The Two-Ion Hybrid Wave in an Inhomogeneous Plasma

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PREFACE

This is the fourth and last of a series of unclassified articles written for experimentalists to clarify the theoretical basis underlying the plasma separation process. The first paper, Task II-1359 (1978), showed how the plasma wave dispersion curves behaved in the region of the ion cyclotron and two-ion hybrid frequencies. The second paper, Task II-2185 (1979), concerned electrostatic drive and showed how the endplate sheath conditions affected the waves. This paper has been published in Phys. Fluids 22, 2346 (1979). The third paper, Task II-3224 (1980), concerned inductive drive and gave explicit formulas for the field enhancement effect. A short version of this paper, Task II-3552 (1981), was cleared for publication and submitted to Phys. Fluids. After a year's review and rebuttal, this paper has been rejected by three referees who cannot believe that the left-hand circularly polarized wave component can be finite inside the plasma at the ion cyclotron frequency. The manuscript has to be rewritten. The present paper deals with the question of space charge neutralization originally brought up by Ken MacKenzie. We show how standard plasma theory takes space charge effects into account, and we give explicit formulas for the frequency shifts caused by density gradients. There are no plans to publish either Part I or Part IV, which contain relatively trivial calculations.
The Two-Ion Hybrid Wave in an Inhomogeneous Plasma

Francis F. Chen

ABSTRACT

By extending the well-known theory of resistive drift waves to the case of a plasma with two ion species, we compute the frequency shift of two-ion hybrid waves caused by a density gradient. For reasonable experimental parameters, the shift is extremely small because it depends not only on the density gradient but also on the fractional concentration of the minor species and on the fractional mass difference between the two ion species. Specifically, for the two-ion hybrid resonance with $k_{\parallel}^2/k_{\perp}^2 << 1$, $\alpha_2 = n_2/n_e << 1$, and $\Delta \Omega = \Omega_2 - \Omega_1 << \Omega$, we obtain

$$\text{Re}(\Delta \omega) \approx -\alpha_2 \frac{\omega_\perp}{\omega_B} \frac{\Delta \Omega}{\Omega} (\Delta \Omega - \alpha_2 \Omega)$$

$$\text{Im}(\Delta \omega) \approx -2\alpha_2 \frac{\Delta \Omega}{\Omega} \left(\frac{k_{\parallel}}{k_{\perp}}\right)^2 \omega_c \tau_{ei} \Delta \Omega,$$

where $\Omega$ and $\omega_c$ are the ion and electron cyclotron frequencies, $\omega_\perp$ is the electron drift frequency, $\omega_B = k^2 D_B$ is the frequency for Bohm diffusion across a perpendicular wavelength, and $\tau_{ei}$ is resistive collision time.

We also show how drift-wave theory automatically accounts for MacKenzie space charge and how space charge neutralization is related to the turning points of the WKB problem.
I. FUNDAMENTAL EQUATIONS AND ASSUMPTIONS

Following the usual treatment of resistive drift waves,\(^1\) we assume a slab geometry in which the equilibrium plasma density is \(n_0 = n_{0}(x)\) and consider electrostatic perturbations varying as \(\exp i(k \cdot r - \omega t)\), with \(k = k_y \hat{y} + k_z \hat{z}\) \(k_z^2 \ll k_y^2\). We treat the plasma as a fluid with \(T_i = 0\) and \(m/M = 0\), but with a finite resistivity \(\eta\). For each species of ions, the equation of motion is then

\[
M_i n_i \left( \frac{\partial v_i}{\partial t} + v_i \cdot \nabla v_i \right) = Z_i e n_i \left( - \nabla \phi + v_i \times \mathbf{B} \right) - n_i^2 e^2 \eta (v_i - v_e). \tag{1}
\]

For isothermal electrons, we have

\[
m_e \left( \frac{\partial v_e}{\partial t} + v_e \cdot \nabla v_e \right) = -en_e \left( - \nabla \phi + v_e \times \mathbf{B} \right) - kT_e v_n e - n_e^2 e^2 \eta (v_e - v_i) \approx 0. \tag{2}
\]

The system of equations is closed by the condition of charge neutrality

\[
n_e = \sum_i Z_i n_i \tag{3}
\]

and the equations of continuity

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i v_i) = \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v_e) = 0. \tag{4}
\]

The neglect of electron inertia in Eq. (2) is essential for keeping the treatment simple; it is a good approximation. Extension to cylindrical geometry\(^2\) causes no problems unless the plasma rotates so fast that centrifugal and Coriolis forces become appreciable. Extension to finite \(T_i\) can also be done,\(^1\) but the algebra becomes cumbersome because of finite Larmor radius effects. The consequences of \(T_i \neq 0\) are discussed in Sec. V. Note that Eq. (1) allows the ion orbits to be large, but no spread in orbit sizes is included; furthermore, the orbits cannot span more than one wavelength because the ions are pushed only by the periodic E-field, and therefore no cyclotron harmonics are generated in this approximation.

In the equilibrium, we assume \(E_0 = 0\) and \(v_{ox} = 0\) and neglect the small "radial" velocity \(v_{ox}\) due to resistive diffusion. The ions then have \(v_i = 0\), and the electrons have only a diamagnetic drift

\[
v_{oe} = -\frac{kT_e}{eB} \frac{n_e'}{n_0} \hat{y} \equiv v_D \hat{y}, \tag{5}
\]

where \((')\) indicates \(\partial / \partial x\) and \(B\) is a uniform field in the \(z\) direction (in esu).
II. DERIVATION OF THE DISPERSION EQUATION

Linearizing the electron equation (2) about this equilibrium and neglecting the effect resistivity $\eta_\perp$ on $v_{ex}$ and $v_{ey}$, we obtain for the $x$, $y$, and $z$ components of Eq. (2)

$$e n_{e1} v_{ex} + e n_{e1} v_{ex} B = -e n_{l1} v_{dy} B - K T_e n_{i1} = 0 \tag{6}$$

$$e n_{e1} \phi + e n_{e1} v_{ex} B = -K T_e n_{i1} \tag{7}$$

$$e n_{e1} \phi - K T_e n_{i1} n_{l1} - n_{e1}^2 \eta_\parallel v_{ez} = 0. \tag{8}$$

The ion parallel velocity $v_{iz}$ has been neglected; this is important only when $k_z^2$ is so large that $\omega/k_z$ becomes comparable to the ion acoustic speed. We now define the variables

$$v = n_{l1}(x)/n_{e1}(x), \quad \chi = e \phi_1/K T_e \tag{9}$$

and use the relations

$$\omega_c = e B/m, \quad \eta = m/n_e e^2 \tau_{ei}, \quad D_B = K T_e/e B. \tag{10}$$

Eqs. (6)-(8) respectively yield

$$v_{ey} = D_B (\chi - v)' \tag{11}$$

$$v_{ex} = -i k_y D_B (\chi - v), \quad v_{ez} = i k_z D_B \omega_c \tau_{ei} (\chi - v). $$

The linearized electron continuity equation is

$$-i \omega n_{e1} + n_{e1} (v_{ex} + i k_y v_{ey} + i k_z v_{ez}) + v_{ex} n_{i1} + v_{eo} i k_y n_{i1} = 0. \tag{12}$$

Substituting Eq. (11) into Eq. (12), we find that the $x$-derivatives cancel out (a consequence of $m \to 0$) and that $v$ and $\chi$ are algebraically related by the simple expression

$$v = \chi \frac{\omega + i b \sigma_\parallel}{\omega + i b \sigma_\parallel} \tag{13}$$

where

$$\omega = k_y v_c \tag{14}$$

$$b = k_z^2 \tag{15}$$

$$a_c^2 = K T_e / M \lambda_c^2 = D_B / \lambda_c \tag{16}$$

and

$$\sigma_\parallel = (k_z/k_y)^2 \omega_c \tau_{ei} \lambda_c. \tag{17}$$
Note that $\nu = \chi$ is just the Boltzmann relation $v_e = v_o \exp(\phi / K T_e)$ in linearized form; Eq. (13) expresses the deviation from Boltzmann when $\omega \neq \omega_*$ and $\sigma_{||} \neq \infty$. The quantity $a_{||}$ is the ion Larmor radius evaluated with the electron temperature; and $b$ is like the usual FLR parameter, but for cold ions.

The quantity $b \sigma_{||}$ can be expressed as

\[ b \sigma_{||} = k^2_z (K T_e / m v_{ei}) = k^2_z D_{||} v_{ei}, \]  

(18)

which shows that $(b \sigma_{||})^{-1}$ is the time for electrons to diffuse a parallel wavelength against fixed ions. The ion mass does not appear in Eqs. (18) and (13); it appears only in the standard definitions (15)-(17), which will be needed later.

We next linearize the ion equation of motion (1) with the simplifications $v_{o1} = 0$, $v_{i1} = 0$, $v_{iz} = 0$. Solving for $v_{ix}$ and $v_{iy}$ in the usual fashion, we obtain for each ion species

\[ \dot{v}_{xj} = \left( \frac{k_j \phi}{B} - \frac{v}{\Omega_{ej}} \frac{\partial \phi'}{\partial \phi'} \right) \left( 1 - \frac{\omega^2}{\Omega_{ej}^2} \right)^{-1}, \]  

(19)

\[ v_{yj} = \left( \frac{\phi'}{B} - \frac{v}{\Omega_{ej}} \frac{k_j \phi}{\omega} \right) \left( 1 - \frac{\omega^2}{\Omega_{ej}^2} \right)^{-1}, \]

where

\[ \Omega_{ej} = Z_j e B / M_j. \]

(20)

The linearized ion continuity equation is in this case

\[-i \omega n_{oj} (v_x' + i k_j v_{yj}) + v_{xj} n'_{oj} = 0. \]

(21)

Eqs. (19) and (21) yield

\[ \frac{n_j}{n_{oj}} = \left[ \phi'' + \frac{n_j}{n_{oj}} \frac{\partial \phi'}{\partial \phi'} \left( k_j + k_j \frac{n_{oj}}{\Omega_{ej}} \frac{\partial \phi'_j}{\partial \phi} \right) \right] \left( \Omega_{ej} B \right)^{-1} \left( 1 - \frac{\omega^2}{\Omega_{ej}^2} \right)^{-1}. \]

(22)

We now specialize to the case of two ion species $j = 1, 2$. Let

\[ \alpha_j = Z_j n_{oj}/n_{oe}, \text{ so that } \alpha_1 + \alpha_2 = 1, \]

(23)

and let

\[ \delta(x) = n'_{o2}/n_{o1} = n_{o2} / n_{o2}. \]

(24)

The quasineutrality condition, Eq. (3), now reads

\[ \nu = \sum_j \frac{\alpha_j}{B \Omega_{ej}} \left( 1 - \frac{\omega^2}{\Omega_{ej}^2} \right)^{-1} \left[ \phi'' + \frac{\delta \phi'}{\partial \phi'} \left( k_j + k_j \delta \frac{\partial \phi'_j}{\partial \phi} \right) \phi \right]. \]

(25)
Upon normalizing $\phi$ to $\chi$, the coefficient $B^{-1}$ becomes $D_B$ [Eq. (10)]. The rest of the coefficient has the dimensions of time and may be defined as

$$\tau = \frac{\alpha_i}{\Omega_c} \left( 1 - \frac{\omega^2}{\Omega_c^2} \right)^{-1}.$$  \hspace{1cm} (26)

Finally, using the electron equation (13) to eliminate $\nu$, we obtain

$$(\tau_1 + \tau_2)(\chi'' + \delta \chi') - \left[ \tau_1 \left( k^2 + \delta \Omega_c l_1 \frac{k}{\omega} \right) + \tau_2 \left( k^2 + \delta \Omega_c l_2 \frac{k}{\omega} \right) + \frac{1}{D_B} \frac{\omega_b + i \sigma_{\parallel}}{\omega + i \sigma_{\parallel}} \right] \chi = 0.$$  \hspace{1cm} (27)

This is the dispersion equation for drift-type waves in a two-ion-species plasma with a density gradient $\delta(x)$. For simplicity we have dropped the subscript on $k_y$, since $k_y$ and total $k$ are essentially the same.

III. RECOVERY OF FAMILIAR RESULTS

1. Resistive drift wave. Let $\alpha_1 = 1$, $\alpha_2 = 0$, and $\omega^2 \ll \Omega_c^2$, so that $\tau = \Omega_c^{-1}$. The "local" dispersion relation is obtained by setting $\chi'' = \chi' = 0$. Eq. (27) then becomes

$$D_B \frac{k^2}{\Omega_c} + D_B \frac{k \delta}{\omega} + \frac{\omega_b + i \sigma_{\parallel}}{\omega + i \sigma_{\parallel}} = 0.$$  \hspace{1cm} (28)

From Eq. (16), the first term is $b$; and from Eqs. (14) and (5), the second term is $-\omega_b/\omega$. Clearing the denominators, we obtain

$$b \omega (\omega + i \sigma_{\parallel}) - \omega_b (\omega + i \sigma_{\parallel}) + \omega (\omega_b + i \sigma_{\parallel}) = 0,$$

$$\omega^2 + i \sigma_{\parallel} \omega + i \sigma_{\parallel} (\omega - \omega_b) = 0.$$  \hspace{1cm} (29)

$$\omega = -\omega_b + i \omega^2/\sigma_{\parallel}.$$  \hspace{1cm} (30)

One root has $\text{Im} (\omega) = -i \sigma_{\parallel}$ and is heavily damped. The unstable root can be found most easily in the limit $\sigma_{\parallel} \gg |\omega_b|$, so that $\omega - \omega_b$ must be small. Solving for $\omega - \omega_b$ and approximating $\omega$ by $\omega_b$ on the r.h.s., we obtain the usual expression for asymptotic growth rate of the resistive drift wave:

$$\omega - \omega_b = -b \omega_b + i \omega^2/\sigma_{\parallel}.$$  \hspace{1cm} (31)

2. Electrostatic ion cyclotron wave. Let $\alpha_1 = 1$, $\alpha_2 = 0$ and make the local approximation $\chi'' = \chi' = 0$. Further, let $k_z^2$ be large enough that the electrons follow the Boltzmann relation, so that the fraction in the last term of
Eq. (27) is unity ($b \sigma_{||} \rightarrow \infty$). Eq. (27) then becomes, for $\omega = \Omega_c$,

\[
\frac{b_{\rho}}{n_c} \left( 1 - \frac{\omega^2}{\Omega_c^2} \right)^{-1} \left( k^2 + k \delta \frac{n_c}{\omega} \right) + 1 = 0, \tag{32}
\]

\[
\frac{kT_e}{m} \left( k^2 + k \delta \frac{n_c}{\omega} \right) = \omega^2 - \Omega_c^2,
\]

\[
\omega^2 = \Omega_c^2 + k^2 c_s^2 \left( 1 + \frac{\omega}{k} \right) \frac{\omega_c + i b \sigma_{||}}{\omega_*) - \omega_c} \tag{33}.
\]

This is the usual formula for the ESIC wave except for the addition of the $\delta / k$ term for an inhomogeneous plasma. This correction term comes from the charge separation caused by the $E_y \times B$ drift in the $x$ direction along $\nabla n_0$.

3. Drift-cyclotron instability. We now make the same approximations as in the previous case for the e.s.i.c. wave, but we allow $b \sigma_{||}$ to be finite. Eq. (33) then becomes (for $\omega = \Omega_c$)

\[
\omega^2 = \Omega_c^2 + k^2 c_s^2 \left( 1 + \frac{\omega}{k} \right) \frac{\omega_c + i b \sigma_{||}}{\omega_*) - \omega_c} \tag{34}.
\]

We may drop the $\delta / k$ term, which gives a numerical correction of no interest to us. We might expect drift excitation of the cyclotron wave when there is a matching of the frequencies $\omega_*$ and $\Omega_c$. It is clear that the threshold occurs at $\omega_* = \Omega_c$, when $\omega^2$ is real. Rationalizing the denominator of Eq. (34), we obtain

\[
\omega^2 = \Omega_c^2 + k^2 c_s^2 \frac{\omega_c \omega_* + b^2 \sigma_{||}^2 + i b \sigma_{||} (\omega_* - \omega_c)}{\omega_*^2 + b^2 \sigma_{||}^2} \tag{35}.
\]

It is clear that there is instability only if $\omega_* > \Omega_c$. From Eqs. (5), (14), and (16), we see that this implies

\[- kD_B \delta / \Omega_c > 1, \text{ or } |k a_1| \delta a_1 | > 1. \tag{36}\]

The Larmor radius has to be large with respect to both the wavelength and the density scale length, and this formulation cannot be expected to give accurate results for the drift-cyclotron instability.

4. Two-ion hybrid resonance. The standard two-ion hybrid resonance frequency $\omega_*$ can be recovered from Eq. (27) by assuming uniform $n_o$ ($\delta = \omega_* = 0$),
perpendicular propagation \( k_z = \sigma_{\parallel} = 0 \), and an infinite plasma \( (\chi'' = \chi' = 0) \). Eqs. (26) and (27) then give \( \tau_1 + \tau_2 = 0 \), or

\[
\frac{a_1}{\Omega_1} \left(1 - \frac{\omega^2}{\Omega_1^2}\right) + \frac{a_2}{\Omega_2} \left(1 - \frac{\omega^2}{\Omega_2^2}\right) = 0, \tag{35}
\]

where the subscript \( \epsilon \) in \( \Omega_\epsilon \) has been suppressed. Thus

\[
\omega^2 \left(\frac{a_1}{\Omega_1 \Omega_2^2} + \frac{a_2}{\Omega_2 \Omega_1^2}\right) = \frac{a_1}{\Omega_1} + \frac{a_2}{\Omega_2},
\]

\[
\omega^2 = \frac{a_1 \Omega_2}{\Omega_1 \Omega_2} + \frac{a_2 \Omega_1}{a_1 \Omega_1 + a_2 \Omega_2} \equiv \omega_r. \tag{36}
\]

\section*{IV. EFFECT OF THE DENSITY GRADIENT}

We now make the local approximation on Eq. (27) to evaluate the effect of the density inhomogeneity on the modes in a two-ion-species plasma. Defining

\[
\epsilon_1 = (\delta/k) (\Omega_1/\omega) \text{ and } \epsilon_2 = (\delta/k) (\Omega_2/\omega), \tag{37}
\]

we can write Eq. (27) as

\[
\tau_1 (1 + \epsilon_1) + \tau_2 (1 + \epsilon_2) + \frac{1}{k^2 V_B} \frac{\omega_s + i b \sigma_{\parallel}}{\omega + i b \sigma_{\parallel}} = 0. \tag{38}
\]

Inserting the definition of \( \tau_j \) from Eq. (26), we have

\[
\frac{\omega_s + i b \sigma_{\parallel}}{\omega + i b \sigma_{\parallel}} (\omega^2 - \Lambda_1^2) (\omega^2 - \Lambda_2^2) = k^2 V_B \left[ \alpha_1 \alpha_2 (1 + \epsilon_1) (\omega^2 - \Lambda_2^2) + \alpha_1 \alpha_2 (1 + \epsilon_1) (\omega^2 - \Lambda_2^2) \right]. \tag{39}
\]

Note that \( V_B^\epsilon c = K_T^e M = c_s^2 \), so that we can define

\[
\omega_{s_j} = k c_{s_j} \tag{40}
\]

and write Eq. (39) in the alternate form

\[
\frac{\omega_s + i b \sigma_{\parallel}}{\omega + i b \sigma_{\parallel}} (\omega^2 - \Lambda_2^2) (\omega^2 - \Lambda_1^2) = \alpha_1 \omega_1^2 (1 + \epsilon_1) (\omega^2 - \Lambda_2^2) + \alpha_2 \omega_2^2 (1 + \epsilon_1) (\omega^2 - \Lambda_1^2). \tag{41}
\]

It is not possible to find a simple expression for the frequency shift for
arbitrary values of \( b \sigma \| \) covering the transition from the two-ion hybrid to the two-ion electrostatic ion cyclotron waves. We must treat the small \( b \sigma \| \) and large \( b \sigma \| \) cases separately.

1. **Damping of the two-ion hybrid.** We first take the case \( b \sigma \| \approx O(\omega_*) \) \(< \omega_r = \Omega_o \) to examine the effect of \( \Omega_o \) on the imaginary part of \( \omega \). Let

\[
\omega = \omega_r + \Delta \omega, \quad \Delta \omega \ll \omega_r, \quad \omega^2 = \omega_r^2 + 2\omega_r \Delta \omega,
\]

where the homogeneous two-ion hybrid frequency \( \omega_r \) is given by Eqs. (35) and (36), which we repeat here for convenience:

\[
\omega_r^2 - \Omega_o^2 = \frac{\alpha_2^2 \Omega_2}{\alpha_1 \Omega_1} (\omega_r^2 - \Omega_1^2)
\]

\[
\omega_r = \Omega_1 \Omega_2 \frac{\alpha_1 \Omega_2 + \alpha_2 \Omega_1}{\alpha_1 \Omega_1 + \alpha_2 \Omega_2}.
\]

Since \( |\omega_*| \ll \omega_r \) and \( b \sigma \| \) is also small, the l.h.s. of Eq. (41) is small, and we can replace \( \omega \) by \( \omega_r \) there. We also set \( \omega = \omega_r \) in \( \epsilon_1 \) and \( \epsilon_2 \) [Eq. (37)].

Eq. (39) then becomes

\[
\frac{\omega_* + ib \sigma_h \omega_r^2 - \Omega_2^2}{\omega_r^2 - \Omega_2^2} (\omega_r^2 - \Omega_1^2) = \frac{k^2 D_B}{\omega} \left[ \alpha_1 \frac{\alpha_2}{\alpha_1} \left( \omega_r^2 - \Omega_2^2 \right) + \alpha_2 \frac{\alpha_1}{\alpha_2} \left( \omega_r^2 - \Omega_1^2 \right) + 2\omega_r \Delta \omega \left( \alpha_1 \frac{\alpha_2}{\alpha_1} \epsilon_1 (\omega_r^2 - \Omega_2^2) + \alpha_2 \frac{\alpha_1}{\alpha_2} \epsilon_2 (\omega_r^2 - \Omega_1^2) \right) \right] + O(\epsilon \Delta \omega).
\]

The first two terms on the l.h.s. cancel by virtue of Eq. (43). Substituting for \( \omega_r^2 - \Omega_2^2 \) from Eq. (43) in the remaining terms, we obtain

\[
2\omega_r (\alpha_1 n_1 + \alpha_2 n_2) \Delta \omega = \frac{\alpha_1 n_2}{\alpha_1 n_1} \frac{(\omega_r^2 - \Omega_2^2)^2}{k^2 D_B} \frac{\omega_* + ib \sigma_h \omega_r}{\omega + ib \sigma_h}.
\]

Separating into real and imaginary parts and noting that \( |\omega_*| = b \sigma || \ll \omega_r \), we have

\[
\frac{\omega_* + ib \sigma ||}{\omega + ib \sigma ||} = \frac{\omega_*}{\omega_r} + i \frac{b \sigma ||}{\omega_r} \frac{\omega_r - \omega_*}{\omega_r}.
\]
The frequency shift $\Delta \omega$ is then approximately

$$\Delta \omega = \frac{\alpha_2 \omega_2 (\omega_r^2 - \lambda_1^2)(\epsilon_1 - \epsilon_2) - \frac{\alpha_2 \omega_2}{\alpha_1 \lambda_1} \frac{(\omega_r^2 - \lambda_1^2)}{2k^2 \rho_B} \lambda_1 \omega_r - \frac{\alpha_2 \omega_2}{\lambda_1} \frac{(\omega_r^2 - \lambda_1^2)}{2k^2 \rho_B} \lambda_1 \omega_r}{2 \omega_r (\alpha_1 \lambda_1 + \alpha_2 \lambda_2)}.$$

(48)

The damping rate is given by

$$-\text{Im}(\omega) = \frac{b \sigma_{II}}{2k^2 \rho_B} \frac{\alpha_2 \omega_2}{\alpha_1 \lambda_1} \frac{(\omega_r^2 - \lambda_1^2)}{2k^2 \rho_B \lambda_1} \frac{1}{\alpha_1 \lambda_1 + \alpha_2 \lambda_2} \frac{1}{\alpha_1 \lambda_1 + \alpha_2 \lambda_2} = \frac{b \sigma_{II}}{2k^2 \rho_B} \frac{\alpha_2 \omega_2}{\alpha_1 \lambda_1} \frac{(\omega_r^2 - \lambda_1^2)}{2k^2 \rho_B \lambda_1} \frac{1}{\alpha_1 \lambda_1 + \alpha_2 \lambda_2}.$$

(49)

where we have used Eq. (44).

From Eq. (18), we see that

$$\frac{b \sigma_{II}}{2k^2 \rho_B} = \frac{k^2 T_e}{k^2 m \nu_{ei}} \frac{eB}{k^2 T_e} = \frac{k^2}{k^2} \omega_c \tau_{ei},$$

(50)

so that

$$-\text{Im}(\omega) = \frac{1}{2} \frac{\alpha_2 \omega_2}{\alpha_1 \lambda_1} \frac{k^2}{k^2} \frac{(\omega_r^2 - \lambda_1^2)}{\lambda_1 \lambda_2 (\alpha_1 \lambda_2 + \alpha_2 \lambda_1)} \omega_c \tau_{ei}.$$

(51)

In particular, for a small concentration of the species 2, so that $\alpha_2 \ll \alpha_1$, and $\omega_r = \Omega_2$, we have

$$-\text{Im}(\omega) \approx \frac{\alpha_2 \omega_2}{2} \frac{k^2}{k^2} \frac{(\omega_r^2 - \lambda_1^2)}{\lambda_1 \lambda_2} \omega_c \tau_{ei}.$$

(52)

The drift terms do not appear here, since resistive damping dominates over drift-wave growth. This is not surprising, since we found in the treatment of the drift-cyclotron wave that instability required $\omega_\star > \omega$. We see the dependence on $\omega - \omega_\star$ again in Eq. (47). There is a simple physical reason for $\text{Im}(\omega) \propto \omega - \omega_\star$. In the physical picture of how a drift wave is destabilized, a phase shift between the density oscillation $\nu$ and the potential oscillation $\chi$ must exist such that $\nu$ leads $\chi$. Let the phase of $\chi$ be zero so that $\chi = \chi \exp \, i(ky - \omega t)$. 
and $\nu = \bar{\nu} \exp i(\kappa y - \omega t + \delta)$. For $\nu$ to lead $\chi$, we must have $\delta < 0$, or $\text{Im} \ (\nu/\chi) < 0$. The modified Boltzmann relation, Eq. (13), and Eq. (47) then tell us that $\omega_\times > \omega_\tau$ is required for instability. When $\nu$ lags $\chi$, as occurs for $|\omega_\times| < \omega_\tau = \Omega_c$, the drift terms are stabilizing.

2. Numerical estimates. Consider a cylindrical plasma column with radius $a = 10$ cm and length $L = 100$ cm, immersed in a 20 kG magnetic field. The plasma has $n_e = 10^{12}$ cm$^{-3}$, $T_e = 2$ eV, $T_i = 0$. Let the local dispersion relation be evaluated at $r = a/2$, where the density scale length is $\Lambda = r = 5$ cm, and let the mode have $\lambda || = 2L = 200$ cm and $\lambda_\perp = 3.9$ cm, corresponding to azimuthal mode number $m = 8$. The two ion species have $M_1 = 238 \ M_H$ and $M_2 = 235 \ M_H$. The plasma parameters are then as follows:

$$k = m/r = 1.6 \ \text{cm}^{-1}, \ k_z = 2\pi/\lambda || = 3.142 \times 10^{-2} \ \text{cm}^{-1}$$

$$\delta = -1/\Lambda = -0.2 \ \text{cm}^{-1}, \ \delta/k = -0.125$$

$$c_s = (kT_e/M_1)^{1/2} = 8.97 \times 10^4 \ \text{cm/sec}$$

$$a_1 = c_s/\Omega_c = 0.111 \ \text{cm}$$

$$b = k^2 a_1^2 = 3.18 \times 10^{-2}$$

$$\nu_{th} = (kT_e/m)^{1/2} = 5.93 \times 10^7 \ \text{cm/sec}$$

$$\omega_c = 3.516 \times 10^{11} \ \text{sec}^{-1}$$

$$\nu_{ei} = 1.5 \times 10^{-6} \ Zn \ln \Lambda/T_e^{3/2} = 5.3 \times 10^6 \ \text{sec}^{-1}$$

$$\omega_{cei} = 6.6 \times 10^4$$

$$D_B = 10^8 T_e V_B / B = 10^4 \ \text{cm}^2/\text{sec}$$

The relevant frequencies are then ordered as follows (in rad/sec):

$$\omega_* = -k\delta D_B = 3.2 \times 10^3$$

$$k^2 D_B = 2.56 \times 10^4$$

$$\kappa c_s = 1.436 \times 10^5$$

$$b\sigma_1 = k_z^2 \nu_{th}^2 \tau_{ei} = 6.55 \times 10^5$$

$$\Omega_1 = 8.05 \times 10^5, \ \Omega_2 = 8.154 \times 10^5$$

$$\nu_{ei} = 5.3 \times 10^6$$
\( \omega_c = 3.5 \times 10^{11} \)

Frequency ratios are:

\( (\Omega_2 - \Omega_1) = \Delta\Omega/\Omega_1 = 1.28 \times 10^{-2} \)

\( \Delta \equiv (\Omega_2^2 - \Omega_1^2)/\Omega_1^2 = 2.575 \times 10^{-2} \)

\( \omega_*/\Omega_c = 4.0 \times 10^{-3} \)

\( \omega_*/b\sigma || = 4.9 \times 10^{-3} \)

\( b\sigma ||/\Omega_c = 0.81 \)

Using these numbers in the damping rate of Eq. (52) for \( \alpha_2 \ll \alpha_1 \), we find

\[-\frac{\text{Im}(\omega)}{\nu_r} \sim \frac{\alpha_2}{2} \frac{k^2}{k^2} \left( \frac{\Omega_1}{\Omega_r} \right)^2 \Delta^2 \omega_c \tau_{\text{e}1} = 8.2 \times 10^{-3} \alpha_2, \tag{53} \]

which is entirely negligible.

3. Frequency shift of the two-ion hybrid. The real part of Eq. (48)
gives for the frequency shift

\[ \Delta \omega = \frac{\frac{\alpha_2}{\nu_r} \frac{(\omega_r - \alpha_2^2)(\epsilon_1 - \epsilon_2)}{\alpha_1 \nu_1} - \frac{\alpha_2}{\nu_r} \frac{\omega_*/\nu_r}{\Omega_r} \left( \frac{(\omega_r - \alpha_2^2)}{\alpha_1 \nu_1} \right)^2}{2 \nu_r (\alpha_1 \nu_1 + \alpha_2 \nu_2)} \]

\[ = \frac{\alpha_2 \nu_2 (\omega_r^2 - \alpha_2^2)}{2 \nu_r (\alpha_1 \nu_1 + \alpha_2 \nu_2)} \left[ \frac{k}{\nu_r} \frac{\Omega_1 - \Omega_r}{\nu_r} - \frac{\omega_r^2 - \alpha_2^2}{\alpha_1 \nu_1} \frac{\omega_*/\nu_r}{\Omega_r} \right] \]

\[ = \frac{\alpha_2 \nu_2 (\omega_r^2 - \alpha_2^2)}{2 \nu_r (\alpha_1 \nu_1 + \alpha_2 \nu_2)} \frac{\delta}{k} \left[ \frac{\Omega_1 - \Omega_r}{\alpha_1 \nu_1} + \frac{\omega_r^2 - \alpha_2^2}{\alpha_1 \nu_1} \right], \tag{54} \]

where we have used \( \omega_* = -k\delta \nu_r \). Replacing \( \omega_r^2 \) in the denominator by Eq. (144), we obtain

\[ \Delta \omega = \frac{1}{2} \frac{\delta}{k} \frac{\omega_r^2 - \alpha_2^2}{\alpha_1 \nu_1^2} \frac{\alpha_2 \nu_2 (\Omega_1 - \Omega_r) + \omega_r^2 - \alpha_2^2}{\alpha_1 \nu_1 + \alpha_2 \nu_2} \]
Specializing to the case \( \alpha_2 \ll \alpha_1 \), we may replace \( \alpha_1 \) by 1 and \( \omega_r \) by \( \Omega_2 \), obtaining

\[
\Delta \omega = \frac{\alpha_2}{2} \frac{\delta}{k} \frac{\Delta}{\Omega_2} \left( \Omega_2^2 - \Omega_1 \Omega_2 - \alpha_2 \Omega_1^2 \right). \tag{56}
\]

If \( \Delta \) is also small, let \( \Delta \Omega = \Omega_2 - \Omega_1 \) and \( \Delta \approx 2 \Delta \Omega / \Omega \). We then have

\[
\Delta \omega = \alpha_2 \frac{\delta}{k} \Delta \Omega \left( \frac{\Delta \Omega}{\Omega} - \alpha_2 \right). \tag{57}
\]

The sign of the shift depends on the relative size of \( \alpha_2 \) and \( \Delta \Omega / \Omega \). For the parameters of Sec. IV-2 and \( \alpha_2 < 1\% \), we have

\[
\frac{\Delta \omega}{\Delta \Omega} = -1.6 \times 10^{-3} \alpha_2, \tag{58}
\]

which is extremely small. However, for \( \alpha_2 \gg 0.1 \), we have

\[
\frac{\Delta \omega}{\Delta \Omega} = \frac{\alpha_2}{2}, \tag{59}
\]

which may be appreciable. Our choices of \( \Lambda = a/2 \) and \( m = 8 \) were quite arbitrary. For \( m = 1 \) or 2 and \( \Lambda \) smaller than \( a/2 \), \( \Delta \omega \) can be an order of magnitude larger than in Eq. (59).

Note that \( \Delta \omega \) is independent of \( KT_e \), and its sign is not necessarily related to that of \( \omega_k \). The two terms in Eq. (57) do not have simple physical interpretations, since they are combinations of other terms. In Eq. (54), however, one can trace the first term \( (\Omega_1 - \Omega_2)^2 \) back to the charge separation in the difference of ion \( E_y/B \) drifts along \( V_n \), and the last term to the electron \( E_y/B \) drift along \( V_n \). To lowest order in \( \omega/\Omega \), the electron and ion \( E_y/B \) drifts of course are equal, so that there is no net charge separation. When there are two ion species, the electron charge cannot cancel the ion charge for all ion mixtures; this is probably why the second term in Eq. (57) depends on \( \alpha_2 \).

4. Frequency shift of the ESIC waves. The regime of electrostatic ion cyclotron waves is obtained in the limit \( b \parallel \omega \ll \omega \parallel \omega_k \parallel \), when the Boltzmann relation holds and

\[
\frac{\omega_k + ib\sigma}{\omega + ib\sigma} = 1. \tag{60}
\]
The dispersion relation Eq. (41) is then approximately

\[
(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = \alpha_1 \omega_{s1}^2 (1 + \varepsilon_1)(\omega^2 - \omega_2^2) + \alpha_2 \omega_{s2}^2 (1 + \varepsilon_2)(\omega^2 - \omega_1^2).
\]  
(61)

Let \(\omega_0\) be the solution of the homogeneous-plasma equation

\[
\frac{\alpha_1 \omega_{s1}^2}{\omega_0^2 - \Omega_1^2} + \frac{\alpha_2 \omega_{s2}^2}{\omega_0^2 - \Omega_2^2} = 1.
\]  
(62)

We replace \(\omega\) by \(\omega_0\) in the small terms containing \(\varepsilon_j\) and assume a small shift \(\Delta \omega\) such that

\[
\omega = \omega_0 + \Delta \omega \quad \omega^2 = \omega_0^2 + 2\omega_0 \Delta \omega.
\]  
(63)

The first-order part of Eq. (61) then reads

\[
2 \omega_0 \Delta \omega (\omega_0^2 - \omega_1^2 + \omega_0^2 - \omega_2^2) = 2 \omega_0 \Delta \omega (\alpha_1 \omega_{s1}^2 + \alpha_2 \omega_{s2}^2) + \\
+ \alpha_1 \omega_{s1}^2 \varepsilon_1 (\omega_0^2 - \omega_2^2) + \alpha_2 \omega_{s2}^2 \varepsilon_2 (\omega_0^2 - \omega_1^2),
\]  
(64)

or

\[
\Delta \omega = \frac{1}{2\omega_0} \frac{\alpha_1 \omega_{s1}^2 \varepsilon_1 (\omega_0^2 - \omega_2^2) + \alpha_2 \omega_{s2}^2 \varepsilon_2 (\omega_0^2 - \omega_1^2)}{2 \omega_0^2 - \omega_1^2 - \omega_2^2 - \alpha_1 \omega_{s1}^2 - \alpha_2 \omega_{s2}^2}.
\]  
(65)

Call the denominator \(D\), and add and subtract a term in the numerator:

\[
\Delta \omega = \frac{(\omega_0^2 - \omega_1^2)(\omega_0^2 - \omega_2^2)}{2\omega_0 D} \left[ \frac{\alpha_1 \omega_{s1}^2}{\omega_0^2 - \omega_1^2} \varepsilon_1 + \frac{\alpha_2 \omega_{s2}^2}{\omega_0^2 - \omega_2^2} \varepsilon_2 + \frac{\alpha_1 \omega_{s1}^2}{\omega_0^2 - \omega_1^2} \varepsilon_1 - \frac{\alpha_2 \omega_{s2}^2}{\omega_0^2 - \omega_2^2} \varepsilon_2 \right].
\]

The first and third terms cancel by virtue of Eq. (62). Thus,

\[
\Delta \omega = \frac{\alpha_2 \omega_{s2}^2}{2\omega_0 D} (\omega_0^2 - \Omega_1^2) (\varepsilon_2 - \varepsilon_1).
\]  
(66)

Since \(\varepsilon_2 - \varepsilon_1 = (\delta/k) (\Delta \Omega/\omega_0)\) and \(\omega_{s2}^2 = k^2 c_s^2 = k^2 D_B \Omega_2 = -(k/\delta)\omega_0 \Omega_2\), we have the frequency shift

\[
\Delta \omega = -\frac{\delta_2}{2} \frac{\omega_0 \Delta \Omega_2 \omega_1}{\omega_0^2} \frac{\omega_0^2 - \Omega_1^2}{\left[ \omega_0^2 - (\omega_2^2 + \alpha_1 \omega_{s1}^2) \right] + \left[ \omega_0^2 - (\omega_1^2 + \alpha_2 \omega_{s2}^2) \right]}.
\]  
(67)
For the particular case $\alpha_2 \ll \alpha_1$, the root of Eq. (62) that is close to $\Omega_2$ is given by
\[
\omega_0^2 - \Omega_2^2 \approx \omega_{s1}^2 \left(1 - \frac{\omega_{s1}^2}{\omega_{s1}^2 - \Omega_2^2}\right)^{-1} \tag{68}
\]

The last fraction is equal to $k^2 c_s^2 / \Omega_1^2 = k^2 D_B / \Omega_1 \Delta \approx 1.2$ for the numerical parameters of Sec. IV-2. Unless there is a fortuitous coincidence between the acoustic shift $\omega_{s1}^2$ and the frequency difference $\omega_0^2 - \Omega_1^2$, the denominator of Eq. (68) is of order unity, and $\omega_0$ differs from $\Omega_2$ only by a term of order $\alpha_2$. With $\omega_0 \approx \Omega_2$, Eq. (67) becomes
\[
\Delta \omega \approx -\frac{\alpha_1}{2} \frac{\omega_x}{\sqrt{\Lambda}} \Delta \Lambda \frac{\omega_{x1}^2 - \Omega_2^2}{\omega_{x1}^2 - \omega_{s1}^2}
\]
\[
\approx -\frac{\alpha_1}{2} \frac{\omega_x}{\sqrt{\Lambda}} \Delta \Lambda \left(1 - \frac{k^2 D_B}{\Omega_1 \Delta}\right)^{-1} \tag{69}
\]

Again, there is a possible resonant denominator, and the general case requires a more accurate approximation for $\omega_0$. However, we have taken the $m = 8$ mode; for smaller $m$-numbers, the $k^2 D_B / \Omega_1 \Delta$ term would be negligibly small. We then have the simple formula
\[
\Delta \omega \approx -\frac{\alpha_1}{2} \frac{\omega_x}{\sqrt{\Lambda}} \Delta \Lambda \tag{70}
\]

V. EFFECTS WE HAVE NEGLECTED

1. Radial eigenfunctions. In making the "local" approximation, we dropped the terms $(\tau_1 + \tau_2)$ $(\chi'' + \phi')$ in Eq. (27). These terms are related to the radial electric field $E_r$ which must exist in order for $\phi$ to satisfy a boundary condition. The $\delta X'$ term represents the ion space charge due to polarization drift along $\nabla n_o$. The $\chi''$ term is due to the nonuniformity of this drift if $E_r$ varies radially; this effect would exist in a finite cylinder even if $\nabla n_0 = 0$. If there is a strong shear in equilibrium quantities such as $B_o$, $E_o$, or $n_0'$, the modes would be localized between closely spaced turning points, and the radial wave equation must be solved carefully. However, in a non-sheared system, the radial ($x$) gradients are generally smaller than the azimuthal ($y$) gradients unless $m$ is as small as 1 or 2. In any case, since the frequency shifts are so small, it is not worthwhile to evaluate them more accurately by removing the local approximation.

2. Finite ion temperature. In fluid theory, finite $T_1$ has two main
effects: there is a Doppler shift due to the ion diamagnetic drift, and there
is a charge separation because the ions $E \times B$ drift more slowly than electrons,
due to the ions' finite Larmor radius. The latter effect can be taken into
account by adding the viscosity term $\nabla \cdot \mathbf{u}$ to the equation of motion.\footnote{1} The
effect is to increase the pressure-driven effects proportional to $\omega_k$ by a factor
$(1 + T_i/T_e)$. Collisional viscosity can also be included in $\mathbf{u}$. The algebra
becomes much more cumbersome, even with the local approximation, and no new
effects or order-of-magnitude changes are expected. In kinetic theory, finite
$T_i$ introduces $\Omega_c$ harmonics, and there will be new physical effects.

3. \textit{Finite radial electric Field $E_\phi$.} If $E_\phi$ is uniform, there is a simple
Doppler shift of the frequency, unless $E_\phi$ is so large that the centrifugal
force\footnote{2} in a cylindrical system causes a Rayleigh-Taylor instability. If $E_\phi$
is nonuniform, the azimuthal $E \times B$ drift is sheared, and there are Kelvin-Helmholtz
instabilities that can be excited. The treatment of K-H instabilities in an
inhomogeneous plasma, including finite Larmor radius effects, has been treated
in only a few papers.\footnote{3,4} The analysis is necessarily complicated, because no
local approximation can be made; the K-H instability depends on the radial gra-
dients of first-order quantities. If nonuniform E-fields are present, it would
not pay to attempt an analytic treatment.

VI. HOW DRIFT WAVE THEORY ACCOUNTS FOR MACKENZIE SPACE CHARGE

Some years ago, Ken MacKenzie suggested that ion oscillations near $\Omega_c$
could never become large in an inhomogeneous plasma, because there is no adequate
mechanism for neutralizing the space charge as ions gyrated into regions of
different density. In the $k_\parallel = 0$ case, neutralizing electron currents must come
from a modulation of the endplate currents, and these are limited in magnitude
to the ion losses at the acoustic velocity. By exciting finite-$k_\parallel$ modes, one is
able to provide neutralizing currents by allowing electrons to flow along B between
wave crests and troughs within the plasma. Actually, there are many other mechan-
isms for generating neutralizing currents, and drift wave theory takes all these
into account, though generally only to order $k_\parallel^2$. In particular the ions them-
selves can effect charge neutralization by changing the wave amplitude profile.
We shall show how this mechanism is related to the location of the turning points
in the WKB problem.

With the help of Eqs. (5), (14), (16), and (24), the radial wave equation
(27) for a single-ion-species plasma with $T_i = 0$ can be written as follows:
\( \chi'' + \left( \frac{1}{r} + \frac{n'_o}{n_o} \right) \chi' - \frac{1}{a^2} \left[ \frac{m^2}{r^2} q_i^2 - \frac{\omega}{\omega_c} + \left( 1 - \frac{\omega^2}{\omega_c^2} \right) \left( \frac{\omega'}{\omega_c} + \frac{i}{\omega_c} \sigma_n \right) \right] \chi = 0 \) (71)

Here, we have converted from plane geometry to cylindrical geometry, and the prime indicates \( \partial / \partial r \). Eq. (71) is an equation of continuity, and each term can be traced back to its origin in the equations of motion to find the physical effect that caused a separation of charge. The last term is easily recognized as coming from the modified electron Boltzmann equation (13), the \( \sigma \| \) part representing electron diffusion along \( B \), and the \( \omega \) part representing \( E_\theta \times B \) drift along \( \nabla n_o \). The factor \( (1 - \omega^2/\omega_c^2) \) causes the electron contribution to space charge to be negligible when \( \omega = \omega_c \), because then all the other terms (which are due to ion space charge) become large. The \( -\omega \lambda / \omega \) term is the ion accumulation due to \( E_\theta \times B \) drift along \( \nabla n_o \). Since this is equal to the electron \( E \times B \) drift at low frequencies, it normally cancels the \( \omega \lambda / \omega \) part of the last term when \( k_\| = 0 \). The \( (m/r)^2 a^2 \) term is the polarization drift of the ions along \( E_\theta \). There is a divergence of ions from azimuthal locations where electrons accumulate because of \( E_\theta \times B \) drift; hence, the polarization term adds to the electron space charge.

The remaining terms in Eq. (71) depend on the radial variation of the potential \( \chi \) and are neglected in the local approximation. The \( (n'_o/n_o) \chi' \) term is due to the \( E_r \)-driven polarization drift alone \( \nabla n_o \), and the \( \chi'' + \chi'/r \) terms come from the radial inhomogeneity of this drift, which causes ion space charge to pile up. The \( n'_o \chi \) term is the lowest-order MacKenzie space charge effect. It causes ions to "stick out" radially past the electrons when the ions are pulled out by the radial electric field.

If electron currents are insufficient to cancel ion space charge when \( \chi \) is constant with radius, then a radial variation in \( \chi \) develops so that the \( E_r \)-driven ion currents can provide neutralization. The more severe the charge imbalance, the faster \( \chi \) must vary with radius. This implies either a high-order radial mode or closely spaced turning points.

To see how space charge imbalance is removed by adjustment of the turning points, let us examine the particularly simple case of constant \( \omega \). The radial density profile is then a Gaussian:

\[ n_o = n_{oo} e^{-r^2/a^2} \] (72)

\[ -15- \]
We further consider low frequencies such that \( \omega^2 \ll \Omega_c^2 \), and \( k_z^2 \) large enough that \( b_0 \gg \omega, \omega_* \). Eq. (71) then simplifies to

\[
\chi'' + \left( \frac{1}{r} \frac{n_0}{r \Omega_c} \right) \chi' - \frac{1}{2} \left( \frac{a_i^2 m^2}{r^2} - \frac{\omega_*}{\omega} + 1 \right) \chi = 0. \tag{73}
\]

In terms of the dimensionless variables

\[
s = r/a, \quad \lambda = a_i / a, \tag{74}
\]

this can be written

\[
\lambda^2 \left( \chi_{ss} + \frac{1}{s} - 2s \chi_s \right) - \left( 1 - \frac{\omega_*}{\omega} + \lambda^2 \frac{m^2}{s^2} \right) \chi = 0. \tag{75}
\]

The first derivative \( \chi_s \) can be transformed away with the substitution

\[
W(s) = \sqrt{s} e^{-s^2/2} \chi, \tag{76}
\]

giving the WKB equation

\[
\frac{d^2 W}{ds^2} + \left( 2 - s^2 + \frac{1}{4} \frac{m^2}{s^2} - \frac{1}{\lambda^2} \left( 1 - \frac{\omega_*}{\omega} \right) \right) W = 0. \tag{77}
\]

The turning points are determined by

\[
Q(s_o) = \frac{1}{4} \frac{m^2}{s_o^2} + 2 - \frac{1}{\lambda^2} \left( 1 - \frac{\omega_*}{\omega} \right) - s_o^2 = 0. \tag{78}
\]

Since \( \omega_* - \omega = O(\lambda^2 \omega_* \omega) \), let

\[
\omega_* - \omega = 2\lambda^2 \omega_* f, \tag{79}
\]

where \( f = O(1) \). Then Eq. (78) becomes, approximately,

\[
s_o^4 - 2(1 + f)s_o^2 + m^2 - \frac{1}{4} = 0. \tag{80}
\]
This quadratic is easily solved to give the turning point locations \( s_0 \) as a function of frequency shift \( f \) and azimuthal mode number \( m \), as shown in Fig. 1.

For a given value of \( m \) and \( f \), there are two turning points, \( s_1 \) and \( s_2 \), between which \( W(s) \) oscillates, and beyond which \( W(s) \) exponentially decays. [Inclusion of the \( \mathbf{b}_0 \parallel \) terms in Eq. (71) would give rise to complex turning points, which simply means that the interior solution is not a pure sine wave and the exterior solution not a pure exponential.] As \( f \) is varied, the ratios among the various polarization and \( \mathbf{E} \times \mathbf{B} \) drifts change, and a charge imbalance can occur. Moving the turning points changes the drifts dependent on \( \chi' \) and \( \chi'' \) and therefore adjusts the charge accumulations due to these drifts. However, charge neutrality
cannot be achieved for all values of $f$, since the WKB quantization condition must also be satisfied:

$$\int_{s_2}^{s_1} Q_k^2(s) \, ds = (n + \frac{1}{2}) \pi.$$  

(81)

This gives the eigenvalues of $f$ (or $\omega$) for which charge balance can occur everywhere. These are entirely analogous to energy levels in a potential well, but here the levels are set by space charge. There is usually one value of $f$ for each set of mode numbers $(m,n)$. For other frequencies, the radial E-field cannot be adjusted so as to cancel all space charges arising from unequal ion and electron drifts.

In practice, the second turning points in Fig. 1 would lie well outside the plasma. The radial eigenmode is in this case determined by the boundary condition at the wall plus the inner turning point. When $\omega_\star$ is not constant, the shear in $\omega_\star(s)$ would bring the turning points closer together. In that case, $s_2$ can lie inside the plasma, and the radial mode profile would be insensitive to the wall boundary condition. In tokamaks the shear in $B$ produces a large shear in $k_\|$, so that $b_\|_\|$ is a function of $s$, and it is this effect that localizes the drift wave. Thus we see that standard drift wave theory takes MacKenzie space charge into account, though the way it is done is often obscured by the mathematical details. When the plasma has a sharp boundary, the wave function contains many radial harmonics, and analytic solutions are not useful because the short radial wavelengths involve very large finite Larmor radius corrections. The plasma column then behaves like a crankshaft, and this has been described in computer models.

REFERENCES