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Saturation Ion Currents to Langmuir Probes

FRANCIS F. CHEN
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey
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The parabolic variation of saturation ion current with probe potential observed in dense plasmas is fortuitous and is not directly related to the effects of orbital motion. Agreement between measured and computed saturation ion characteristics is illustrated. The discussion is in the framework of collisionless, magnetic-field-free theories; they apply to the experiments only if the ion Larmor radius is much larger than the probe radius.

IN dense plasmas for which the Debye length h is less than the probe radius r_p, it is often found that the saturation ion flux I_i to a Langmuir probe varies as $(-V_p)^{\frac{1}{2}}$, where V_p is the (negative) probe voltage. This is the dependence expected for cylindrical probes drawing orbital-motion-limited current but not for those drawing space-charge-limited current. In the experiment of Gardner *et al.*,[1] for instance, a linear $I_i^2-V_p$ dependence was observed even though r_p/h was of order 10 and the probe current was almost certainly space-charge limited. That such a relationship sometimes holds even for thin sheaths was pointed out by Langmuir[2] himself; he also pointed out that the erroneous application of the orbital theory to such a case would lead to a spurious value of the space potential.

Gardner *et al.*[1] conjectured that a linear $I_i^2-V_p$ relation might come about because the potential at the sheath edge might be proportional to V_p, so that orbital-motion limitation might occur in the quasineutral region. We believe this view to be misleading or, at least, not useful. Our reasons are summarized in the Appendix. It is a better physical picture to think of the sheath edge as having a constant potential and that the increase in I_i with $|V_p|$ is caused by an increase in sheath radius. The recent work of Lam[3] on a problem previously treated by Wenzl[4] supports this picture. For a cylindrical probe in a plasma with a monoenergetic ion distribution of energy E_i, it is found that the potential η_s of the sheath edge is independent of η_p and that η_s varies only from 1.0 at $\beta=0$ to $\ln 2=0.69$ at $\beta=\infty$, where $\beta\equiv E_i/kT_e$ and $\eta\equiv -eV/kT_e$. The insensitivity to E_i indicates that the results would not be greatly different for Maxwellian distributions. The absorption radius, or effective probe radius, inside of which all ions are collected, always occurs for cylindrical probes at the radius where $\eta=\ln 2$. Therefore, the absorption radius always lies outside the sheath and is always quite close to the sheath radius. The picture is, then, that these two radii move outwards together as η_p is increased, while the potentials at these two radii remain constant.

We now wish to present some numerical results to support the view that the linear dependence of I_i^2 on V_p is fortuitous but is approximately true for certain ranges of parameters. In Figs. 1 and 2 are shown log–log plots of η_p versus dimensionless current J or $J\xi_p$ for spherical and cylindrical probes, respectively, for various values of $\xi_p=r_p/h$. It is seen that J^2 varies approximately as η_p only for large η_p and small ξ_p. These curves were computed by the method of Allen, Boyd, and Reynolds,[5] which is valid for $\beta=0$. For cylinders, this method yields a different result from the $\beta=0$ limit of a finite-β theory. The reason is that the angular momentum L is assumed to be zero at $r=\infty$ in this theory, while L is finite in the $\beta=0$ limit of a finite-β theory because of nonuniform convergence.

In Figs. 3 and 4 we show similar curves for $\beta=0.1$ for spheres and cylinders, respectively. Again one finds that

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1 A. L. Gardner, W. L. Barr, R. L. Kelly, and N. L. Oleson, Phys. Fluids **5**, 794 (1962).
2 I. Langmuir, *The Collected Works of Irving Langmuir*, edited by Guy Suits (Pergamon Press, Inc., New York, 1961), Vol. 4, p. 64.
3 S. H. Lam, Princeton Univ. Gas Dynamics Laboratory Rept. 681, AFOSR 64-0353 (1964).
4 F. Wenzl, Z. Angew. Phys. **2**, 59 (1950).
5 J. E. Allen, R. L. F. Boyd, and P. Reynolds, Proc. Phys. Soc. (London) **70B**, 297 (1957).

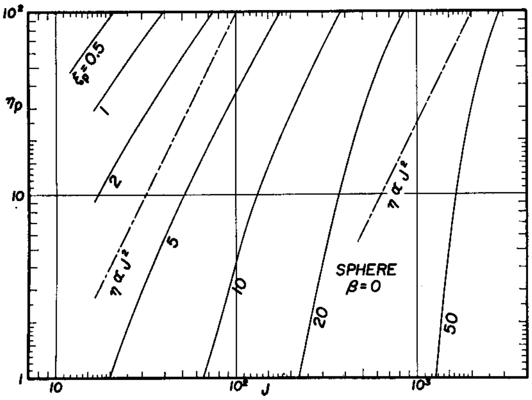

FIG. 1. Curves of $\log\eta_p$ vs $\log J$ for various values of $\xi_p=r_p/h$, for spherical probes and zero ion temperature. Here J is defined by $J=ZI_i(e^2/kT_e)(m_i/2ZkT_e)^{1/2}$, η_p by $\eta_p=-eV_p/kT_e$, and h by $h^2=kT_e/4\pi ne^2$.

the $J^2-\eta_p$ relation holds only for certain values of ξ_p and η_p. These curves were computed by the method of Bernstein and Rabinowitz,[6] which is valid for monoenergetic ion distributions. The variation of these results with β is not great.

In the limit of large ξ_p, that is, of dense plasmas, the shape of the I_i-V_p curve can be expressed in terms of a single universal function by proper scaling of the variables. This result was obtained by Lam[3] in a rigorous boundary-layer analysis of the Bernstein–Rabinowitz equations. Results of the theory of Lam are summarized in Figs. 5 and 6, which show normalized I_i^2 as a function of normalized V_p for both cylinders and spheres. Here τ is I_i/I_B, where I_B is the current predicted by Bohm[7] by neglecting the sheath thickness; and A is essentially a constant but has a weak dependence on β. One sees that the $I_i^2-V_p$ relation can be approximated by a straight line in all cases except

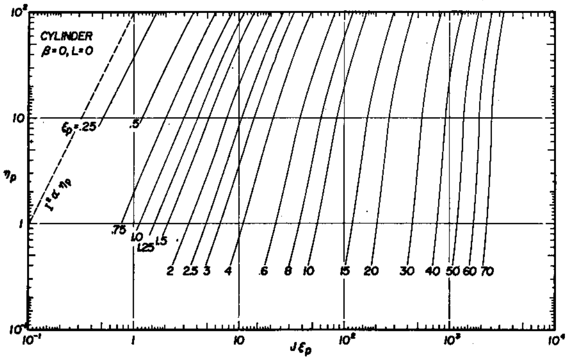

FIG. 2. Curves of $\log\eta_p$ vs $\log J\xi_p$ for various values of ξ_p, for cylindrical probes, zero ion temperature, and zero ion angular momentum. Here $J\xi_p$ is equal to $ZI_ir_p(e/kT_e)^2(2m_ikT_e/Z)^{1/2}$, Z being the ion charge number.

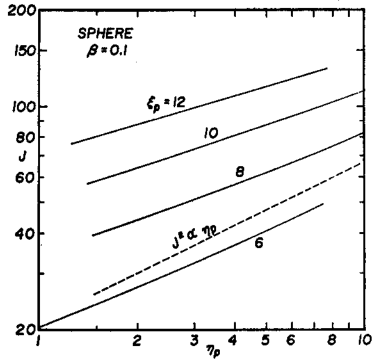

FIG. 3. Curves of $\log J$ vs $\log\eta_p$ for spherical probes and finite ion energies $E_i=\beta kT_e$. Symbols are as in Fig. 1.

for spherical probes under large voltages. The good fit of the cylindrical probe curve in Fig. 6 to a straight line is entirely accidental; the curve actually has an inflection point in this range of the variables. In terms of normal variables, the equations for the dotted straight lines give the following useful approximate formulas:

$$\text{Sphere: low } V_p \quad \frac{d(ZeI_i)^2}{-dV_p}=2.6\times10^{-20}\frac{Z}{N}[n^2r_p(kT_e)_{\text{eV}}]^{2/3}; \qquad (1)$$

$$\text{Cylinder: low } V_p \quad \frac{d(ZeI_i)^2}{-dV_p}=3.3\times10^{-21}\frac{Z}{N}[n^2r_p(kT_e)_{\text{eV}}]^{2/3}; \qquad (2)$$

$$\text{Cylinder: high } V_p \quad \frac{d(ZeI_i)^2}{-dV_p}=2.6\times10^{-21}\frac{Z}{N}[n^2r_p(kT_e)_{\text{eV}}]^{4/3}. \qquad (3)$$

Here Z and N are, respectively, the charge number and atomic weight of the ions; I_i is the particle flux per centimeter length in the cylindrical case and the left-hand side is in A²/V.

In Figs. 7 and 8 we show the comparison of Lam's theoretical curves with some cylindrical probe measurements made by Kuckes[8] in a thermally ionized cesium plasma with $Z=1$. Since the magnetic field was about 10 kG in this experiment, the ratio of r_L to r_p was only 3 or 4. In spite of this, it is seen that an excellent fit with theory is obtained for large ξ_p. For $\xi_p\approx8$ the theory is not expected to be very accurate, and indeed one can

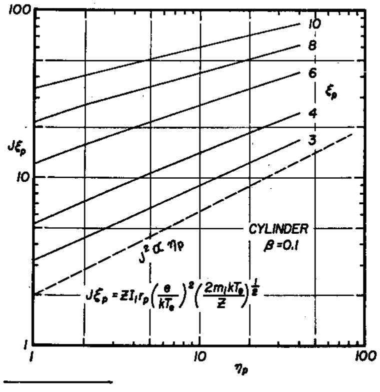

FIG. 4. Curves of $\log J\xi_p$ vs $\log\eta_p$ for cylindrical probes and finite ion energies $E_i=\beta kT_e$. Symbols are as in Fig. 2.

[6] I. B. Bernstein and I. Rabinowitz, Phys. Fluids 2, 112 (1959).
[7] D. Bohm, *The Characteristics of Electrical Discharges in Magnetic Fields*, edited by A. Guthrie and R. K. Wakerling (McGraw-Hill Book Company, Inc., New York, 1949), p. 45.
[8] A. F. Kuckes, Princeton Plasma Physics Laboratory (private communication).

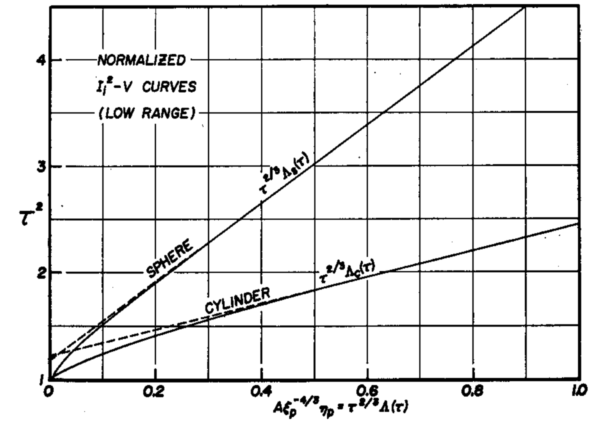

FIG. 5. Curves of τ^2 vs $A\,\xi_p^{-4/3}\eta_p$, from the theory of Lam (Ref. 3), which is valid for large ξ_p. For spheres,

$$\tau\approx I_i/[1.5\pi r_p^2 n(2ZkT_e/m_i)^{1/2}].$$

For cylinders,

$$\tau\approx I_i/[1.9 r_p n(2ZkT_e/m_i)^{1/2}].$$

A range of τ corresponding to relatively thin sheaths is covered.

discern a difference in slope between theory and experiment in Fig. 7. The value of ξ_p found from the fit with theory yields a value of the plasma density n. I both cases this value was within 18% of that found by ordinary microwave interferometry.

In Fig. 9 we show the data of Gardner *et al.*,[1] taken from Fig. 7 of Ref. 1. In reducing the data to dimensionless form we have assumed $r_p=9\times10^{-3}$ cm, $l_p=0.32$ cm, $kT_e=7.5$ eV, and $V_s=129$ V, where l_p is the probe length and V_s the space potential relative to the cathode. The ratio r_L/r_p was of order 10^2 in this case. Also shown in Fig. 9 are four points from Kuckes' data of Fig. 7 and theoretical curves from Bernstein and Rabinowitz[6] (BR) for $\beta=0.1$ and from Lam[3] for $\beta=1$. The experimental value of β was about 1 in both cases. The theory of Lam is not very accurate for such low values of ξ_p; on the other hand, BR calculations for $\beta=1$ are not available. Both theories suffer from the

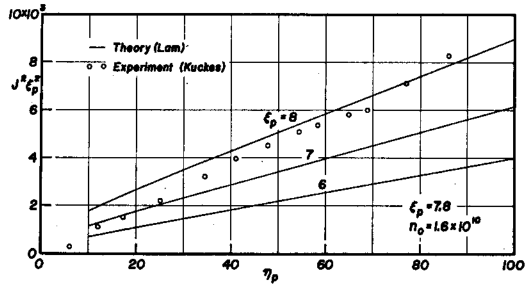

FIG. 7. Normalized $I_i^2-V_p$ curves compared with experimental measurements, for a low-density plasma.

neglect of a spread in ion energies, an effect which should become noticeable at $\beta=1$.

The Gardner data have been plotted for assumed effective charge numbers Z of 1.0, 1.5, and 2.0, since an independent measurement of Z is not available. It is seen that although the data fall in a straight line, the slope is larger than theory would predict for any value of Z. Part of the discrepancy may be due to the inexactness of the theories, and part to errors in the assumed values of r_p, l_p, kT_e, and V_s. The value of kT_e was taken from a measurement (Fig. 15, Ref. 1) at the same port and the same radius ($r=1.14$ cm) as the probe curve. Reasonable adjustments in r_p, l_p, and kT_e do not remove the discrepancy between the absolute magnitude of I_i and the slope. Adjustments in V_s can bring agreement, but there are limits to the value of V_s: it must lie between the floating potential V_f and the intercept V_i of the straight line. In this particular case V_i-V_f was too small for I in Fig. 7, Ref. 1, to be the total current; hence we assumed that the electron part had already been subtracted, and what was plotted was I_i. We then chose V_s to be the potential at which I_i was approximately equal to the random ion current in the plasma. If one ignores the slope and uses only the absolute magnitude of I_i, one finds $\xi_p\approx10$ for $Z=1.5$. This yields $n_0\approx5\times10^{12}$ cm^{-3}, about 60% higher than indicated by microwave measurements (Fig. 17, Ref. 1). If one regards the slope alone, use of Eq. (2) leads to a

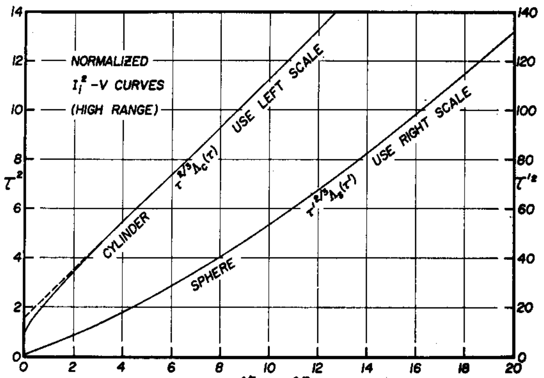

FIG. 6. The extension of the curves of Fig. 5 to a range of τ corresponding to thick sheaths.

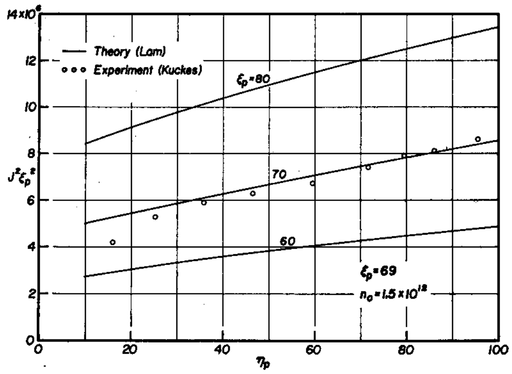

FIG. 8. Normalized $I_i^2-V_p$ curves compared with experiment, for a high-density plasma.

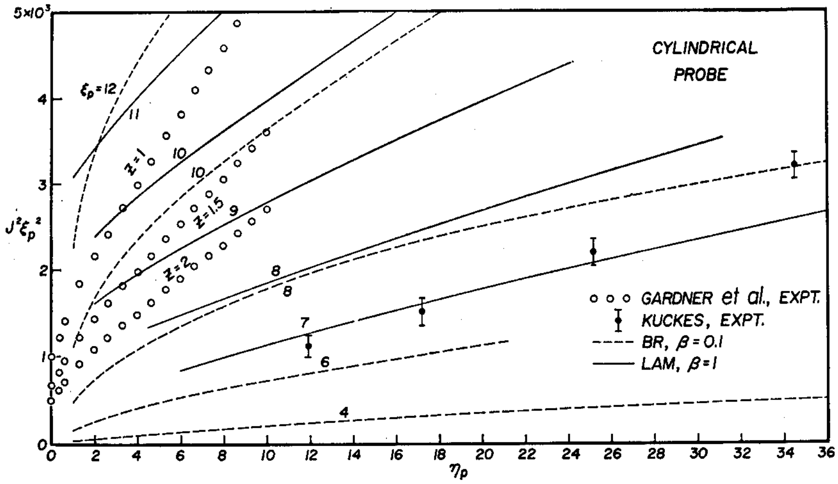

FIG. 9. Normalized $I_i^2 - V_p$ curves compared with the experiments of Gardner *et al.*[1] and of Kuckes.[8] Since in Ref. 1 the effective charge number Z is unknown, the data have been reduced for three assumed values of Z.

larger discrepancy: $n_0 \approx 1.3 \times 10^{13}$ cm^{-3}. This is still unresolved.

Note that use of Eqs. (1), (2), and (3) gives only the product $Z^3 n^{\frac{1}{2}}$, since information on the intercept has been discarded. If kT_e and V_s are known accurately enough, the use of the $I_i^2 - V_p$ plot can, in principle, give Z and n separately, but the accuracy is not great.

We are indebted to Dr. A. F. Kuckes for access to his data, to Professor S. H. Lam for helpful conversations, to H. Fishman for some of the numerical computations, and to K. P. Mann for help with the drawings. Further numerically computed ion probe characteristics may be found in another report.[9]

APPENDIX

We give here four reasons for the statement that the $I_i^2 - V_p$ dependence is not caused by Langmuir's orbital-motion theory operating in the quasineutral region. Unless otherwise specified, the discussion concerns cylindrical probes, for which this dependence is predicted by Langmuir. Although we consider only monoenergetic distributions, the insensitivity of the theories to ion energy makes it difficult to see how these arguments can be greatly different for Maxwellian distributions.

(1) An $I_i^2 - V_p$ dependence occurs for certain ranges of parameters even when all ions are assumed to move radially, so that there is no question of orbital motion. This can be seen in Figs. 1 and 2.

(2) Figures 1, 3, and 5 show that $I_i^2 \alpha V_p$ is not a bad approximation even for spheres. Orbital theory would predict $I_i \alpha V_p$.

(3) The parameter τ in the theory of Lam[3] is essentially the ratio of the sheath area to the probe area. It is clear from Figs. 5 and 6, then, that it is the change of sheath area which gives rise to the approximately linear $I_i^2 - V_p$ relation.

(4) For $\beta = 1$, Lam[3] gives $\bar{\eta}_s = 0.83$, independent of η_p. Here $\bar{\eta}_s$ is the potential at the sheath edge, defined in the usual manner as the radius at which the quasineutral solution of Poisson's equation turns back on itself. The true value of η_s is somewhat less than the value $\bar{\eta}_s$ given by the quasineutral solution; therefore, for $\beta = 1$ the value of η_s always lies below 0.83 as η_p is varied. Although it is unlikely that η_s lies far below $\bar{\eta}_s$, it is, in principle, possible for η_s to vary with η_p, giving rise to orbital-motion limitation outside the sheath. However, for such small values of η_s, the orbital-motion theory no longer predicts that I_i^2 should be proportional to V_p; instead, the second term in the more exact formula $I_i \propto 2\pi^{-\frac{1}{2}}\chi^{\frac{1}{2}} + e^{\chi}(1 - \operatorname{erf}\chi^{\frac{1}{2}})$, where $\chi \equiv \eta_s/\beta$, gives rise to considerable curvature in the $I_i^2 - V_p$ plot. It is possible that in a Maxwellian distribution the ions with small β have large χ even though $\eta_s < 0.83$; but it would then be quite inconceivable that the slow ions should contribute enough to the ion current to make the $I_i^2 - V_p$ plot linear over a range of a factor 4 in I_i^2 (Ref. 1, Fig. 7).

It is of course possible in principle to define the "sheath edge" at a point r^* where η is much larger than 0.83. This meets with two difficulties: first, it is not easy to find a good alternative definition; second, the deviation from quasineutrality would be so large at r^* that the potential would fall off more rapidly than r^{-2} at r^*, and the orbital theory would no longer be valid everywhere outside r^*. Alternatively, one could try to define a "sheath edge" r^* far out in the quasineutral region. Aside from the aforementioned difficulties of small η and lack of a good definition, this approach would also suffer from the fact that r^* could lie outside the absorption radius as η_p is varied; then the orbital-motion theory cannot be applied to the region outside r^*.

[9] F. F. Chen, Princeton Plasma Physics Laboratory Rept. MATT-252 (1964), J. Nucl. Energy, Pt. C. (to be published).