Saturation Ion Currents to Langmuir Probes

FRANCIS F. CHEN
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey
(Received 15 June 1964; in final form 8 September 1964)

The parabolic variation of saturation ion current with probe potential observed in dense plasmas is fortuitous and is not directly related to the effects of orbital motion. Agreement between measured and computed saturation ion characteristics is illustrated. The discussion is in the framework of collisionless, magnetic-field-free theories; they apply to the experiments only if the ion Larmor radius is much larger than the probe radius.

In dense plasmas for which the Debye length \(k\) is less than the probe radius \(r_p\), it is often found that the saturation ion flux \(I_i\) to a Langmuir probe varies as \((-V_p)^4\), where \(V_p\) is the (negative) probe voltage. This is the dependence expected for cylindrical probes drawing orbital-motion-limited current but not for those drawing space-charge-limited current. In the experiment of Gardner et al., it was observed that a linear \(I_i^\beta-V_p\) dependence was observed even though \(r_p/k\) was of order 10 and the probe current was almost certainly space-charge limited. That such a relationship sometimes holds even for thin sheaths was pointed out by Langmuir himself; he also pointed out that the erroneous application of the orbital theory to such a case would lead to a spurious value of the space potential.

Gardner et al. conjectured that a linear \(I_i^\beta-V_p\) relation might come about because the potential at the sheath edge might be proportional to \(V_p\), so that orbital-motion limitation might occur in the quasineutral region. We believe this view to be misleading or, at least, not useful. Our reasons are summarized in the Appendix. It is a better physical picture to think of the sheath edge as having a constant potential and that the increase in \(I_i\) with \(V_p\) is caused by an increase in sheath radius. The recent work of Lam on a problem previously treated by Wenzl supports this picture. For a cylindrical probe in a plasma with a monoenergetic ion distribution of energy \(E_i\), it is found that the potential \(\eta_\beta\) of the sheath edge is independent of \(\eta_p\) and that \(\eta_\beta\) varies only from 1.0 at \(\beta=0\) to \(\ln 2=0.69\) at \(\beta=\infty\), where \(\beta=E_i/kT_e\) and \(\eta=-eV/kT_e\). The insensitivity to \(E_i\) indicates that the results would not be greatly different for Maxwellian distributions. The absorption radius, or effective probe radius, inside of which all ions are collected, always occurs for cylindrical probes at the radius where \(\eta=\ln 2\). Therefore, the absorption radius always lies outside the sheath and is always quite close to the sheath radius. The picture is, then, that these two radii move outwards together as \(\eta_p\) is increased, while the potentials at these two radii remain constant.

We now wish to present some numerical results to support the view that the linear dependence of \(I_i^\beta\) on \(V_p\) is fortuitous but is approximately true for certain ranges of parameters. In Figs. 1 and 2 are shown log-log plots of \(\eta_\beta\) versus dimensionless current \(J_i^\beta\) for spherical and cylindrical probes, respectively, for various values of \(\xi_p=r_p/k\). It is seen that \(J_i^\beta\) varies approximately as \(\eta_\beta\) only for large \(\eta_\beta\) and small \(\xi_p\). These curves were computed by the method of Allen, Boyd, and Reynolds, which is valid for \(\beta=0\). For cylinders, this method yields a different result from the \(\beta=0\) limit of a finite-\(\beta\) theory. The reason is that the angular momentum \(L\) is assumed to be zero at \(r=\infty\) in this theory, while \(L\) is finite in the \(\beta=0\) limit of a finite-\(\beta\) theory because of nonuniform convergence.

In Figs. 3 and 4 we show similar curves for \(\beta=0.1\) for spheres and cylinders, respectively. Again one finds that

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Fig. 1. Curves of \( \log \eta_0 \) vs \( \log J \) for various values of \( \xi_0 = r_p/k \), for spherical probes and zero ion temperature. Here \( J \) is defined by \( J = ZI_1(e^2/kT_e)/(m_i/2ZkT_e)^{1/2} \), \( \eta_0 = -eI_1/3\pi kT_e \), and \( h \) by \( h^2 = kT_e/4\pi e^2 \).

The \( J^2 - \eta_0 \) relation holds only for certain values of \( \xi_0 \) and \( \eta_0 \). These curves were computed by the method of Bernstein and Rabinowitz,\(^6\) which is valid for monoenergetic ion distributions. The variation of these results with \( \beta \) is not great.

In the limit of large \( \xi_0 \), that is, of dense plasmas, the shape of the \( I_1 - V_p \) curve can be expressed in terms of a single universal function by proper scaling of the variables. This result was obtained by Lam\(^7\) in a rigorous boundary-layer analysis of the Bernstein–Rabinowitz equations. Results of the theory of Lam are summarized in Figs. 5 and 6, which show normalized \( I_1^2 \) as a function of normalized \( V_p \) for both cylinders and spheres. Here \( \tau \) is \( I_1/I_B \), where \( I_B \) is the current predicted by Bohm\(^7\) by neglecting the sheath thickness; and \( A \) is essentially constant but has a weak dependence on \( \beta \). One sees that the \( I_1^2 - V_p \) relation can be approximated by a straight line in all cases except for spherical probes under large voltages. The good fit of the cylindrical probe curve in Fig. 6 to a straight line is entirely accidental; the curve actually has an inflection point in this range of the variables. In terms of normal variables, the equations for the dotted straight lines give the following useful approximate formulas:

\[
\text{Sphere: } \frac{d(ZeI_1)^2}{dV_p} = 2.6 \times 10^{-30} \frac{Z^n}{N} (n^2 r_p (kT_e)_0)^{1/2} \quad (1)
\]

\[
\text{Cylinder: } \frac{d(ZeI_1)^2}{dV_p} = 3.3 \times 10^{-31} \frac{Z^n}{N} (n^2 r_p (kT_e)_0)^{1/2} \quad (2)
\]

\[
\text{Cylinder: } \frac{d(ZeI_1)^2}{dV_p} = 2.6 \times 10^{-31} \frac{Z^n}{N} (n^2 r_p (kT_e)_0)^{1/2} \quad (3)
\]

Here \( Z \) and \( N \) are, respectively, the charge number and atomic weight of the ions; \( I_1 \) is the particle flux per centimeter length in the cylindrical case and the left-hand side is in \( A^2/V \).

In Figs. 7 and 8 we show the comparison of Lam's theoretical curves with some cylindrical probe measurements made by Kuckes\(^8\) in a thermally ionized cesium plasma with \( Z = 1 \). Since the magnetic field was about 10 kG in this experiment, the ratio of \( r_L \) to \( r_p \) was only 3 or 4. In spite of this, it is seen that an excellent fit with theory is obtained for large \( \xi_0 \). For \( \xi_0 = 8 \) the theory is not expected to be very accurate, and indeed one can

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\(^8\) A. F. Kuckes, Princeton Plasma Physics Laboratory (private communication).
Fig. 5. Curves of $\tau^2$ vs $A\xi_p^3n_c$ from the theory of Lam (Ref. 3), which is valid for large $\xi_p$. For spheres,

$$\tau = I_1\sqrt{1.57} n (2\pi kT_e/m_e)^{1/2}$$

For cylinders,

$$\tau = I_2/[1.975 (2\pi kT_e/m_e)^{1/2}]$$

A range of $\tau$ corresponding to relatively thin sheaths is covered.

discern a difference in slope between theory and experiment in Fig. 7. The value of $\xi_p$ found from the fit with theory yields a value of the plasma density $n$. In both cases this value was within 18% of that found by ordinary microwave interferometry.

In Fig. 9 we show the data of Gardner et al. taken from Fig. 7 of Ref. 1. In reducing the data to dimensionless form we have assumed $r_p = 9 \times 10^{-3}$ cm, $l_p = 0.32$ cm, $kT_e = 7.5$ eV, and $V_s = 129$ V, where $l_p$ is the probe length and $V_s$ the space potential relative to the cathode. The ratio $r_L/r_p$ was of order 10 in this case. Also shown in Fig. 9 are four points from Kuczes' data of Fig. 7 and theoretical curves from Bernstein and Rabinowitz (BR) for $\beta = 0.1$ and from Lam for $\beta = 1$. The experimental value of $\beta$ was about 1 in both cases. The theory of Lam is not very accurate for such low values of $\xi_p$; on the other hand, BR calculations for $\beta = 1$ are not available. Both theories suffer from the neglect of a spread in ion energies, an effect which should become noticeable at $\beta = 1$.

The Gardner data have been plotted for assumed effective charge numbers $Z$ of 1.0, 1.5, and 2.0, since an independent measurement of $Z$ is not available. It is seen that although the data fall in a straight line, the slope is larger than theory would predict for any value of $Z$. Part of the discrepancy may be due to the inexactness of the theories, and part to errors in the assumed values of $r_p$, $l_p$, $kT_e$, and $V_s$. The value of $kT_e$ was taken from a measurement (Fig. 13, Ref. 1) at the same point and the same radius ($r = 1.14$ cm) as the probe curve. Reasonable adjustments in $r_p$, $l_p$, and $kT_e$ do not remove the discrepancy between the absolute magnitude of $I_1$ and the slope. Adjustments in $V_s$ can bring agreement, but there are limits to the value of $V_s$; it must lie between the floating potential $V_f$ and the intercept $V_1$ of the straight line. In this particular case $V_f - V_1$ was too small for $I_1$ to be the total current; hence we assumed that the electron part had already been subtracted, and what was plotted was $I_1$. We then chose $V_s$ to be the potential at which $I_1$ was approximately equal to the random ion current in the plasma. If one ignores the slope and uses only the absolute magnitude of $I_1$, one finds $\xi_p = 10$ for $Z = 1.5$. This yields $n_s = 5 \times 10^{12}$ cm$^{-3}$, about 66% higher than indicated by microwave measurements (Fig. 17, Ref. 1). If one regards the slope alone, use of Eq. (2) leads to a
larger discrepancy: \( n_0 = 1.3 \times 10^{13} \text{ cm}^{-3} \). This is still unresolved.

Note that use of Eqs. (1), (2), and (3) gives only the product \( Zn_0 \), since information on the intercept has been discarded. If \( kT_\alpha \) and \( V_\alpha \) are known accurately enough, the use of the \( I_i^2 - V_p \) plot can, in principle, give \( Z \) and \( n \) separately, but the accuracy is not great.

We are indebted to Dr. A. F. Kuckes for access to his data, to Professor S. H. Lam for helpful conversations, to H. Fishman for some of the numerical computations, and to K. P. Mann for help with the drawings. Further numerically computed ion probe characteristics may be found in another report.\(^9\)

**APPENDIX**

We give here four reasons for the statement that the \( I_i^2 - V_p \) dependence is not caused by Langmuir's orbital-motion theory operating in the quasineutral region. Unless otherwise specified, the discussion concerns cylindrical probes, for which this dependence is predicted by Langmuir. Although we consider only monoenergetic distributions, the insensitivity of the theories to ion energy makes it difficult to see how these arguments can be greatly different for Maxwellian distributions.

(1) An \( I_i^2 - V_p \) dependence occurs for certain ranges of parameters even when all ions are assumed to move radially, so that there is no question of orbital motion. This can be seen in Figs. 1 and 2.

(2) Figures 1, 3, and 5 show that \( I_i^2 V_p \) is not a bad approximation even for spheres. Orbital theory would predict \( I_m V_p \).

(3) The parameter \( \tau \) in the theory of Lam\(^8\) is essentially the ratio of the sheath area to the probe area. It is clear from Figs. 5 and 6, then, that it is the change of sheath area which gives rise to the approximately linear \( I_i^2 - V_p \) relation.

(4) For \( \beta = 1 \), Lam\(^8\) gives \( \eta_0 = 0.83 \), independent of \( \eta_0 \). Here \( \eta_\alpha \) is the potential at the sheath edge, defined in the usual manner as the radius at which the quasineutral solution of Poisson's equation turns back on itself. The true value of \( \eta_0 \) is somewhat less than the value \( \eta_0 \) given by the quasineutral solution; therefore, for \( \beta = 1 \) the value of \( \eta_0 \) always lies below 0.83 as \( \eta_0 \) is varied. Although it is unlikely that \( \eta_0 \) lies far below \( \eta_\alpha \), it is, in principle, possible for \( \eta_0 \) to vary with \( \eta_0 \), giving rise to orbital-motion limitation outside the sheath. However, for such small values of \( \eta_0 \), the orbital-motion theory no longer predicts that \( I_i^2 \) should be proportional to \( V_p \); instead, the second term in the more exact formula \( I_i \approx 2\pi^{-1}x^4 + \varepsilon(1 - erf(x)) \), where \( x = \eta_0 / \beta \), gives rise to considerable curvature in the \( I_i^2 - V_p \) plot. It is possible that in a Maxwellian distribution the ions with small \( \beta \) have large \( x \) even though \( \eta_0 < 0.83 \); but it would then be quite inconceivable that the slow ions should contribute enough to the ion current to make the \( I_i^2 - V_p \) plot linear over a range of a factor 4 in \( I_i^2 \) (Ref. 1, Fig. 7).

It is of course possible in principle to define the "sheath edge" at a point \( r^* \) where \( \eta \) is much larger than 0.83. This meets with two difficulties: first, it is not easy to find a good alternative definition; second, the deviation from quasineutrality would be so large at \( r^* \) that the potential would fall off more rapidly than \( r^{-2} \) at \( r^* \), and the orbital theory would no longer be valid everywhere outside \( r^* \). Alternatively, one could try to define a "sheath edge" \( r^* \) far out in the quasineutral region. Aside from the aforementioned difficulties of small \( \eta \) and lack of a good definition, this approach would also suffer from the fact that \( r^* \) could lie outside the absorption radius as \( \eta_\alpha \) is varied; then the orbital-motion theory cannot be applied to the region outside \( r^* \).