Fixed Point Effects in Digital Filters

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Finite-precision Problems

- Quantizers are nonlinear devices
  - Characteristics may be significantly different from the ideal linear filter
- Overflow
- Coefficient quantization
- Limit-Cycle Oscillations

Quantizers

- Nonlinear effects make it extremely difficult to precisely analyze the filter's performance.
- How do we model a fixed-point filter then?
  - Adopt a statistical model of the quantization effects
  - Results in a linear model for the filter

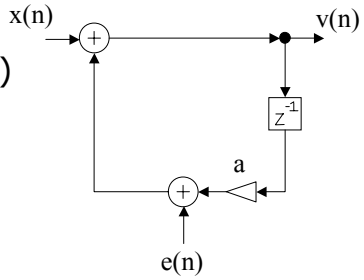
Statistical Characterization

v(n) = Q[av(n-1)]+x(n)

Q[av(n-1)] = av(n-1)+e(n)

v(n) = av(n-1)+x(n)+e(n)

- We can now view the response of the filter as coming from two inputs.



Basic Assumptions

- The noise source is stationary white noise.
  - The sequence e(n) is uncorrelated with the sequence e(m) for n≠m.
  - Sequence is mean ergodic and correlation ergodic.
- The error sequence e(n) is uncorrelated with the input sequence x(n).

Mean

- The mean of the output generated by the filter with impulse response h(n) when excited by e(n) is

$$m_q = m_e \sum_{n=-\infty}^{\infty} h(n)$$

equivalently put

$$m_q = m_e H(0)$$

Autocorrelation

The autocorrelation is computed to be

$$\gamma_{qq}(n) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(k)h(l)\gamma_{ee}(k-l+n)$$

This reduces to

$$\sigma_q^2 = \sigma_e^2 \sum_{k=-\infty}^{\infty} h^2(k)$$

By Parseval's theorem $\sigma_q^2 = \frac{\sigma_e^2}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 d\omega$

Types of Quantization

b fractional bits

- Rounding $m_e = 0$ $\sigma_e^2 = \frac{2^{-2b}}{12}$
- Truncation $m_e = -\frac{2^{-b}}{2}$ $\sigma_e^2 = \frac{2^{-2b}}{12}$
- Magnitude Truncation $m_e = 0$ $\sigma_e^2 = \frac{2^{-2b}}{3}$

Section Ordering is Important

- Example

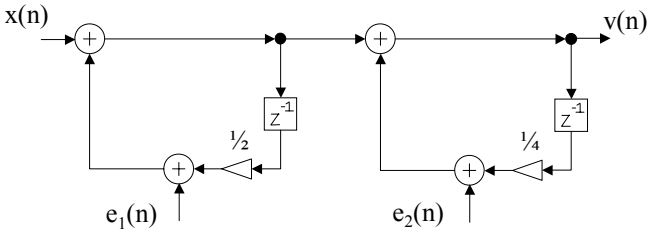
$$H(z) = H_1(z)H_2(z)$$

Where

$$H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad H_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$$

and their corresponding impulse responses are given by h(n), $h_1(n)$ and $h_2(n)$

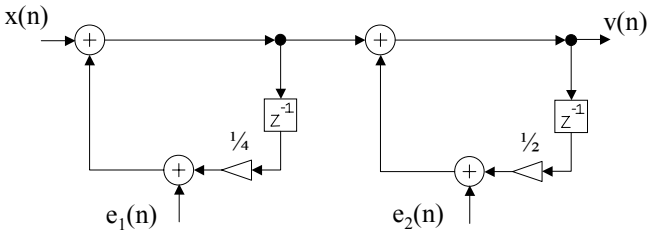
Realization 1



With rounding

$$\sigma_{q1}^2 = \sigma_e^2\left[\sum_{n=0}^{\infty} h^2(n) + \sum_{n=0}^{\infty} h_2^2(n)\right] \approx 2.90\sigma_e^2$$

Realization 2



With rounding

$$\sigma_{q2}^2 = \sigma_e^2 \left[\sum_{n=0}^{\infty} h^2(n) + \sum_{n=0}^{\infty} h_1^2(n) \right] \approx 3.16\sigma_e^2$$

Section Ordering Comparison

- The overall noises were found to be

$$\sigma_{q1}^2 \approx 2.90\sigma_e^2$$

$$\sigma_{q2}^2 \approx 3.16\sigma_e^2$$

$$\frac{\sigma_{q2}^2}{\sigma_{q1}^2} \approx 1.09$$

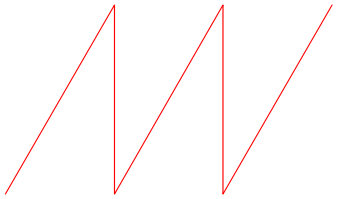
- Thus, the second realization leads to 9% more noise power than the first.

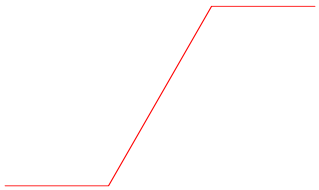
Overflow

- Wrap Around
  - Ex.

  7 (0111) + 1 (0001) = -8 (1000)
- Saturation
  - Ex.

  5 (0101) + 4 (0100) = 7 (0111)





Scaling to Prevent Overflow

- Pessimistic Scaling
- Narrowband Scaling

$$A_x < \frac{1}{\sum_{m=-\infty}^{\infty} |h_k(m)|}$$

$$A_x < \frac{1}{\max_{0 \le \omega \le 2\pi} |H_k(\omega)|}$$

Practical Round-Off Effects on Digital Filters

- Coefficient Quantization
  - Frequency Response Characteristics
- Internal Wordlength Quantization
  - Dynamic Range
  - Signal-to-Noise Ratio
  - Limit Cycles

Coefficient Quantization

- Coefficient quantization alters the values of the coefficients => changes your frequency response
- A filter designed in floating point arithmetic to meet certain specs may not meet those specifications after coefficient quantization

Coefficient Quantization Example

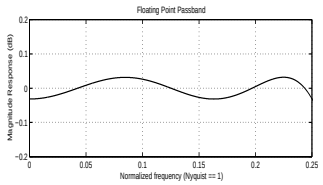
- We used the Parks-McClellan optimal FIR filter algorithm to design a 21-tap filter
- Designed the filter in floating point format
- Took each of the coefficients and rounded to the nearest 8-bit two's complement number
- Ex:

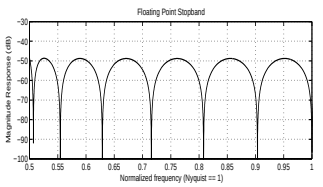
  0.2011929 => .2031250

  (00011010 two's complement)
- Lesson: Over-design the filter and/or use an optimization algorithm to meet the spec

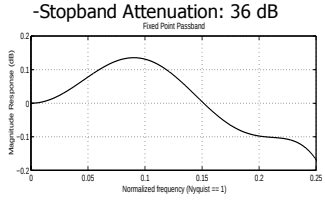
Frequency Response

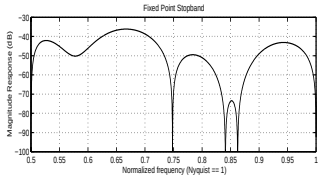
Floating Point
-Passband Ripple: .003 dB
-Stopband Attenuation: 48 dB





Fixed Point
-Passband Ripple: .14 dB
-Stopband Attenuation: 36 dB

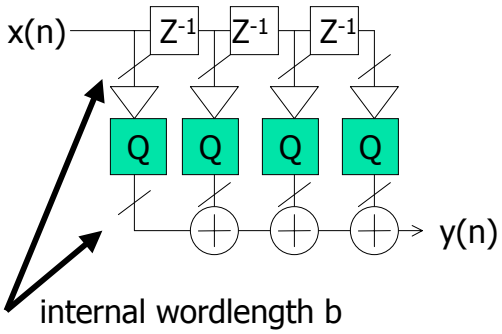




Coefficient Quantization => Frequency Response Degradation

Internal Wordlength Quantization

- Quantization of internal wordlength leads to finite-wordlength effects



1) Dynamic Range
2) Signal-to-Noise Ratio
3) Limit Cycles

Dynamic Range Constraints

- Dynamic Range is defined as:

$$20 \log_{10} \frac{\text{| range of representable numbers |}}{\text{| smallest non-zero representable number |}}$$

- The larger the dynamic range specification, the larger internal wordlength b required
  - Ex. b-bit number ($x_0.\ x_1\ x_2\ x_3\ x_4 \ldots x_{b-1}$)
  - DR: $20 \log |2 - 2^{-(b-1)}| \ / \ | 2^{-(b-1)} |$
    $= 20 \log (2^{-b} - 1)$
    ~ 6 dB / bit

Dynamic Range (cont'd)

- b = 8 => DR ~ 48 dB
- b = 12 => DR ~ 72 dB
- b = 16 => DR ~ 96 dB

Nowadays, most hi-fi audio systems have a DR in the range of 80-100 dB

Signal-to-Noise Ratio

- The signal-to-noise ratio is defined as:

  SNR $= 10 \log_{10}$ signal power / noise power
  $= 10 \log_{10} \sigma_x^2 / \sigma_e^2$

- For our system (wordlength b):

  SNR $= 10 \log_{10} \sigma_x^2 / (2^{-(b-1)}/12)$
  $= 10 \log_{10} \sigma_x^2 + 6.02b + 4.77$

- Ex. $x(n) = .75 \sin(\omega n)$ => SNR = 6.02b - .739
  b = 8 => SNR = 47 dB

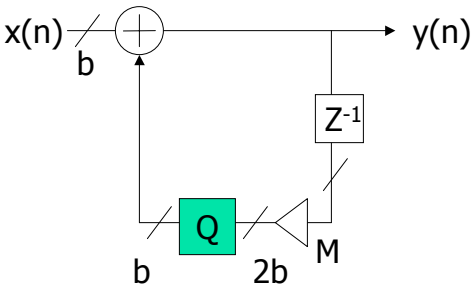
Signal-to-Noise Ratio (cont'd)

- Thus, SNR increases by ~ 6 dB / bit
- To maximize SNR, you want to scale the signal x(n) as large as possible (to increase σ_x^2)
- However, there is a tradeoff—you need to keep all internal signals small enough to prevent overflow / saturation (use normalization and scaling techniques)

Limit Cycles

- Limit cycles occur when the output of a digital filter does not decay to zero when the input goes to zero
- Limit cycles occur only in IIR filters and never occur in FIR filters
- They are caused by quantizing the data after a feedback multiplier in a recursive loop

Limit-Cycle Example



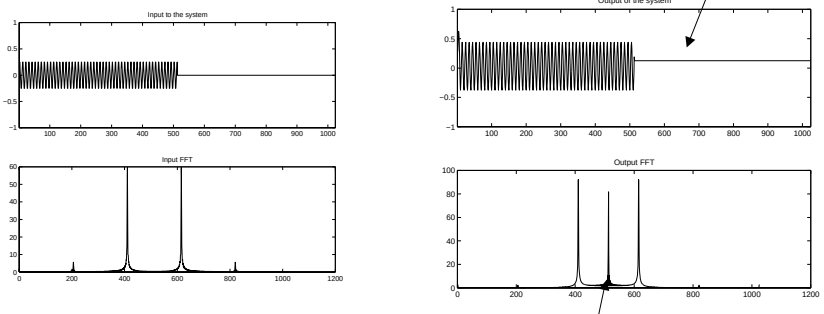
- let M = -.96, y(-1) = 14, x(n) = 0 and rounding to the nearest integer

n	-.96 * y(n-1)	y(n)
0	-13.44	-13
1	12.48	12
2	-11.52	-12
3	11.52	12

=> Limit cycle puts energy at Fs/2 which is detrimental

Limit Cycles in Matlab

Limit cycle of .125 …



INPUT

OUTPUT

…causes a spike at DC

Eliminating Limit Cycles

- Use magnitude truncation (which always decreases the energy of the signal)
- Use a filter for which a Lyapunov function exists
- Use controlled rounding
- Use novel filter structures designed to eliminate limit cycles

References

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