new corr. on p.15 p.14

LANDAU DAMPING OF HELICON WAVES

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ABSTRACT

Efficient coupling of rf power to a plasma without the generation of fast particles is shown to be possible with helicon waves because of the good match between parallel phase velocity and electron thermal speed.

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1. INTRODUCTION

Recent experiments by Boswell et al.[1,2] have shown that nearly fully ionized plasmas of density $> 10^{12}$ cm^{-3} in A, He, and H can be produced in a 10-cm diam. chamber 120 cm long with only 180 W of 9-MHz power and a confining magnetic field B_o of only 750 G. Magnetic probe measurements indicated wave profiles consistent with those expected of helicon waves, but the rate of energy absorption by wave damping was more than 10^3 times that expected from collisions. At the measured parallel wavelength of 50 cm and temperature of 3 eV, the value of $\zeta = \omega/k_z v_{th}$ was > 4, indicating negligible Landau damping. However, the latter is so sensitive to ζ that a factor of two change in k_z or factor of 4 change in T_e is all that is necessary for Landau damping to become efficient.

The classical helicon wave[3,4] is governed by the equations

$$\nabla \times \underline{E} = -\partial \underline{B}/\partial t \tag{1}$$

$$\underline{\nabla} \times \underline{B} = \mu_o \underline{j} \tag{2}$$

$$\underline{\nabla} \cdot \underline{B} = 0 \tag{3}$$

$$\underline{E} = \underline{j} \times \underline{B}_o / en_o \ , \tag{4}$$

where all quantities are first-order except B_o and n_o. From Eqs.(2) and (4) we also have

$$\underline{\nabla} \cdot \underline{j} = 0 \tag{5}$$

$$E_z = 0 \ . \tag{6}$$

The helicon is an electromagnetic wave with $\underline{\nabla} \cdot \underline{E} = 0$. This is clearly not possible in Cartesian geometry, because Eqs.(2) and (3) state that $\underline{k}$, $\underline{B}$, and $\underline{j}$ are mutually perpendicular. Since $\underline{E} \perp \underline{B}$ by Eq.(1) and $\underline{E} \perp \underline{j}$ by Eq.(5), $\underline{E}$ cannot also be perpendicular to $\underline{k}$. However, we shall show that $\underline{\nabla} \cdot \underline{E} = 0$ is possible in cylindrical geometry in the absence of damping.

Taking the cross-product of Eq.(4) with $\underline{B}_o$ and using Eq.(5), we find that

$$\underline{j}_\perp = -en_o \, \underline{E} \times \underline{B}_o / B_o^2 \ . \tag{7}$$

Thus the current perpendicular to $\underline{B}_o$ is entirely due to the $\underline{E} \times \underline{B}_o$ drift of the electrons. The frequency has to be high enough that the $\underline{E} \times \underline{B}_o$ and polarization drifts of the ions can be neglected and low enough that the electrons' cyclotron gyrations can be neglected. Furthermore, the neglect of displacement current in Eq.(2) gives another upper limit to ω. Eq.(6) implies that the parallel conductivity is infinite. As we show later, these conditions are not really restrictive, and helicons can exist over a wide parameter range.

II. STANDARD DERIVATION OF THE DISPERSION RELATION

We assume perturbations of the form exp $i(m\theta + kz - \omega t)$. Eqs.(1), (4) and (5) then give

$$i\omega \underline{B} = \nabla \times \underline{E} = \nabla \times (\underline{j} \times \underline{B}_o)/en_o = (\underline{B}_o \cdot \nabla)\, \underline{j}/en_o = \frac{B_o}{en_o}\frac{\partial \underline{j}}{\partial z} \tag{8}$$

Using Eq.(2) gives

$$\underline{B} = \frac{k}{\omega}\frac{B_o}{en_o}\underline{j} = \frac{kB_o}{\omega e n_o \mu_o} \nabla \times \underline{B} \tag{9}$$

We define

$$\gamma^2 \equiv \frac{\omega e n_o \mu_o}{B_o} = \frac{\omega}{\omega_c}\frac{\omega_p^2}{c^2} = \frac{\omega}{\Omega_c}\frac{\Omega_p^2}{c^2} \tag{10}$$

$$K = \gamma^2/k\ . \tag{11}$$

Eq.(9) can then be written

$$\nabla \times \underline{B} = K\,\underline{B} \tag{12}$$

The curl of this, with Eq.(3), gives

$$-\nabla \times \nabla \times \underline{B} = \nabla^2 \underline{B} = -K\,\nabla \times \underline{B}\ . \tag{13}$$

Substituting for $\nabla \times \underline{B}$ from Eq.(12), we obtain

$$\nabla^2 \underline{B} + K^2 \underline{B} = 0\ . \tag{14}$$

The components of this equation in cylindrical geometry are:-

$$\nabla^2 B_r - \frac{2}{r^2} \frac{\partial B_\theta}{\partial \theta} - \frac{B_r}{r^2} + K^2 B_r = 0 \tag{15}$$

$$\nabla^2 B_\theta + \frac{2}{r^2} \frac{\partial B_r}{\partial \theta} - \frac{B_\theta}{r^2} + K^2 B_\theta = 0 \tag{16}$$

$$\nabla^2 B_z + K^2 B_z = 0 \quad , \tag{17}$$

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

and the additional terms arise from the differentiation of the unit vector in taking the Laplacian of a vector. Fourier analyzing in θ and z, we obtain

$$B_r'' + \frac{1}{r} B_r' + \left(K^2 - k^2 - \frac{m^2+1}{r^2}\right) B_r - \frac{2im}{r^2} B_\theta = 0 \tag{18}$$

$$B_\theta'' + \frac{1}{r} B_\theta' + \left(K^2 - k^2 - \frac{m^2+1}{r^2}\right) B_\theta + \frac{2im}{r^2} B_r = 0 \tag{19}$$

$$B_z'' + \frac{1}{r} B_z' + \left(K^2 - k^2 - \frac{m^2}{r^2}\right) B_z = 0 \tag{20}$$

We define a new variable ρ such that

$$\rho^2 = (K^2 - k^2) r^2 \tag{21}$$

and write the Bessel differential operator as

$$L_{m^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \left(1 - \frac{m^2}{\rho^2}\right) \tag{22}$$

Eqs.(18)-(20) can then be written

$$L_{m^2+1}\,(B_r) - \frac{2im}{\rho^2} B_\theta = 0 \tag{23}$$

$$L_{m^2+1}\,(B_\theta) + \frac{2im}{\rho^2} B_r = 0 \tag{24}$$

$$L_{m^2}\,(B_z) = 0 \tag{25}$$

Eqs.(23) and (24) can be separated if we transform to left- and right-handed circularly polarized components B^L, B^R :

$$\sqrt{2}\; B^L = B_r + i\, B_\theta\,, \qquad \sqrt{2}\, B^R = B_r - i\, B_\theta \quad . \tag{26}$$

We then obtain

$$L_{(m-1)^2}\,(B^R) = 0 \tag{27}$$

$$L_{(m+1)^2}\,(B^L) = 0 \tag{28}$$

$$L_{m^2}\,(B_z) = 0 \tag{29}$$

Solutions of $L_{m^2} = 0$ are the Bessel functions $J_m(\rho)$ and $Y_m(\rho)$; we retain only the J_m functions, which are finite at $\rho = 0$. Defining the transverse wave number T so that

$$T^2 \equiv K^2 - k^2 = (\gamma^4/k^2 - k^2)\ , \tag{30}$$

we can write the solutions as

$$B^R = C_1 J_{m-1}(Tr)\,, \quad B^L = C_2 J_{m+1}(Tr)\,, \quad B_z = C_3 J_m(Tr)\ . \tag{31}$$

We note that, so far, γ, k, T, and ρ are all real.

Having found $\underline{B}$, we may now express the other quantities in terms of $\underline{B}$. From the first and last parts of Eq.(8), we obtain

$$\underline{j} = \frac{\omega}{k}\ \frac{en_o}{B_o}\ \underline{B} = \mu_o\ K\ \underline{B}\ . \tag{32}$$

Eq.(4) then yields

$$E_r = (B_o/en_o)j_\theta = (\omega/k)B_\theta \tag{33}$$

$$E_\theta = -(B_o/en_o)j_r = -(\omega/k)B_r \,. \tag{34}$$

Transforming to circular polarization by

$$\sqrt{2}\,E_r = E^R + E^L \,, \qquad \sqrt{2}\,E_\theta = i(E^R - E^L) \,, \tag{35}$$

we obtain

$$\sqrt{2}\,E^R = i(\omega/k)(B_r - iB_\theta) = i\sqrt{2}\,(\omega/k)B^R \tag{36}$$

$$\sqrt{2}\,E^L = -i(\omega/k)(B_r + iB_\theta) = -i\,\sqrt{2}(\omega/k)B^L \tag{37}$$

Thus

$$E^R = i(\omega/k)B^R = i(\omega/k)C_1\,J_{m-1}(Tr) \tag{38}$$

$$E^L = -i(\omega/k)B^L = -i(\omega/k)\,C_2\,J_{m+1}(Tr) \,. \tag{39}$$

We next evaluate $\underline{\nabla}.\underline{B}$ and $\underline{\nabla}.\underline{E}$:

$$\frac{1}{T}\,\underline{\nabla}\cdot\underline{B} = B_r{}' + \frac{1}{\rho}B_r + \frac{im}{\rho}B_\theta + \frac{ik}{T}B_z = 0 \,, \tag{40}$$

where the (') stands for $\partial/\partial\rho$. Converting to B^R and B^L by Eq.(35) and substituting Eq.(31), we obtain

$$C_1(J_{m-1}' + \frac{1-m}{\rho}J_{m-1}) + C_2(J_{m+1}' + \frac{1+m}{\rho}J_{m+1}) + i\sqrt{2}\,C_3(k/T)J_m = 0 \tag{41}$$

We make use of the recursion relation

$$J_n' = J_{n-1} - \frac{n}{\rho}J_n = -J_{n+1} + \frac{n}{\rho}J_n \tag{42}$$

to obtain

$$[\,-C_1 + C_2 + i\,\sqrt{2}\,C_3(k/T)\,]J_m(Tr) = 0 \tag{43}$$

To evaluate $\underline{\nabla}\cdot\underline{E}$, we note that the result is the same except that, from Eqs.(38) and (39), the sign of C_2 is changed and that, from Eq.(6), there is no corresponding C_3 term. Thus $\underline{\nabla}\cdot\underline{E}$ would vanish if

$$(C_1 + C_2)\ J_m(Tr) = 0 \quad . \tag{44}$$

This is satisfied everywhere if we set $C_2 = -C_1$. Eq.(43) now gives

$$C_3 = -i\sqrt{2}(T/k)C_1 \quad . \tag{45}$$

It is therefore possible to have a purely electromagnetic helicon wave in cylindrical coordinates. The components of $\underline{B}$ and $\underline{E}$ are specified by Eqs.(44) and (45), and the current $\underline{j}$ is parallel to $\underline{B}$, as given by Eq.(32). The complete solution is summarized as follows:

$$B^R = C_1 J_{m-1}(Tr), \qquad E^R = i(\omega/k)C_1\ J_{m-1}(Tr) \tag{46}$$

$$B^L = -C_1 J_{m+1}(Tr), \qquad E^L = i(\omega/k)C_1\ J_{m+1}(Tr) \tag{47}$$

$$B_z = -i\ \sqrt{2}(T/k)C_1 J_m(Tr) \qquad E_z = 0 \tag{48}$$

$$\underline{j} = (\omega/k)(en_o/B_o)\underline{B} \ . \tag{49}$$

The dispersion relation is determined by the boundary conditions. If the uniform plasma is contained in a conducting cylinder of radius a, then we require $E_\theta = 0$ at r=a. Eqs.(35), (46), and (47) then give

$$J_{m-1}(Ta) - J_{m+1}(Ta) = 0 \quad . \tag{50}$$

The recursion relation (42) simplifies this to

$$J_m'\ (Ta) = 0 \quad . \tag{51}$$

On the other hand, if the plasma is bounded by an insulator, we require $j_r = 0$ at r=a. Eq.(49) then shows that $B_r = 0$ or $B^R = -B^L$. Eqs.(46) and

(47) then give $J_{m-1}(Ta) = J_{m+1}(Ta)$, leading to the same boundary condition, Eq.(51). If we define

$$P'_{mn} \equiv n^{th} \text{ zero of } J'_m(\rho), \tag{52}$$

the eigenvalues of T are $T = P'_{mn}/a$. Eq.(30) yields the dispersion relation

$$T^2 = K^2 - k^2 = (P'_{mn}/a)^2 \tag{53}$$

with

$$K = (\omega/k)(en_o\mu_o/B_o) \quad . \tag{54}$$

If the plasma is nonuniform, K would be a function of r proportional to $n_o(r)/B_o(r)$. The differential equations (18)-(20) would then be changed, but it may be possible to choose functions of r that lead to other known functions, or to use a WKB analysis.

III. VALIDITY OF THE HELICON EQUATIONS

Neglect of the displacement current $-i\omega\,\mu_o\varepsilon_o\underline{E}$ in Eq.(2) requires that the $\mu_o\underline{j}$ term be much larger. Since $\underline{j}$ is given by Eq.(7), we have

$$\left|\frac{\omega\varepsilon_o E}{\mu_o j}\right| = \left|\frac{\omega\mu_o\varepsilon_o\, E}{\mu_o e n_o E/B_o}\right| = \left|\frac{\omega e B_o}{m}\,\frac{\varepsilon_o m}{n_o e^2}\right| = \frac{\omega\omega_c}{\omega_p{}^2} = \frac{\omega\Omega_e}{\Omega_p{}^2} \tag{55}$$

Therefore, displacement current is negligible if

$$\frac{\omega}{\Omega_c} \ll \frac{\Omega_p{}^2}{\Omega_c{}^2} = \frac{c^2}{V_A{}^2} \quad , \quad \text{or} \quad \frac{\omega}{\omega_c} \ll \frac{\omega_p{}^2}{\omega_c{}^2} \quad . \tag{56}$$

To neglect all electron motions except $\underline{E} \times \underline{B}_o$ drift, consider the velocity components in a cold plasma:

$$V_{ex} = -\frac{ie}{m\omega}\,\frac{E_x - i(\omega_c/\omega)E_y}{1 - \omega_c{}^2/\omega^2} \simeq \frac{E_y}{B_o} + \frac{i\omega}{\omega_c}\,\frac{E_x}{B_o} \simeq \frac{E_y}{B_o} \tag{57}$$

$$V_{ey} = \frac{-ie}{m\omega} \frac{E_y + i(\omega_c/\omega)E_x}{1 - \omega_c^2/\omega^2} \simeq -\frac{E_x}{B_o} + \frac{i\omega}{\omega_c}\frac{E_y}{B_o} \simeq -\frac{E_x}{B_o} \tag{58}$$

The approximations require $\omega/\omega_c \ll 1$, and what is left is the $\underline{E} \times \underline{B}_o$ drift. This condition is more stringent than Eq.(56), since ω_p^2/ω_c^2 usually exceeds unity.

To ~~reflect~~ neglect all the ion motions, and hence the $\underline{V} \times \underline{B}$ term that usually appears on the left-hand side of Eq.(4), consider the ion velocity components:

$$V_{ix} = \frac{ie}{M\omega} \frac{E_x + i(\Omega_c/\omega)E_y}{1 - \Omega_c^2/\omega^2} \simeq \frac{i\Omega_c}{\omega}\frac{E_x}{B_o} - \frac{\Omega_c^2}{\omega^2}\frac{E_y}{B_o} \tag{59}$$

$$V_{iy} = \frac{ie}{M\omega} \frac{E_y - i(\Omega_c/\omega)E_x}{1 - \Omega_c^2/\omega^2} \simeq \frac{i\Omega_c}{\omega}\frac{E_y}{B_o} + \frac{\Omega_c^2}{\omega^2}\frac{E_x}{B_o} , \tag{60}$$

where we have taken $\omega^2 \gg \Omega_c^2$ in the denominator. The contribution of $\underline{V}_i$ to $\underline{j}_i$ is therefore less than the electron contribution by a factor of Ω_c^2/ω^2 in the $\underline{E} \times \underline{B}_o$ drift (second term) and by a factor of Ω_c/ω in the polarization drift (first term). Hence, the ion motion can be neglected if $\omega/\Omega_c \gg 1$. The helicon equations are therefore valid for

$$\Omega_c \ll \omega \ll \omega_c , \tag{61}$$

which gives a range of over 1000 to 1 in ω for argon waves.

IV. HELICONS WITH FINITE RESISTIVITY

In this case, Eq.(4) is modified to

$$\underline{E} = \eta\underline{j} + \underline{j} \times \underline{B}_o/en_o , \tag{62}$$

and Eq.(6) to

$$E_z = \eta j_z . \tag{63}$$

Now Eqs.(1), (62), and (2) give, by analogy with Eq.(8) ,

$$i \omega \underline{B} = \underline{\nabla} \times \underline{E} = \frac{B_o}{en_o} \frac{\partial \underline{j}}{\partial z} + \eta \underline{\nabla} \times \underline{j}$$

$$= \frac{ikB_o}{en_o \mu_o} \underline{\nabla} \times \underline{B} + \frac{\eta}{\mu_o} \underline{\nabla} \times (\underline{\nabla} \times \underline{B}). \tag{64}$$

Defining K as in Eqs.(11) and (10), we have

$$\underline{B} = \frac{1}{K} \underline{\nabla} \times \underline{B} + \frac{i\eta}{\omega\mu_o} \nabla^2 \underline{B} . \tag{65}$$

To make further progress, we must assume that the damping due to η is small, so that in the last term $\nabla^2 \underline{B}$ can be approximated by the $\eta = 0$ value, $-K_o^2 \underline{B}$ [Eq.(14)]. Eq.(65) then becomes

$$(1 + i\delta)\underline{B} = K^{-1} \underline{\nabla} \times \underline{B} , \tag{66}$$

where we have defined

$$\delta \equiv \eta K_o^2/\omega\mu_o \ll 1, \tag{67}$$

K_o^2 being the real value of K in the $\eta = 0$ problem.

The curl of Eq.(66) yields

$$(1 + i\delta) \underline{\nabla} \times \underline{B} = -K^{-1} \nabla^2 \underline{B} , \tag{68}$$

and substituting Eq.(66) for $\underline{\nabla} \times \underline{B}$ gives

$$\nabla^2 \underline{B} + K^2 (1 + i\delta)^2 \underline{B} = 0 . \tag{69}$$

Let ω be real and k complex to represent a spatially damped wave. Eq.(11) states that

$$Kk = \gamma^2 , \tag{70}$$

where $\gamma^2 = \omega_e \mu_o (n_o/B_o)$ is real. Since k is now complex, K will necessarily be complex, as evident from the form of Eq.(69). Comparing Eq.(69) with Eq.(14), we see that the entire analysis is unchanged if we simply replace K by K(1 + iδ). We therefore define the variable ρ, in analogy with Eq.(21), by

$$\rho^2 = [K^2(1+i\delta)^2 - k^2]\ r^2 = T^2r^2 \tag{71}$$

To satisfy real boundary conditions (assuming that they are not significantly changed by the small damping), T_r must be real, and hence ρ must be real. The dispersion relation, Eq.(53), now becomes

$$T^2 = K^2(1+i\delta)^2 - k^2 = (p'_{mn}/a)^2 \quad . \tag{72}$$

With Eq.(70), this can be written

$$T^2 = \gamma^4(1+i\delta)^2/k^2 - k^2 \ . \tag{73}$$

Since T and γ are real, k must be complex in such a way as to make T real. In previous literature, the k^2 in the first term of Eq.(73) was replaced by its real part and Im(k) was found from the second term of Eq.(73). This yielded the spurious result

$$I_m(k) = \frac{K_o^4\eta}{\mu_o\omega}\ \frac{1}{R_e(k)} \quad .$$

In fact, it is the k^2 in the first term that determines the damping, the second term usually being small. To see this, we solve the quadratic

$$k^4 + T^2k^2 - \gamma^4(1 + i\delta)^2 = 0, \tag{74}$$

obtaining

$$2k^2 = -T^2 \pm T^2\ [1 + (2\gamma^2/T^2)^2\ (1+i\delta)^2]^{\frac{1}{2}} \quad . \tag{75}$$

To simplify this, we note from Eq.(73) that $T^2 = (\gamma^4/k^2) - k^2$ in the $\delta = 0$ case, so that $\gamma^4 = k^2(T^2+k^2)$. In the usual case $k^2 << T^2$, then, we have $\gamma^4 \simeq k^2T^2$, and $2\gamma^2/T^2 \simeq 2k/T << 1$. We therefore expand the square root, keeping the + sign, which gives the least-damped root:

$$2k^2 = \frac{2\gamma^4}{T^2}(1 + i\delta)^2 \tag{76}$$

$$k = \frac{\gamma^2}{T}(1 + i\delta) \tag{77}$$

Comparing Eq.(76) with Eq.(73), we see that our solution is tantamount to dropping the last term in Eq.(73).

In summary, the approximate solution for $k^2 << T^2$ can be written with the help of Eqs.(10), (53), (67), and (70) as follows:

$$\mathrm{Re}(k) = \frac{\gamma^2}{T} = \frac{\omega e n_o \mu_o}{B_o}\frac{a}{P'_{mn}} , \quad \frac{\omega}{k_z} = \frac{B_o}{e n_o \mu_o}\frac{P'_{mn}}{a} \tag{78}$$

$$\frac{\mathrm{Im}(k)}{\mathrm{Re}(k)} = \delta = \frac{\eta K_o^2}{\omega\mu_o} = \frac{\eta\gamma^4}{\omega\mu_o}\frac{1}{[\mathrm{Re}(k)]^2} = \frac{\eta T^2}{\omega\mu_o} \tag{79}$$

In terms of the electron-ion collision frequency ν_{ei} given by

$$\nu_{ei} = \frac{n_o e^2}{m}\eta , \tag{80}$$

we can also write the damping rate as

$$\frac{\mathrm{Im}(k)}{\mathrm{Re}(k)} = \frac{\eta T^2}{\omega\mu_o} = \frac{\nu_{ei}}{\omega}\frac{c^2T^2}{\omega_p^2} = \frac{\nu_{ei}}{\omega_c}\frac{\gamma^2}{k^2} . \tag{81}$$

To account for electron neutral collisions, we merely need to replace ν_{ei} with $\nu_{ei} + \nu_{eo}$.

We next consider the effect of η on the fields and currents given by Eqs.(46)-(49). If we assume that B^R and B^L are relatively unchanged, we see that B_z, which contains 1/k, is now multiplied by $\simeq(1-i\delta)$; it is no longer 90^o out of phase with B^R and B^L but now has a small component in phase with B^L. The current $\underline{j}_\perp$ also contains $1/k \propto (1 - i\delta)$ and now is slightly advanced in phase relative to $\underline{B}$. The field $\underline{E}_\perp$, which was 90^o out of phase with $\underline{B}$, now contains an in-phase component. The component $E_z = \eta j_z$ is

$$E_z \simeq -i\sqrt{2}\ \frac{\eta T^3}{\mu_o \gamma^2}\ C_1 J_m(Tr) \quad . \tag{82}$$

It is the j_z due to this field that causes the main part of the damping. If one evaluates $\underline{\nabla} \cdot \underline{E}$, the perpendicular components cancel as before, but E_z now contributes the amount

$$\underline{\nabla} \cdot \underline{E} \simeq \sqrt{2}(\eta/\mu_o) T^2 C_1 J_m(Tr) \quad , \tag{83}$$

giving rise to a small electrostatic component of the wave.

The magnitude of resistive damping can be estimated from Eq.(81). For a small tube with $T \simeq 1\ \mathrm{cm}^{-1}$ and a 3-eV plasma driven at 8 MHz, we find that

$$\frac{\mathrm{Im}(k)}{\mathrm{Re}(k)} = 5.2 \times 10^{-5}\ \frac{Z\ \ln\Lambda}{T_{ev}^{3/2}} \cdot \frac{(100)^2}{(2\pi)(8\mathrm{x}10^6)} \cdot \frac{10^7}{4\pi} = 1.7\% \tag{84}$$

V. HELICONS WITH LANDAU DAMPING

With some care, Landau damping can be added to the cold plasma equations by a trivial modification. We now write Eq.(2) as

$$c^2 \underline{\nabla} \mathrm{x} \underline{B} = -i\omega\, \underline{\underline{\varepsilon}} \cdot \underline{E} \quad , \tag{85}$$

where the plasma currents $\underline{j}$ have been incorporated in the dielectric tensor $\underline{\underline{\varepsilon}}$. For a cold plasma, $\underline{\underline{\varepsilon}}$ can be written in Stix notation as

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \tag{86}$$

where $S = (R+L)/2$, $D = (R-L)/2$, and

$$R = 1 - \frac{\omega_p^2}{\omega(\omega-\omega_c)} - \frac{\Omega_p^2}{\omega(\omega+\Omega_c)} \simeq 1 + \frac{\Omega_p^2}{\omega\Omega_c} - \frac{\Omega_p^2}{\omega^2} \qquad (87)$$

$$L = 1 - \frac{\omega_p^2}{\omega(\omega+\omega_c)} - \frac{\Omega_p^2}{\omega(\omega-\Omega_c)} \simeq 1 - \frac{\Omega_p^2}{\omega\Omega_c} + \frac{\Omega_p^2}{\omega^2} \qquad (88)$$

$$P = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\Omega_p^2}{\omega^2} \simeq 1 - \frac{\omega_p^2}{\omega^2} \quad . \qquad (89)$$

We have made the helicon approximation of Eq.(61) that $\Omega_c \ll \omega \ll \omega_c$. In this regime, we see that S = 1 and

$$D = \frac{\Omega_p^2}{\omega^2} \left(\frac{\omega}{\Omega_c} - 1\right) \simeq \frac{\gamma^2 c^2}{\omega^2} \quad . \qquad (90)$$

The plasma current $\underline{j} = \nabla x B/\mu_o$ is given by

$$j_x = -i\omega\varepsilon_o (E_x - i\frac{\gamma^2 c^2}{\omega^2} E_y) \qquad (91)$$

$$j_y = -i\omega\varepsilon_o (E_y + i\frac{\gamma^2 c^2}{\omega^2} E_x) \qquad (92)$$

$$j_z = -i\omega\varepsilon_o \, PE_z \qquad (93)$$

The first terms of Eqs.(91) and (92) are the displacement current, which we have shown to be negligible, and the second terms are the $\underline{E}x\underline{B}_o$ drifts. To incorporate kinetic effects in the parallel direction, Eq.(93) has to be replaced by the solution of the Vlasov equation for electron motion along $\underline{B}_o$. The result is[5]

$$P = \varepsilon_{zz} = 1 - \frac{\omega_p^2}{k^2 V_{th}^2} Z' \left(\frac{\omega}{k\, V_{th}}\right) = 1 - \frac{\omega_p^2}{\omega^2} \zeta^2 Z'(\zeta) \qquad (94)$$

where $k = k_z$, $V_{th}^2 = 2k_B T_e/m$, and $Z'(\zeta)$ is the plasma dispersion function derivative. Electron collisions with ions or neutrals with a frequency ν

can also be incorporated by adding a Krook collision term to Vlasov's equation. In that case we define ζ to be

$$\zeta \equiv \frac{\omega + i\nu}{kV_{th}} \quad , \tag{95}$$

and P becomes

$$P = \varepsilon_{zz} = 1 - \frac{\omega_p^2}{\omega^2} \, \frac{1 - i(\nu/\omega)}{1 - i(\nu/2\omega) Z'(\zeta)} \quad \zeta^2 Z'(\zeta) \tag{96}$$

The "1" again represents displacement current, which we can neglect. Furthermore, since we can solve the helicon equations analytically only for small damping, we make the same assumption here and expand $Z'(\zeta)$ for large ζ :

$$\zeta^2 Z'(\zeta) \simeq 1 + 3/2\zeta^2 - 2i\sqrt{\pi}\, \zeta^3 e^{-\zeta^2} + \ldots . \tag{97}$$

In the collisionless case, Eqs.(93) and (94) then give

$$j_z = i\omega\varepsilon_o \, E_z \, \frac{\omega_p^2}{\omega^2} \left(1 + \frac{3}{2} \, \frac{k^2 V_{th}^2}{\omega^2} - 2i\sqrt{\pi} \quad \zeta^3 \, e^{-\zeta^2} \right) . \tag{98}$$

This contains non-dissipative terms which correspond to the Bohm-Gross dispersion relation for plasma waves and which we do not need. To see how these can be neglected, consider the generalized Ohm's law including electron inertia:

$$\underline{E} + \underline{V} \times \underline{B}_o = \eta \underline{j} + \frac{1}{en} \underline{j} \times \underline{B}_o + \frac{1}{\varepsilon_o \omega_p^2} \, \frac{\partial \underline{j}}{\partial t} \quad . \tag{99}$$

For $\partial \underline{j}/\partial t \rightarrow -i\,\omega\, \underline{j}$, the z-component becomes

$$E_z = \left(\eta - \frac{i\omega}{\varepsilon_o \omega_p^2} \right) j_z \tag{100}$$

To compare this with Eq.(98), we solve Eq.(98) for E_z, assuming that the Landau term is small:

$$E_z = -\frac{i\omega}{\varepsilon_o \omega_p^2}\left(1 - \frac{3}{2}\frac{k^2 V_{th}^2}{\omega^2}\right)^{-1}\left(1 + 2i\sqrt{\pi}\,\zeta^3 e^{-\zeta^2}\right) j_z \quad . \quad . \qquad (101)$$

The imaginary part of this is identical with that of Eq.(100) except for the small Bohm-Gross thermal correction, which we may neglect. Since the inertial term in Ohm's law is negligible at our low frequencies, we may also neglect that term in Eq.(101), we identify the real part of the coefficient in Eq.(101) with an equivalent resistivity due to Landau damping:

$$\eta_{LD} = 2\sqrt{\pi}\,\frac{\omega}{\varepsilon_o \omega_p^2}\,\zeta^3 e^{-\zeta^2} \quad . \qquad (102)$$

From Eq.(80), the equivalent collision frequency takes an even simpler form

$$\nu_{LD}/\omega = 2\sqrt{\pi}\,\zeta^3 e^{-\zeta^2} \quad , \qquad (103)$$

where $\zeta = \omega/kV_{th}$. Inserting this into Eq.(81), we obtain the Landau damping rate

$$\frac{\mathrm{Im}(k)}{\mathrm{Re}(k)} = 2\sqrt{\pi}\,\frac{c^2 T^2}{\omega_p^2}\,\zeta^3 e^{-\zeta^2} \quad . \qquad (104)$$

This is such a steep function of ζ that a small change in T_e or k_z, which will not greatly affect the eigenmodes, can appreciably increase the damping. In our previous example, taking $n_o = 10^{12}$ cm^{-3} , we have $\nu_{ei}/\omega = 6.5 \times 10^{-2}$. Evaluating Eq.(103) for T_e = 3eV and λ_z = 25 cm, we have $\nu_{LD}/\omega = 0.59$, though at λ_z = 50 cm it would be only 5.4×10^{-5}. A jump in T_e or k_z can explain a discontinuous change in absorption.

VI. PHYSICAL MECHANISMS

Although there can be no Landau damping when $E_z = 0$, a finite value of E_z arises when there is a dissipative mechanism of any sort, and this leads to the Landau acceleration of electrons, which, in turn increases the value of E_z. From Eq.(49), we see that a j_z must exist in the wave even in the absence of damping in order to keep the components of $\underline{E}$ and $\underline{B}$ in proper balance. When a dissipative mechanism impedes the flow of electrons that provides this current, an electrostatic field E_z must arise to push the electrons along $\underline{B}_o$, and it is this E_z which causes Landau damping.

To see this quantitatively, consider the value of j_z given by Eqs. (49) and (48), leading to the expression (82) for $E_z = \eta j_z$:

$$j_z = -i\sqrt{2}\,(T^3/\mu_o\gamma^2)\; C_1 J_m(Tr) \quad . \tag{105}$$

Here we have approximated k by γ^2/T Eq.(77) . The energy dissipated per m^3 is

$$\frac{dW}{dt} = \eta j_z{}^2 = -2\eta(T^3/\mu_o\gamma^2)^2\; C_1{}^2 J_m{}^2 \; . \tag{106}$$

The wave energy is $|B|^2/2\mu_o + \varepsilon_o|E|^2/2$, the kinetic energy of the drifting electrons being negligible. From Eqs.(46)-(48), we see that $\varepsilon_o|E|^2$ is smaller than $|B|^2/\mu_o$ by a factor ω^2/c^2k^2, and that B_z is the dominant part of $|B|$ because of the large factor T/k. Thus the wave energy is approximately

$$W \simeq |B_z|^2/2\mu_o = 2(TC_1J_m/k)^2/2\mu_o \simeq (T^2C_1J_m/\gamma^2)^2/\mu_o \tag{107}$$

The wave energy decays as $W \simeq \exp[-\,2\mathrm{Im}(k)z]$, so that

$$\frac{dW}{dz} = -2\mathrm{Im}(k)W, \tag{108}$$

and the group velocity V_g is $\simeq \omega/k$, from Eq.(78). Thus, the wave decays in time as

$$\frac{dW}{dt} = V_g\frac{dW}{dz} = -2\,\frac{\omega}{k}\;\mathrm{Im}(k)W \; . \tag{109}$$

Using Eq.(81) for Im(k) and Eq.(107) for W, we obtain

$$\frac{dW}{dt} = -\frac{2\omega}{k}\frac{k}{\omega}\frac{\eta T^2}{\mu_o}\frac{1}{\mu_o}\frac{T^4}{\gamma^4}C_1^{\ 2}J_m^{\ 2} = -2\eta(T^3/\mu_o\gamma^2)^2\ C_1^{\ 2}J_m^{\ 2}\ , \tag{110}$$

in agreement with Eq.(106).

We next investigate the structure of the wave, particularly for the lowest possible mode, m=1, n=1. Fig.1 shows the Bessel functions J_o, J_1, and J_2 representing the radial variation of B^R(or E^R), B_z, and B^L (or E^L), respectively, showing that $P_{11}' = 1.84$. Eqs.(46)-(49) give for this mode:

$$E_R \simeq iJ_o\ , \qquad E^L \simeq iJ_2, \qquad B^R \simeq J_o, \qquad B^L \simeq -J_2, \tag{111}$$

$$B_z \simeq -iJ_1\ , \qquad j_z \simeq -iJ_1\ . \tag{112}$$

We have assumed $\underline{B}_o$ and $\underline{k}$ to be in the $+\hat{z}$ direction. Eq.(35) then gives

$$E_r \simeq i(J_o + J_2)\quad , \qquad E_\theta \simeq -(J_o - J_2) \tag{113}$$

$$B_r \simeq J_o - J_2\quad , \qquad B_\theta \simeq i(J_o + J_2) \tag{114}$$

We see that $|E_r|$ falls slightly from 1 at r=0 to ≃0.6 at r=a, while $|E_\theta|$ falls from 1 to 0 at the edge; the opposite is true for B_r and $|B_\theta|$. Furthermore, E_r and E_θ are 90° out of phase, as expected for a rotating pattern, and E and B are also 90° out of phase, as expected for an electromagnetic wave. The current j_z is in phase with B_z. These phase relations show that the mode has the structure shown in Fig.2a on two planes 90° apart. The solid lines represent $\underline{E}$ and are seen to be purely radial at the conducting boundary. The dashed lines represent $\underline{B}$ and are perpendicular to the $\underline{E}$ lines. Since $\underline{j}$ is parallel to $\underline{B}$, these are also lines of $\underline{j}$. These lines are purely azimuthal at the boundary, showing the reason that insulating and conducting boundaries have the same boundary condition. The dashed lines seem to be not divergence-free, but that is because the z-component has not been drawn. The direction of $\underline{B}_\perp$ is easily found, since it is the same as the direction of $\underline{j}_\perp$, which is $-\underline{E}\times\underline{B}_o$. The entire pattern is translated in the $+\hat{z}$ direction, so that an observer at a given z will see a clockwise rotation when looking along $\underline{B}_o$. It is clear that $\underline{E}$ is purely circularly polarized in the R direction at the center and is purely plane polarized at the edge, consistent with Eqs.(111) and (113).

The continuity of the $\underline{B}$ and $\underline{j}$ lines is shown in Fig.2b, where the middle, elliptical $\underline{B}_\perp$ surface is drawn for one whole wavelength in z. We see that the $\underline{B}$ and $\underline{j}$ lines are more or less helical, flowing primarily in the z-direction. What we see in the cross-sectional planes of Fig.2a is only the perpendicular component necessary for the lines to follow the twisting ellipse. In Fig.2b, two real lines of $\underline{B}$ or $\underline{j}$ are shown dashed; the solid ellipses are only the projections of the helical lines onto the cross-sections. The reference arrow is the direction of $\underline{E}$.

From Fig.2, one can see that an antenna with segments parallel to $\hat{z}$ and $\hat{\theta}$ can couple to the wave in two ways. The z-legs can induce a plasma current j_z opposing the j_z in the antenna, and this can couple to the j_z of the wave. In addition, electrons responding to the induced E_z will move along $\underline{B}_o$ until they pile up and form space-charge clouds of opposite charge 180° apart in θ. These charges will give an electrostatic field $\underline{E}$ like the one passing through the center of Fig.2a, thus exciting the radial E-field of the wave. We suspect that the latter mechanism is the more efficient.

The way a plasma responds to the requirements of the wave is illustrated in Fig.3. The frequency and magnetic field are usually fixed; let them be f=7 MHz and B_o = 1kG. Let $p'_{mn} = p'_{11} = 1.84$, and let a = 2.5 or 10 cm. Eqs.(53) and (54) then give

$$n_o = 1.13 \times 10^{14}\ k(T^2 + k^2)^{\frac{1}{2}}\ \text{cm}^{-3}\ , \tag{115}$$

where k and T are in cm^{-1}, and n is plotted in Fig.3 vs. $\lambda = 2\pi/k$.

$$d \equiv \frac{\mathrm{Im}(k)}{\mathrm{Re}(k)} = 10^{12} T^2 n_o^{-1} \zeta^3 e^{-\zeta^2} \tag{116}$$

where $\zeta = 0.118 \lambda T_{ev}^{-1/2}$. (117)

The damping rate d is also shown in Fig.3, ~~compared~~ computed with the value of n_o given by Eq.(115). The roll-off of d at small λ is not real; it is due to the breakdown of the expansion, Eq.(97).

When the rf is first turned on, non-resonant heating causes a low-density plasma. The wavelength is roughly set by twice the antenna length - say $\lambda = 50$ cm and a = 10 cm. The resonant density is $\simeq 3 \times 10^{12}$ cm^{-3}. When the density reaches this, a resonant helicon wave can be excited, but it is not efficiently absorbed, d being only 5×10^{-5} at 3 eV. However, the plasma can jump to a state of higher entropy increase by either a temperature change or a density change. Suppose the temperature jumps to 10 eV. Then d suddenly jumps to $\simeq 10^{-1}$, and the plasma becomes highly absorbing. The temperature, however, can be damped by some inelastic process in the gas. In that case, the plasma can jump to a higher n_o by ionizing the neutral gas until it is all ionized. To match the jump in n_o, the wavelength must decreases for the wave to stay in tune. If λ jumps from 50 cm to 25 cm, for instance, the damping d increases from $\simeq 5 \times 10^{-5}$ to $\simeq 2 \times 10^{-2}$. In practice, the heating and ionization will be localized near walls, where E_z and j_z are maximum, and so is the energy deposition. This should lead to relatively flat n_o and T_e profiles, thus making a solution of the inhomogeneous-plasma problem unnecessary. However, the energy confinement, which is determined by the end plate sheaths and the radial electric field, may be better on axis, leading to the appearance of a hot, dense core. The energy balance problem in a helicon-excited plasma is a complicated one which is outside the scope of this paper.

VII. DESIGN OF HELICON SOURCES

We neglect collisions and assume $T^2 >> k^2$; Eqs.(53), (54), and (104) then give

$$n_o = \frac{B_o}{\omega e \mu_o} k(T^2 + k^2)^{\frac{1}{2}} \simeq \frac{B_o}{\omega e \mu_o} Tk \qquad (118)$$

$$\frac{\mathrm{Im}(k)}{\mathrm{Re}(k)} = 2\sqrt{\pi} \frac{c^2 T^2}{\omega_p^2} \zeta^3 e^{-\zeta^2} \qquad (119)$$

where $\zeta = \omega/kV_{th}$. We now change to cgs units, assume the lowest mode so that T = 1.84/a, and use the following notation:

$B = B_o$ in kG

$f = \omega/2\pi$ in MHz

$\lambda = 2\pi/k \simeq 2\pi/\mathrm{Re}(k)$ in cm

$n_{14} = n_o$ in units of 10^{14} cm^{-3}

$L = 1/\mathrm{Im}(k)$ in cm

$T_e = k_B T_e$ in eV

Our equations become

$$n_{14} = 7.90 \ BTk/f \qquad (120)$$

$$\frac{1}{kL} = 10^{-2}(T^2/n_{14}) \ \zeta^3 e^{-\zeta^2} . \qquad (121)$$

To design a source with large Landau damping, let us set $\zeta^3 e^{-\zeta^2}$ at its maximum value 0.41, which occurs for $\zeta = (3/2)^{\frac{1}{2}}$. (This value is not particularly significant, since it comes from the asymptotic expansion for $Z'(\zeta)$, but what matters is that $\zeta^3 e^{-\zeta^2}$ and ζ are both of order unity.) We now have

$$n_{14} = 91.3 \ B/\lambda af \qquad (122)$$

$$L/\lambda = 11.5 \ n_{14} a^2 . \qquad (123)$$

The condition $\zeta = 1.22$ gives the further relation

$$f\lambda = 72.4 \ T_e^{\frac{1}{2}} \qquad (124)$$

Eliminating f from Eqs.(122) and (123) yields

$$n_{14} = 1.26 \ B/a \ T_e^{1/2} \tag{125}$$

$$L/\lambda = 14.5 \ Ba/T_e^{1/2} \ . \tag{126}$$

To get a reasonable number of wavelengths in a damping length, the ratio L/λ should be set at some number like 5. To be specific, let us set

$$L/\lambda = 7.2 \tag{127}$$

Then Eq.(126) becomes

$$Ba = 0.5 \ T_e^{1/2} \ . \tag{128}$$

Using this in Eq.(125) yields

$$n_{14} = 0.63/a^2 \tag{129}$$

This shows that the achievable density scales as the reciprocal of the cross-sectional area of the plasma column.

The required field is then found from Eq.(125):

$$B = T_e^{1/2}/2a \ . \tag{130}$$

A coil of length $\lambda/2$ and diameter 2a has aspect ratio

$$A = \lambda/4a \ . \tag{131}$$

The frequency f is constrained only by the choice of A (say, A=5) which will satisfy $T^2 >> k^2$ and geometrical considerations in the construction of the coil. Eqs.(124), (128), and (131) yield

$$f = 18.1 \ T_e^{1/2}/aA = 36.2 \ B/A \ . \tag{132}$$

Finally, the absorption length L, which is approximately the length of the plasma, is given by Eqs.(127) and (131):

$$L = 28.8 \ aA \ . \tag{133}$$

As examples, we give in Table I some parameters for different densities n_o, assuming A=5, $L/\lambda = 7$, and T_e = 4 eV, which is within a factor 2 of the T_e in most laboratory plasmas.

TABLE I

$n(cm^{-3})$	a(cm)	B(kG)	f(MHz)	λ(cm)	L(cm)
10^{11}	25.1	.04	.288	500	3500
10^{12}	7.9	.13	.912	159	1100
10^{13}	2.5	.40	2.88	50	350
10^{14}	.79	1.26	9.12	15.9	111
10^{15}	.25	4.0	28.8	5.0	35
10^{16}	.08	12.6	91.2	1.6	11

These values will be slightly modified by other choices of T_e, L/λ, and A. More accurate calculations will require, in order: (a) retaining k^2 relative to T^2, (b) using the exact $Z'(\zeta)$ function instead of the asymptotic expansion, and (c) iterating the solution of the differential equation to include the complex value of K in the η term.

The power required can be estimated from the experimental results of Boswell et al., where 2 x 10^{-14} W per argon ion was absorbed. If this rate remains the same, 44 W of 90-MHz power would be required to generate a 10^{16} cm^{-3} plasma 1.6 mm in diameter and 11 cm long.

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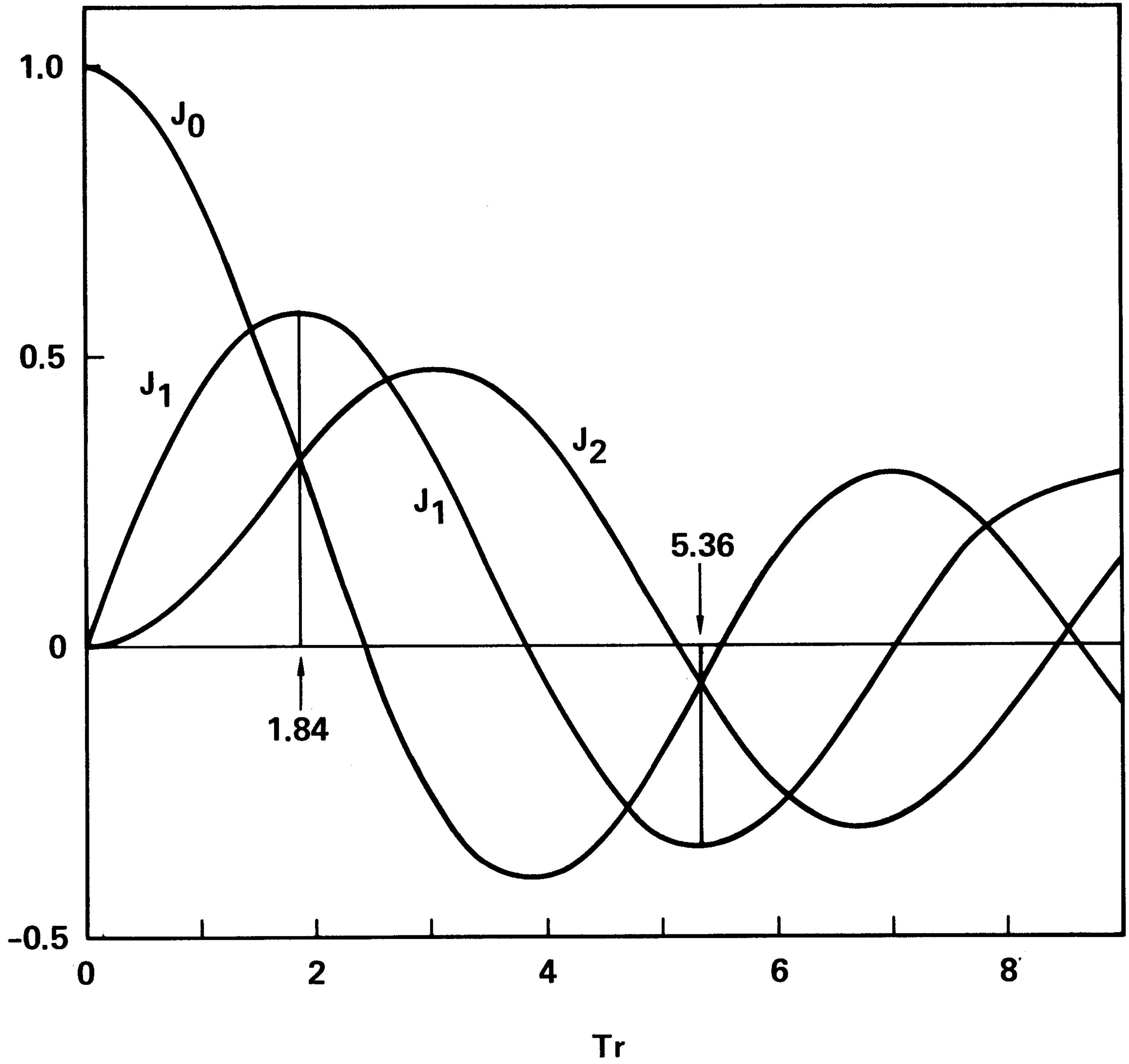


FIG. 1

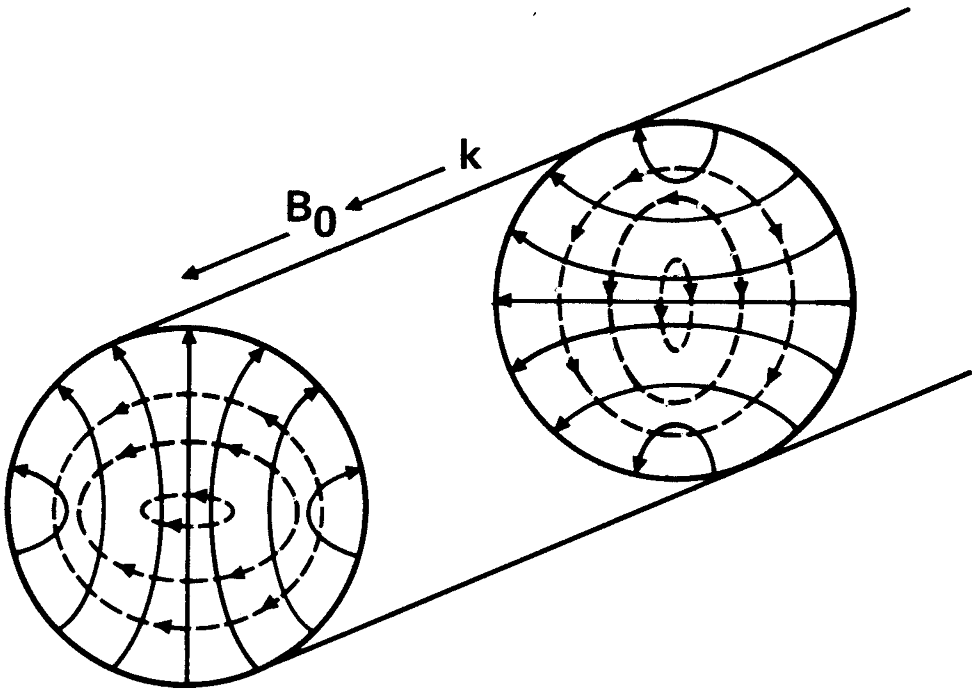


FIG. 2a

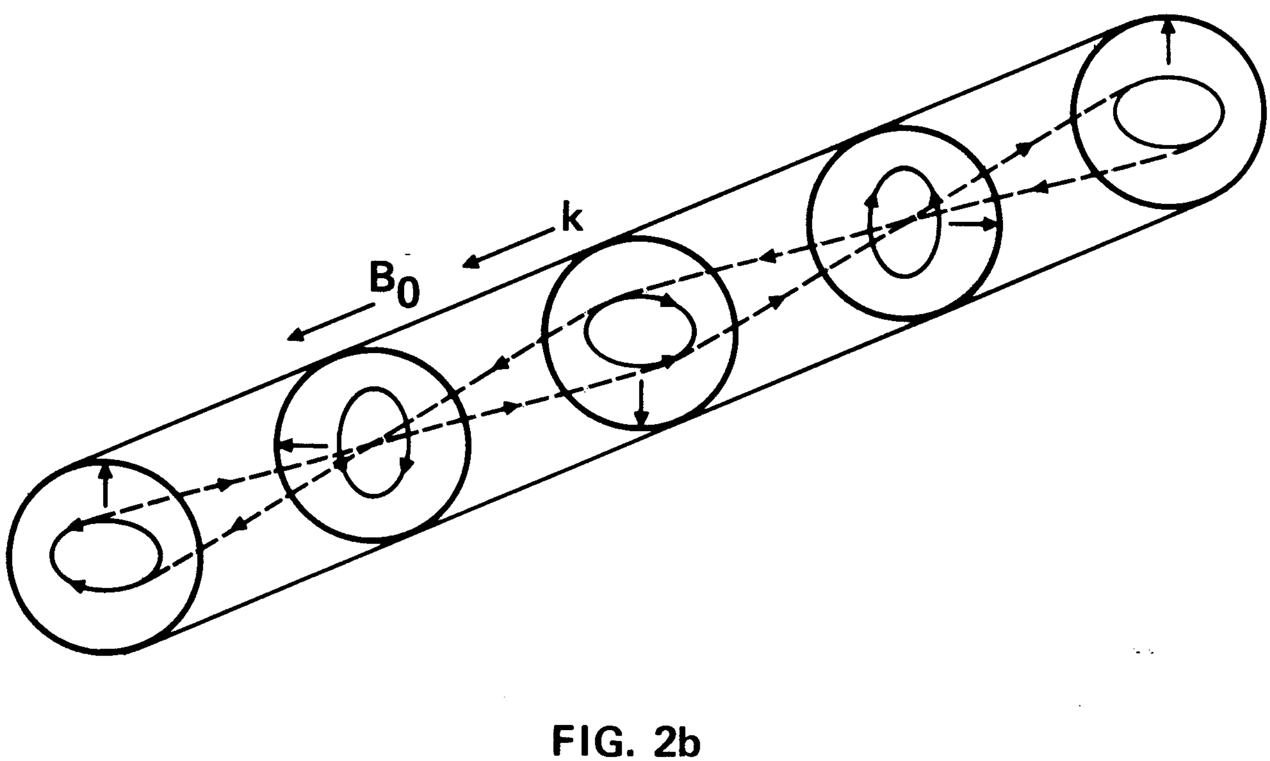
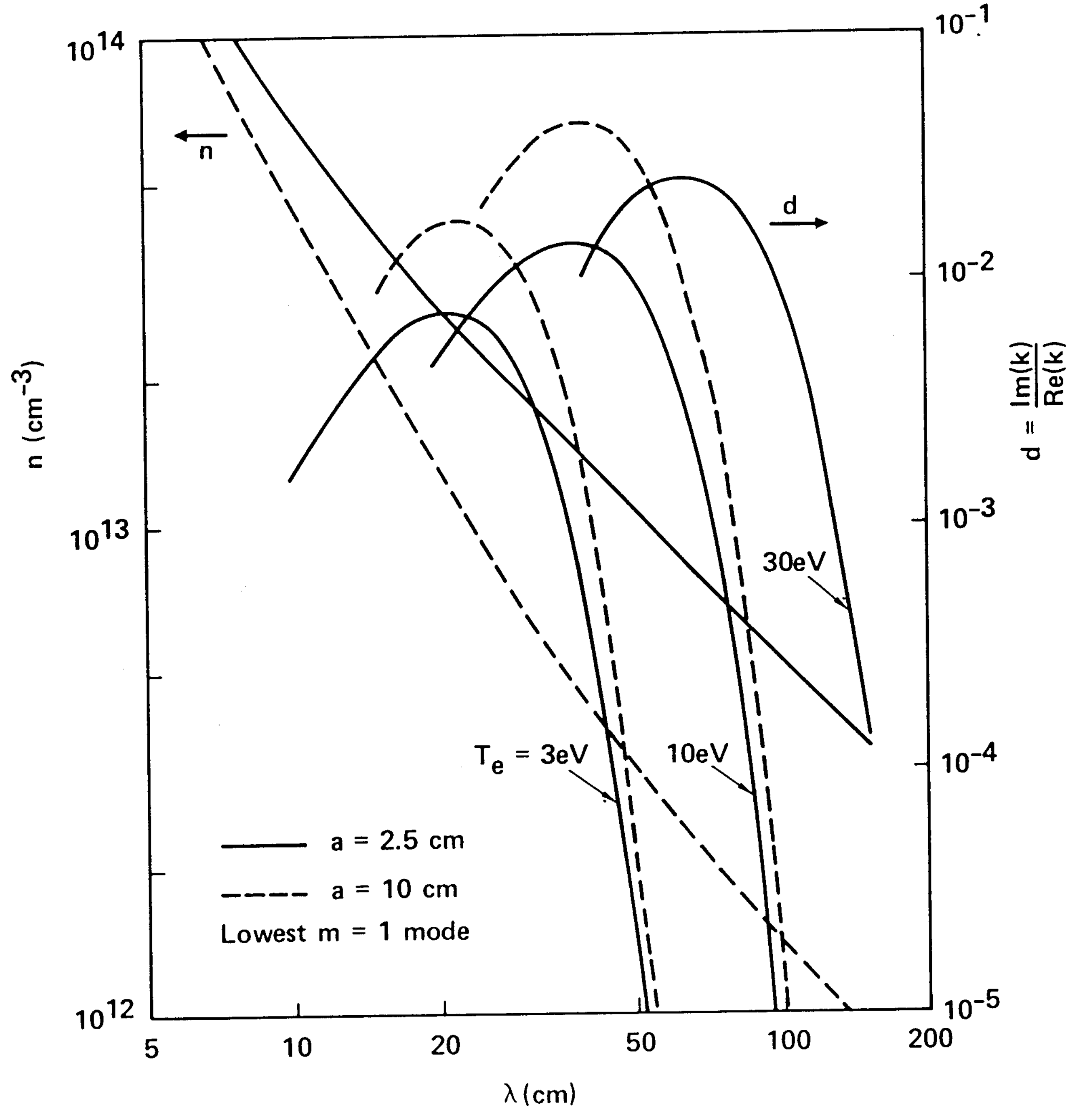


FIG. 2b



10^14
10^13
10^12
n (cm^-3)
n
d
10^-1
10^-2
10^-3
10^-4
10^-5
d = Im(k)/Re(k)
30eV
T_e = 3eV
10eV
a = 2.5 cm
a = 10 cm
Lowest m = 1 mode
5
10
20
50
100
200
λ (cm)

FIG. 3