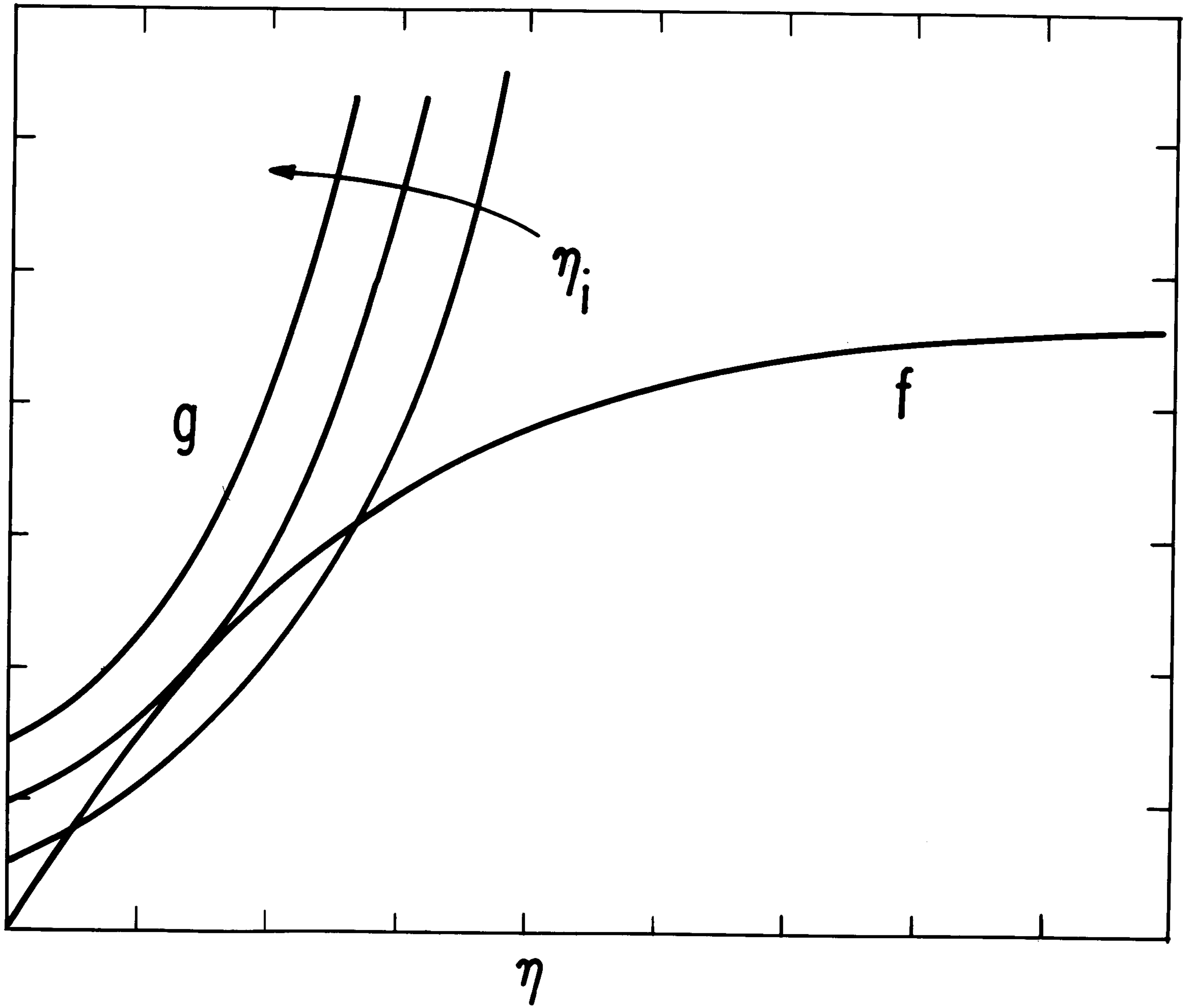

FIG. 4