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Convection Arising from Large-Amplitude Plasma Waves

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It is found that large-amplitude plasma waves can cause dc particle drifts which can have a deleterious effect on plasma confinement during rf heating. A standing wave in an inhomogeneous plasma is likely to cause convective cells with a periodicity equal to half the wavelength of the wave. As examples, ordinary and extraordinary electromagnetic waves and ion cyclotron waves are chosen for detailed treatment.

I. INTRODUCTION

When plasmas are subjected to rf heating, it is often observed that particles are lost rapidly during the rf pulse. We have investigated the possibility that this enhanced loss is caused by nonlinear interactions of the large-amplitude waves. When two waves of frequencies ω_1 and ω_2 interact, the sum and difference frequencies are generated. If $\omega_1 \approx \omega_2$, the difference frequency is zero or nearly zero. Such extremely low frequencies are especially efficient for transporting plasma, and it is these dc drifts arising from high-frequency waves that we have studied.

The method we have followed is a straightforward quasilinear calculation of the second-order corrections to zeroth-order quantities in the linear theory of plasma waves. The mode–mode coupling effects occurring at harmonics and combinations of ω_1 and ω_2 were treated in an earlier paper.[1] In this paper, we eliminate these effects by averaging over time and consider only $\omega \approx 0$. After the completion of this work, we became aware of a similar calculation by Kotsarenko *et al.*[2] Our work differs from theirs in the following respects: (1) Rather than emphasizing the small quasilinear change in the equilibrium profiles, we pay particular attention to the interesting stratified drifts which occur in the presence of standing waves; (2) we follow the development of quasisteady electric fields and convective patterns when the plasma is nonuniform; and (3) we pay particular attention to the case of a strong magnetic field. In this paper, mathematical detail and completeness are sacrificed for simplicity and clarity of physical ideas. A more detailed treatment of the cases in which a steady-state solution exists, including the effects of cylindrical geometry, may be found in another paper.[3]

II. GENERAL RESULTS

The plasma is described by the two-fluid equations of motion and of continuity, together with Maxwell's equations (in esu):

$$\frac{\partial \mathbf{v}_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \nabla \mathbf{v}_\alpha = \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) - \frac{KT_\alpha}{m_\alpha}\frac{\nabla n_\alpha}{n_\alpha} - \nu_{\alpha\beta}(\mathbf{v}_\alpha - \mathbf{v}_\beta), \quad (1)$$

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0, \quad \alpha = e, i, \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3)$$

$$c^2 \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi e(n_i \mathbf{v}_i - n_e \mathbf{v}_e), \quad (4)$$

where $\nu_{\alpha\beta}$ is a collision frequency, and the rest of the notation is standard. We write $\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)} + \mathbf{v}^{(2)} + \cdots$, and similarly for the other variables. The simplest set of equations which will produce the effects we wish to point out can be obtained by setting $\mathbf{E}^{(0)} = 0$, $\nu_{\alpha\beta} = 0$, $T_\alpha = 0$, $\mathbf{v}_\alpha^{(0)} = 0$, and $\mathbf{B}^{(0)} = B_0\hat{\mathbf{z}}$. We are considering, therefore, a cold, collisionless, nonstreaming plasma in a uniform magnetic field. If $\nabla n_0 = 0$, the solution of Eqs. (1)–(4) to first-order results in the waves of the Clemmow–Mullaly–Allis diagram.[4] These known solutions, indicated by the superscript (1), will be assumed to be locally valid when ∇n_0 has a scale length long compared with the wavelength.

The quasilinear correction to the linear solutions is found by assuming $|\mathbf{v}^{(2)}| \ll |\mathbf{v}^{(1)}|$, etc., subtracting the zeroth- and first-order equations from Eqs. (1)–(4), and neglecting higher orders than the second (in the oscillation amplitude). From Eq. (1) we obtain

$$\frac{\partial \mathbf{v}_\alpha^{(2)}}{\partial t} = -\mathbf{v}_\alpha^{(1)} \cdot \nabla \mathbf{v}_\alpha^{(1)} + \frac{q_\alpha}{m_\alpha}(\mathbf{E}^{(2)} + \mathbf{v}_\alpha^{(1)} \times \mathbf{B}^{(1)} + \mathbf{v}_\alpha^{(2)} \times \mathbf{B}_0). \quad (5)$$

In general, $\mathbf{v}_\alpha^{(2)}$ will contain an oscillating part and a dc part. Since we are interested in the dc part, we take a short-time average of Eq. (5) over a few periods of the fundamental frequency. We now have

$$\frac{\partial \mathbf{v}_\alpha^{(2)}}{\partial t} = \frac{q_\alpha}{m_\alpha}(\mathbf{E}^{(2)} + \mathbf{v}_\alpha^{(2)} \times \mathbf{B}_0 + \mathbf{S}_\alpha), \qquad (6)$$

$$\mathbf{S}_\alpha \equiv \langle \mathbf{v}_\alpha^{(1)} \times \mathbf{B}^{(1)} \rangle - \frac{m_\alpha}{q_\alpha}\langle \mathbf{v}_\alpha^{(1)} \cdot \nabla \mathbf{v}_\alpha^{(1)} \rangle, \qquad (7)$$

in which the second-order quantities are understood to be quasi-dc compared with the frequency of the original oscillation. For the ions, we must keep the time derivative in Eq. (6) in order to include the polarization current; otherwise, the buildup of the field $\mathbf{E}^{(2)}$ will be found to occur much too fast. Usually, $\mathbf{v}^{(2)}$ will be different for ions and electrons, hence the possibility of charge separation and dc electric fields.

Note that the nonlinear source term $\mathbf{S}_\alpha$ has the dimensions of an electric field and will give rise to an $\mathbf{S} \times \mathbf{B}_0$ drift $\mathbf{v}^{(2)}$. Equation (7) shows that $q\mathbf{S}$ is a force arising from (a) the Lorentz force due to first-order motion across the *perturbed* line of force, and (b) the viscous drag due to motion along the first-order velocity gradient. The terms in Eq. (7) are mostly oscillatory, but there can be a nonzero time average if, for instance, $\mathbf{v}^{(1)}$ and $\mathbf{B}^{(1)}$ have an in-phase component. Then, the source term $\mathbf{S}$ can be nonvanishing and give rise to a secular drift $\mathbf{v}^{(2)}$.

Proceeding in a similar manner with Eqs. (2)–(4), we obtain

$$\frac{\partial n_\alpha^{(2)}}{\partial t} = \dot{N}_\alpha - n_0 \nabla \cdot \mathbf{v}_\alpha^{(2)} - \mathbf{v}_\alpha^{(2)} \cdot \nabla n_0, \qquad (8)$$

$$\nabla^2 \phi^{(2)} = 4\pi e(n_e^{(2)} - n_i^{(2)}), \qquad (9)$$

where $\mathbf{E}^{(2)} = -\nabla \phi^{(2)}$ and

$$\dot{N}_\alpha \equiv -\langle \nabla \cdot n_\alpha^{(1)} \mathbf{v}_\alpha^{(1)} \rangle. \qquad (10)$$

Equations (6)–(10) give the second-order quantities in terms of the source terms $\mathbf{S}_\alpha$ and $\dot{N}_\alpha$, which are known from the linear solutions. Equations (6), (8), and (9) constitute a set of nine scalar equations for the nine unknown quantities $\mathbf{v}_i^{(2)}$, $\mathbf{v}_e^{(2)}$, n_i, n_e, and ϕ.

The solution of Eq. (6) is

$$\mathbf{v}_{\perp\alpha}^{(2)} = B_0^{-2}[(\mathbf{E}^{(2)} + \mathbf{S}_\alpha) \times \mathbf{B}_0 + (m_\alpha/q_\alpha)\dot{\mathbf{E}}_\perp^{(2)}], \qquad (11)$$

$$\dot{v}_{\parallel\alpha}^{(2)} = (q_\alpha/m_\alpha)(E_\parallel^{(2)} + S_{\parallel\alpha}), \qquad (12)$$

where the dot indicates $\partial/\partial t$. In obtaining Eqs. (11) and (12), we have assumed that the time variation is slow compared with a cyclotron period and have made use of the fact that $\dot{\mathbf{S}}_\alpha = 0$, since $\mathbf{S}$ is a time-averaged quantity. We may now insert Eqs. (11) and (12) into Eq. (8) and use the latter in the time derivative of Eq. (9). After a little manipulation, we obtain

$$\left(1 + \frac{4\pi\rho}{B_0^2}\right)\nabla_\perp^2 \dot{\phi}^{(2)} + \nabla_\parallel^2 \dot{\phi}^{(2)} + \frac{4\pi\rho}{B_0^2}\frac{\nabla n_0}{n_0} \cdot \nabla_\perp \dot{\phi}^{(2)}$$
$$= 4\pi e\left[\dot{N}_e - \dot{N}_i + \frac{n_0}{B_0^2}\mathbf{B}_0 \cdot \left(\frac{\nabla n_0}{n_0} + \nabla\right) \times (\mathbf{S}_i - \mathbf{S}_e) + n_0 \nabla_\parallel (v_{i\parallel}^{(2)} - v_{e\parallel}^{(2)})\right], \qquad (13)$$

where $\rho \equiv n_0 m_i$. If the dielectric constant $1 + 4\pi\rho/B_0^2$ is $\gg 1$, as we shall assume, we may neglect $\nabla^2 \dot{\phi}^{(2)}$ on the left-hand side. This amounts to replacing Poisson's equation (9) with the quasineutrality condition $n_e^{(2)} = n_i^{(2)}$. We then obtain the following equation describing the development of the quasisteady potential distribution $\phi^{(2)}$:

$$\frac{1}{\omega_{ci}}\left(\frac{\nabla n_0}{n_0} + \nabla_\perp\right) \cdot \nabla_\perp \dot{\phi}^{(2)}$$
$$= \frac{\mathbf{B}_0}{B_0} \cdot \left(\frac{\nabla n_0}{n_0} + \nabla\right) \times (\mathbf{S}_i - \mathbf{S}_e) + \frac{B_0}{n_0}(\dot{N}_e - \dot{N}_i) + B_0 \frac{\partial}{\partial z}(v_{i\parallel}^{(2)} - v_{e\parallel}^{(2)}), \qquad (14)$$

where $\omega_{ci} \equiv eB_0/m_i$. We must now distinguish several cases.

(1) $k_\parallel = 0$. If $k_\parallel = 0$, the last term on the right-hand side vanishes; and the remaining terms are independent of time. The potential ϕ [we may now omit the superscript (2)] then grows linearly with time t:

$$\left(\frac{\nabla n_0}{n_0} + \nabla\right) \cdot \nabla \phi = \omega_{ci} t \left[\frac{\mathbf{B}_0}{B_0} \cdot \left(\frac{\nabla n_0}{n_0} + \nabla\right) \times (\mathbf{S}_i - \mathbf{S}_e) + \frac{B_0}{n_0}(\dot{N}_e - \dot{N}_i)\right]. \qquad (15)$$

(2) $k_\parallel \neq 0$. In this case, we can take the time derivative of Eq. (14) and use Eq. (12) for $\dot{v}_{\parallel\alpha}$. Since the source terms $\mathbf{S}_\alpha$, $\dot{N}_\alpha$ are independent of time, we obtain

$$\left(\frac{\nabla n_0}{n_0} + \nabla_\perp\right) \cdot \nabla_\perp \ddot{\phi} = \omega_{ci} \frac{\partial}{\partial z}\left[\omega_{ce}\left(S_{\parallel e} - \frac{\partial \phi}{\partial z}\right) + \omega_{ci}\left(S_{\parallel i} - \frac{\partial \phi}{\partial z}\right)\right]. \qquad (16)$$

(2′) $k_\parallel \neq 0$, $S_{\parallel e} = S_{\parallel i} = 0$. In this case, a possible solution is $\partial\phi/\partial z = 0$. The quasisteady electric field is then perpendicular to $\mathbf{B}_0$ and is described by Eq. (15) for the $k_\parallel = 0$ case. Neglecting $\nabla n_0/n_0$ for the moment, we see that another solution is given by $\ddot{\phi} \cong -(k_z/k_\perp)^2\omega_{ce}\omega_{ci}\phi$, where we have replaced $\partial/\partial z$ by ik_z and $\nabla_\perp$ by $ik_\perp$. This is an oscillation at the lower hybrid frequency which we shall ignore, since we are interested in secularly growing fields.

($2''$) $k_\parallel \neq 0$, $S_{\parallel e} \neq 0$. In this case, the large coefficient ω_{ce} in Eq. (16) requires $\partial\phi/\partial z \approx S_{\parallel e}$; and it is $S_{\parallel e}$ that determines E_z at all times. E_z cannot grow secularly, but $\mathbf{E}_\perp$ is still given by Eq. (15) and can increase with time.

($2'''$) $k_\parallel \neq 0$, $S_{\parallel e} = 0$, $S_{\parallel i} \neq 0$. In this case, for $\omega_{ce} \gg \omega_{ci}$, we have

$$\left(\frac{\nabla n_0}{n_0} + \nabla_\perp\right)\cdot\nabla_\perp\ddot{\phi} \cong \omega_{ci}^2 \frac{\partial}{\partial z}\left(S_{\parallel i} - \frac{m_i}{m_e}\frac{\partial\phi}{\partial z}\right). \quad (17)$$

The source term $S_{\parallel i}$, therefore, requires $\partial\phi/\partial z \approx (m_e/m_i)S_{\parallel i}$. However, this case is rather complicated, because the effect of the perpendicular source terms may be of comparable magnitude; and the problem may not be easily separable.

III. APPLICATION TO MICROWAVE HEATING

A. Ordinary Waves

1. Traveling wave

As specific examples, we consider electromagnetic waves propagating perpendicular to $\mathbf{B}_0(k_\parallel = 0)$. Figure 1 shows the $\mathbf{E}$ and $\mathbf{B}$ vectors of a plane-polarized ordinary wave propagating in the y direction. We investigate the self-interaction of this wave at large amplitudes. If $\mathbf{E}^{(1)}$ is taken to have the form $\hat{\mathbf{z}}\mathcal{E}\cos(ky - \omega t)$, the linear solution of Eqs. (1)–(4) yields

$$\begin{aligned} \mathbf{E}^{(1)} &= \hat{\mathbf{z}}\,\mathcal{E}\cos\Phi, \\ \mathbf{B}^{(1)} &= \hat{\mathbf{x}}\,\mathcal{E}(k/\omega)\cos\Phi, \\ \mathbf{v}_e^{(1)} &= \hat{\mathbf{z}}\,\mathcal{E}(e/m\omega)\sin\Phi, \\ \mathbf{v}_i^{(1)} &= -\hat{\mathbf{z}}\,\mathcal{E}(e/M\omega)\sin\Phi, \\ n^{(1)} &= 0, \end{aligned} \quad (18)$$

where $\Phi \equiv ky - \omega t$. The source terms defined by Eqs. (7) and (10) can now be evaluated for this solution. Since $\langle\sin\Phi\cos\Phi\rangle = 0$, we find $\dot{N}_\alpha = 0$, $\mathbf{S}_\alpha = 0$. The quasilinear effects identically vanish in this particular case.

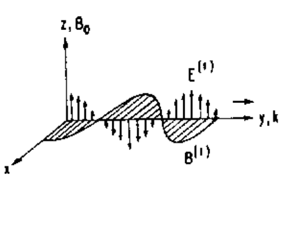

FIG. 1. A traveling ordinary wave.

2. Standing waves

We next consider two ordinary waves of the type shown in Fig. 1, traveling in the $+y$ and $-y$ directions. The real part of the first-order quantities from the linear solution are as follows:

$$\begin{aligned} \mathbf{E}^{(1)} &= \hat{\mathbf{z}}\,\mathcal{E}(\cos\Phi + \cos\Phi'), \\ \mathbf{B}^{(1)} &= \hat{\mathbf{x}}\,\mathcal{E}(k/\omega)(\cos\Phi - \cos\Phi'), \\ \mathbf{v}_e^{(1)} &= \hat{\mathbf{z}}\,\mathcal{E}(e/m\omega)(\sin\Phi + \sin\Phi') = -(M/m)\mathbf{v}_i^{(1)}, \\ n^{(1)} &= 0, \end{aligned} \quad (19)$$

where $\Phi \equiv ky - \omega t$ and $\Phi' \equiv -ky - \omega t$. From this and Eqs. (7) and (10), we can compute the source terms:

$$\begin{aligned} \dot{N}_i &= \dot{N}_e = 0, \\ \mathbf{S}_e &= \hat{\mathbf{y}}\,\mathcal{E}^2\frac{ek}{m\omega^2}\langle(\sin\Phi + \sin\Phi') \\ &\qquad\cdot(\cos\Phi - \cos\Phi')\rangle + 0 \quad (20) \\ &= \hat{\mathbf{y}}\,\mathcal{E}^2\frac{ek}{m\omega^2}(-\sin 2ky), \\ \mathbf{S}_i &= -\frac{m}{M}\mathbf{S}_e, \end{aligned}$$

where we have noted that $\langle\cos\Phi'\sin\Phi\rangle = -\langle\cos\Phi\sin\Phi'\rangle = \frac{1}{2}\sin 2ky$. Thus, the main effect is that the electrons feel a second-order force in the y direction, which gives them an $\mathbf{S}_e \times \mathbf{B}_0$ drift in the x direction. The drift is spatially periodic with half the wavelength of the original waves:

$$\mathbf{v}_e^{(2)} = -\hat{\mathbf{x}}\frac{\mathcal{E}^2}{B_0^2}\frac{\omega_{ce}}{\omega}\frac{k}{\omega}\sin 2ky \qquad (\nabla n_0 = 0). \quad (21)$$

If $\nabla n_0 = 0$, there is simply a stratified pattern of drifts, with the ions moving m/M times more slowly than the electrons and in the opposite direction. If ∇n_0 is in the y direction (that is, in the direction of $\mathbf{k}$), the drifts cause no charge separation; and $\mathbf{E}^{(2)}$ vanishes.

If ∇n_0 is in the x direction (that is, perpendicular to $\mathbf{k}$), however, the difference between electron and ion drift velocities causes a charge separation and thus an electric field, which we can compute from Eq. (15). The terms $\dot{N}_\alpha$ and $\nabla \times \mathbf{S}_\alpha$ vanish identically in this case, and we obtain

$$\mathbf{E}^{(2)} = \hat{\mathbf{y}}\frac{\mathcal{E}^2}{B_0}\left|\frac{\nabla n_0}{n_0}\right|\frac{\omega_{ci}\omega_{ce}}{2\omega^2}\,t\cos 2ky. \quad (22)$$

Inserting this in Eq. (11), we obtain

$$\begin{aligned} \mathbf{v}_e^{(2)} &= -\hat{\mathbf{x}}\left(\frac{\mathcal{E}^2}{B_0^2}\frac{\omega_{ce}}{\omega}\right) \\ &\qquad\cdot\left(\frac{k}{\omega}\sin 2ky - \frac{\omega_{ci}t}{2\omega}\left|\frac{\nabla n_0}{n_0}\right|\cos 2ky\right). \end{aligned} \quad (23)$$

This result shows that when the microwave heating is first applied, a pattern of stratified drifts, given by the first term, is set up by the interaction of the electron velocity in one wave with the perturbed magnetic field in the other. This pattern is shown in Fig. 2(a). The ions have a drift m/M times smaller. The resulting charge separation creates an electric field, which causes both species to drift together with a velocity given by the second term in Eq. (23). After a time $\tau = 2k/\omega_{ci}|\nabla n_0/n_0|$, the second term dominates, and the convective pattern is shifted a quarter wavelength relative to the initial pattern. The drifts tend to smooth out the density gradient. The cross-field transport is very fast relative to classical diffusion rates because the theory does not break down until $E^{(2)}$ becomes comparable to $E^{(1)}$, and $E^{(1)}/B_0$ is fast even for moderate values of $\mathcal{E}$. Standing waves such as we have assumed can arise, for example, when microwave heating is applied by launching a wave with $E \parallel B$ across a plasma with a reflecting wall on the other side.

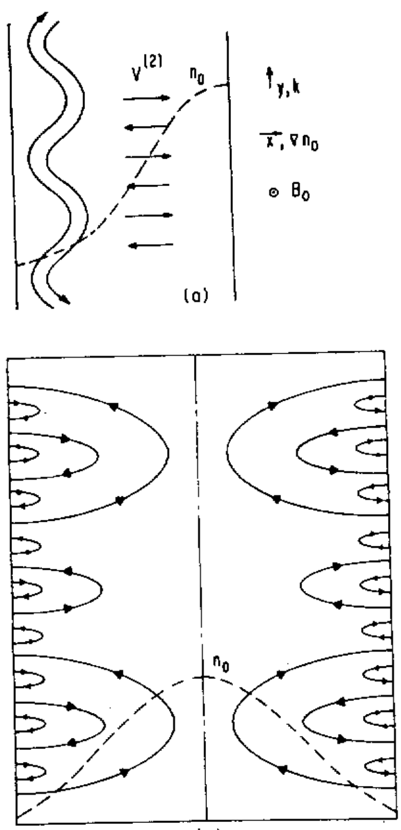

FIG. 2. (a) Stratified drifts $v^{(2)}$ arising when a standing ordinary wave is imposed on a plasma. If the density gradient is in the direction shown, dc electric fields will arise. (b) Convective patterns in an inhomogeneous plasma due to the above effect.

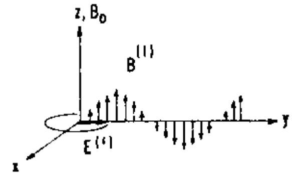

FIG. 3. A traveling extraordinary wave.

Up to now we have neglected the effect of ∇n_0 on the first-order quantities of Eq. (19). From the dispersion relation $c^2k^2/\omega^2 = 1 - \omega_p^2/\omega^2$ for this mode, it is clear that k will increase toward the edge of the plasma. Although we have not treated this problem in detail, Fig. 2(b) shows what the drift pattern would probably look like when this change of wavelength is taken into account.

B. Extraordinary Waves

1. Traveling wave

Next we consider the self-interaction of an elliptically polarized extraordinary wave with $k_\parallel = 0$:

$$\mathbf{E}^{(1)} = \mathcal{E}(\hat{\mathbf{y}} \cos \Phi - a\hat{\mathbf{x}} \sin \Phi), \quad \Phi \equiv ky - \omega t, \qquad (24)$$
$$\mathbf{B}^{(1)} = \hat{\mathbf{z}}\, \mathcal{E} a(k/\omega) \sin \Phi,$$

where

$$a \equiv \omega(\omega_{ce}^2 + \omega_{pe}^2 - \omega^2)/\omega_{pe}^2\omega_{ce}, \qquad (25)$$

ω_{pe} being the electron plasma frequency. The wave vectors are shown in Fig. 3. The first-order quantities from the linear solution are

$$\mathbf{v}_e^{(1)} = -\frac{e\mathcal{E}}{B_0}\frac{\omega^2}{\omega_{pe}^2}\left[\hat{\mathbf{x}}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right) \cos \Phi - \hat{\mathbf{y}}\, \frac{\omega_{ce}}{\omega} \sin \Phi\right],$$

$$\mathbf{v}_i^{(1)} = -\frac{e\mathcal{E}}{M\omega}(a\hat{\mathbf{x}} \cos \Phi + \hat{\mathbf{y}} \sin \Phi), \qquad (26)$$

$$n_e^{(1)} = \frac{k\mathcal{E}}{4\pi e} \sin \Phi,$$

$$n_i^{(1)} = -\frac{e\mathcal{E}kn_0}{M\omega^2} \sin \Phi.$$

From these one can compute the source terms of Eqs. (7) and (10):

$$\mathbf{S}_e = \frac{1}{2}\frac{\mathcal{E}^2}{B_0}\frac{\omega^2\omega_{ce}^2}{\omega_{pe}^4}\frac{k}{\omega}\hat{\mathbf{x}},$$
$$\mathbf{S}_i = \dot{N}_i = \dot{N}_e = 0. \qquad (27)$$

If ∇n_0 is perpendicular to $\mathbf{k}$, the $\mathbf{S}_e \times \mathbf{B}$ drift is perpendicular to ∇n_0, and no electric field is built up. The second-order drifts from Eq. (11) are then

$$\mathbf{v}_e^{(2)} = -\frac{1}{2}\frac{\mathcal{E}^2}{B_0^2}\frac{\omega^2\omega_{ce}^2}{\omega_{pe}^4}\frac{k}{\omega}\hat{\mathbf{y}}, \qquad \mathbf{v}_i^{(2)} \quad 0. \qquad (28)$$

The self-interaction of this wave causes an electron drift which is independent of mass and uniform in space (except for the slow variation of ω_{pe}); the ions do not drift.

If ∇n_0 is parallel to $\mathbf{k}$, one would expect $\mathbf{v}_e^{(2)}$ to give rise to an electric field and a plasma rotation. This case, however, requires a more careful solution of the linear problem in a nonuniform plasma, because the charge separation from $\mathbf{v}_e^{(2)}$ is to a large extent cancelled by the source terms $\dot{N}_e$ and $\dot{N}_i$, which do not vanish when $\nabla n_0 \neq 0$.

2. Standing waves

We now consider the superposition of two extraordinary waves propagating in the $+y$ and $-y$ directions:

$$\mathbf{E}^{(1)} = \mathcal{E}\hat{\mathbf{y}}(\cos\Phi + \cos\Phi') - \mathcal{E}a\hat{\mathbf{x}}(\sin\Phi + \sin\Phi'),$$
$$\mathbf{B}^{(1)} = \hat{\mathbf{z}}\,\mathcal{E}a(k/\omega)(\sin\Phi - \sin\Phi'), \qquad (29)$$
$$\Phi' = -ky - \omega t,$$

with a defined by Eq. (25). The linear solution yields

$$\mathbf{v}_e^{(1)} = -\frac{e\mathcal{E}}{B_0}\frac{\omega^2}{\omega_{pe}^2}\left[\hat{\mathbf{x}}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)(\cos\Phi + \cos\Phi') - \hat{\mathbf{y}}\frac{\omega_{ce}}{\omega}(\sin\Phi + \sin\Phi')\right], \qquad (30)$$

$$n_e^{(1)} = \frac{k\mathcal{E}}{4\pi e}(\sin\Phi - \sin\Phi').$$

As before, the ion drifts will be small, and we omit the ion quantities. The source terms are

$$\mathbf{S}_e = -\hat{\mathbf{y}}\,\mathcal{E}^2\frac{ek}{m}\left[\frac{1}{\omega_{pe}^2} + \frac{1}{\omega_{ce}^2}\left(1 - \frac{\omega^2}{\omega_{pe}^2}\right)^2\right]\sin 2ky, \qquad (31)$$
$$\dot{N}_e = 0.$$

Note that the self-interactions of the waves have cancelled out, but we have the stratified drifts varying as $\sin 2ky$. Comparing this with Eq. (20), we see that the results for standing extraordinary waves are the same as for standing waves if we replace ω^{-2} by $\omega_{pe}^{-2} + \omega_{ce}^{-2}(1 - \omega^2/\omega_{pe}^2)^2$. With this substitution, Eqs. (23) for $\mathbf{v}_e^{(2)}$ may be used; the electric field builds up as in the previous case, and the stratified drifts of Fig. 2 will occur.

3. Mixed waves

If both ordinary and extraordinary waves are present simultaneously, we find no new nonlinear interactions besides those we have already discussed.

IV. APPLICATION TO ION CYCLOTRON HEATING

As an example of the case $k_\parallel \neq 0$, we consider ion cyclotron waves propagating in the $x - z$ plane (Fig. 4). Taking $m/M = 0$, $n_e = n_i$, and $\epsilon \equiv \omega_{pi}^2/c^2k^2 \ll 1$, from Eqs. (1)–(4) we find the dispersion relation

$$\Omega^2 = \left[1 + \frac{\omega_{pi}^2}{c^2}\left(\frac{1}{k^2} + \frac{1}{k_z^2}\right)\right]^{-1}, \qquad (32)$$

where $\Omega \equiv \omega/\omega_{ci}$ and $k^2 \equiv k_x^2 + k_z^2$. If there are two waves with phases Φ and Φ', the first-order quantities are as follows:

$$E_y^{(1)} = \mathcal{E}(\cos\Phi + \cos\Phi'),$$
$$E_x^{(1)} = -\frac{\mathcal{E}}{\Omega}\frac{k^2}{k_z^2}(\sin\Phi + \sin\Phi'), \qquad E_z^{(1)} = 0,$$
$$v_{ix}^{(1)} = \frac{\mathcal{E}}{B_0}\frac{1}{\epsilon}(\cos\Phi + \cos\Phi'),$$
$$v_{iy}^{(1)} = \frac{\mathcal{E}}{B_0}\frac{1}{\Omega}\left(\frac{1}{\epsilon} - 1\right)(\sin\Phi + \sin\Phi'), \qquad v_{iz}^{(1)} = 0,$$
$$v_{ex}^{(1)} = \frac{\mathcal{E}}{B_0}(\cos\Phi + \cos\Phi'),$$
$$v_{ey}^{(1)} = \frac{\mathcal{E}}{B_0}\frac{1}{\Omega}\frac{k^2}{k_z^2}(\sin\Phi + \sin\Phi'), \qquad (33)$$
$$v_{ez}^{(1)} = \frac{\mathcal{E}}{B_0}\frac{k_x}{k_z}\left(\frac{1}{\epsilon} - 1\right)(\cos\Phi - \cos\Phi'),$$
$$B_x^{(1)} = -\mathcal{E}\frac{k_z}{\omega}(\cos\Phi - \cos\Phi'),$$
$$B_y^{(1)} = -\frac{\mathcal{E}}{\Omega}\frac{k^2}{\omega k_z}(\sin\Phi - \sin\Phi'),$$
$$B_z^{(1)} = \frac{\mathcal{E}}{B_0}\frac{k_x}{\Omega}\frac{M}{e}(\cos\Phi + \cos\Phi'),$$
$$n_1 = \frac{\mathcal{E}}{B_0}\frac{n_0}{\epsilon}\frac{k_x}{\omega}(\cos\Phi + \cos\Phi'),$$

where ∇n_0 has been taken as $n_0'\hat{\mathbf{x}}$.

A. Self-Interaction

For a single wave, we take $\Phi = k_x x + k_z z - \omega t$ and omit the Φ' terms in Eq. (33). The source terms work out to be

$$\mathbf{S}_i = \frac{1}{\epsilon}\mathbf{S}_e = -\hat{\mathbf{y}}\frac{1}{2}\frac{\mathcal{E}^2}{B_0}\frac{k_x}{\omega}\frac{1}{\epsilon^2}, \quad \dot{N}_i = \dot{N}_e = 0. \qquad (34)$$

Since $S_{\parallel\alpha}$ vanishes, we have case (2′) of Sec. II; the $S_{\perp\alpha}$ terms determine the drift motions. Initially the $\mathbf{S}\times\mathbf{B}$ drifts are seen to be in the $+\hat{x}$ direction for $k_x < 0$; that is, radially outward if the wave is

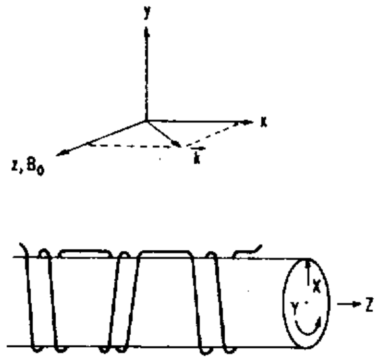

FIG. 4. Coordinate system used for ion cyclotron waves. The cylindrical geometry of a Stix coil has been replaced by Cartesian coordinates.

propagating inward. The resulting charge separation will cause the plasma to "rotate" in the y direction. The rate of buildup of $\mathbf{E}^{(2)}$ requires a more detailed treatment of the plasma inhomogeneity; here again, the $\dot{N}_i$ term tends to cancel the charge separation when $\nabla n_0 \neq 0$.

B. Counterstreaming Ion Cyclotron Waves

Here, we take $\Phi = k_x x + k_z z - \omega t$, $\Phi' = k_x x - k_z z - \omega t$, so that $\langle \sin\Phi \cos\Phi' + \cos\Phi \sin\Phi' \rangle = 0$, $\langle \sin\Phi\cos\Phi' - \cos\Phi \sin\Phi'\rangle = \sin 2k_z z$, $\langle \cos\Phi\cos\Phi'\rangle = \langle \sin\Phi\sin\Phi'\rangle = \frac{1}{2}\cos 2k_z z$. From Eq. (33) we compute the following source terms:

$$\mathbf{S}_i = -\hat{\mathbf{y}}\,\frac{\mathcal{E}^2}{B_0}\frac{k_x}{\omega}\frac{1}{\epsilon^2}(1+\cos 2k_z z) + \hat{\mathbf{z}}\,\frac{\mathcal{E}^2}{B_0}\frac{k_z}{\omega}\frac{1}{\Omega}\left[1-\frac{1}{\epsilon}\left(1+\frac{k^2}{k_z^2}\right)\right]\sin 2k_z z,$$
$$\mathbf{S}_e = -\hat{\mathbf{y}}\,\frac{\mathcal{E}^2}{B_0}\frac{k_x}{\omega}\left[\frac{1}{\epsilon}+\left(2-\frac{1}{\epsilon}\right)\cos 2k_z z\right] - \hat{\mathbf{z}}\,\frac{2\mathcal{E}^2}{B_0}\frac{k}{k_z}\frac{k}{\omega}\frac{1}{\Omega}\sin 2k_z z. \quad (35)$$

Since $S_{\parallel e} \neq 0$, we have case (2″) of Sec. II. The potential is then given by

$$\phi = \frac{\mathcal{E}^2}{B_0}\frac{k^2}{k_z^2}\frac{1}{\omega\Omega}\cos 2k_z z + f(x). \quad (36)$$

The function $f(x)$, and therefore E_x, is determined by the perpendicular drifts. From the y components of $\mathbf{S}_\alpha$, we see that $\mathbf{v}_\perp^{(2)}$ has a constant part (arising from the self-interactions) and a periodic part. For $\epsilon \ll 1$, the periodic parts for ions and electrons are 180° out of phase. These drifts are depicted in Fig. 5. The periodic part of $\mathbf{v}_\perp^{(2)}$ causes a charge separation in the z direction, but this is easily cancelled by electron motion along $\mathbf{B}_0$, and E_z is still given by Eq. (36). The constant part of $\mathbf{v}_\perp^{(2)}$ gives rise to a component E_x and a plasma rotation, as in the case of self-interaction.

C. Standing Waves.

We now take $\Phi = k_x x + k_z z - \omega t$, $\Phi' = -k_x x - k_z z - \omega t$, corresponding to standing waves in both the radial and the z direction. The argument $2k_z z$ in the periodic terms of the previous paragraph is now replaced by $\gamma \equiv 2(k_x x + k_z z)$. The source terms are found to be $\dot{N}_e = \dot{N}_i = 0$ and

$$\mathbf{S}_i = \hat{\mathbf{x}}\,\frac{\mathcal{E}^2}{B_0}\frac{k_x}{\omega_{ci}}\left[\frac{1}{\epsilon^2}-\frac{1}{\Omega^2}\left(\frac{1}{\epsilon}-1\right)\right]\sin\gamma + \hat{\mathbf{z}}\,\frac{\mathcal{E}^2}{B_0}\frac{k_z}{\omega}\frac{1}{\Omega}\left[1-\frac{1}{\epsilon}\left(1+\frac{k^2}{k_z^2}\right)\right]\sin\gamma,$$
$$\mathbf{S}_e = \hat{\mathbf{x}}\,\frac{\mathcal{E}^2}{B_0}\frac{k_x}{\omega}\frac{k^2}{k_z^2}\frac{1}{\Omega}\left(\frac{1}{\epsilon}-2\right)\sin\gamma - \hat{\mathbf{z}}\,\frac{2\mathcal{E}^2}{B_0}\frac{k}{\omega}\frac{k}{k_z}\frac{1}{\Omega}\sin\gamma, \quad (37)$$

where we have assumed $m/M \ll \epsilon^2$. Note that the self-interactions have cancelled out. From the condition $\partial\phi/\partial z = S_{\parallel e}$, we obtain

$$\phi = \frac{\mathcal{E}^2}{B_0}\frac{k^2}{k_z^2}\frac{1}{\omega\Omega}\cos 2(k_x x + k_z z) + f(x). \quad (38)$$

In this case $f(x)$ vanishes because the perpendicular drifts, being perpendicular to the density gradient $\nabla n_0 = n_0'\hat{\mathbf{x}}$, do not cause any charge separation. The derivatives of Eq. (38) are sufficient to give both E_x and E_z everywhere. Adding the $\mathbf{S}\times\mathbf{B}_0$ and $\mathbf{E}^{(2)}\times\mathbf{B}_0$ drifts, we obtain

$$\mathbf{v}_{\perp i}^{(2)} = -\hat{\mathbf{y}}\,\frac{\mathcal{E}^2}{B_0^2}\frac{k_x}{\omega_{ci}}\left[\frac{1}{\epsilon^2}-\frac{1}{\Omega^2}\left(\frac{1}{\epsilon}-1-\frac{2k^2}{k_z^2}\right)\right]\cdot\sin 2(k_x x + k_z z),$$
$$\mathbf{v}_{\perp e}^{(2)} = -\hat{\mathbf{y}}\,\frac{\mathcal{E}^2}{B_0^2}\frac{k^2}{k_z^2}\frac{k_x}{\omega}\frac{1}{\epsilon}\frac{1}{\Omega}\sin 2(k_x x + k_z z). \quad (39)$$

These drifts are perpendicular to ∇n_0 and do not cause any direct particle loss; however, losses can

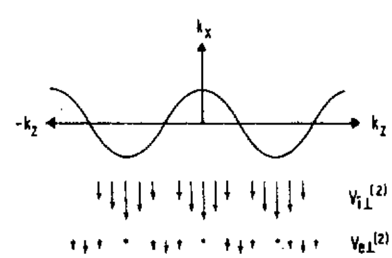

FIG. 5. Quasilinear drifts for ion cyclotron waves propagating in the x direction but standing in the z direction.

occur indirectly because the stratified shear flow would lead to an anomalous viscosity.

In addition to these drifts, the quasilinear electric field will give rise to an acceleration of ions along $\mathbf{B}_0$. From Eqs. (12), (37), and (38) we find

$$\dot{v}_{i\parallel}^{(2)} = \frac{\mathcal{E}^2}{B_0^2}\frac{k_z}{\Omega^2}\left[1 + \frac{2k^2}{k_z^2} - \frac{1}{\epsilon}\left(1 + \frac{k^2}{k_z^2}\right)\right] \cdot \sin 2(k_x x + k_z z). \quad (40)$$

These drifts are also shown in Fig. 6.

D. Numerical Estimates

As typical conditions for an ion cyclotron heating experiment, we take $\mathcal{E} = 50$ V/cm, $B_0 = 16$ kG, $n = 10^{13}$ cm^{-3}, $k_z = 0.5$ cm^{-1}, and $k_x = 0.16$ cm^{-1}. Then, we have $\mathcal{E}/B_0 = 3 \times 10^5$ cm/sec, $\epsilon = 0.1$, and $\omega_{ci} = 1.6 \times 10^8$ sec^{-1} for hydrogen. From Eq. (39) we find

$$|v_{\perp i}^{(2)}| \approx |v_{\perp e}^{(2)}| \approx 3 \times 10^4 \text{ cm/sec}.$$

Since ϵ^{-1} is proportional to $k_x^2 + k_z^2$, we see from Eq. (39) that both $v_{\perp e}^{(2)}$ and $v_{\perp i}^{(2)}$ vary as k_z^5 in the heating region and become larger in the magnetic beach, where $k_z \to \infty$.

Integrating Eq. (40) over half a wavelength of the drift pattern, we find that the maximum energy acquired by the ions is of the order of 10 eV. This is not a large effect in itself, but the local concentrations of density caused by the ion motions may excite ion acoustic waves.

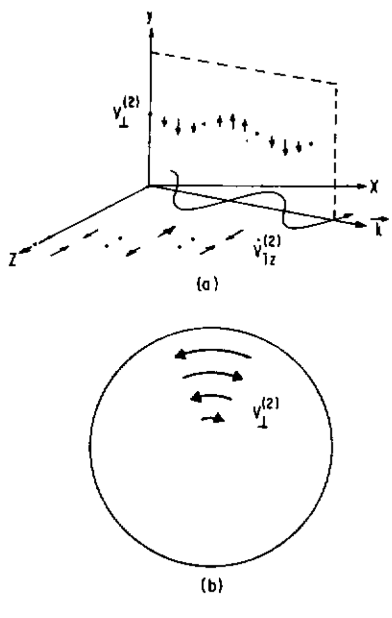

FIG. 6. Pattern of stratified drifts for standing ion cyclotron waves (a) in the idealized plane geometry, and (b) in practice.

V. CONCLUSION

Quasilinear effects in rf plasma heating can give rise to dc drifts of ions and electrons. In an inhomogeneous plasma these drifts can create dc electric fields, which in turn cause convection of both species together. Traveling waves generally can only give rise to a rotation of the plasma; this is dangerous only if the rotation is large enough to cause centrifugal instabilities. Standing waves, however, are particularly treacherous; they lead to stratified drift patterns which can convect plasma out either directly or through secondary effects, such as enhanced viscosity.

In ion cyclotron heating the drift velocities are typically of order 3×10^4 cm/sec and can be many times larger in the magnetic beach. In microwave heating with $k_\parallel = 0$, the dc drifts build up linearly with time and reach a value $|v_\perp^{(2)}| = \mathcal{E}/B_0$ in a time $t = (\omega/\omega_h)^2(2\Lambda B_0/\mathcal{E})$, where $\mathcal{E}$ is the amplitude of the wave electric field, ω_h is the lower hybrid frequency, and Λ is the density scale length.

These convective effects may account in part for enhanced losses observed in previous heating experiments. In the design of future experiments, it would be desirable to minimize these effects by carefully avoiding standing waves.

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