Appendix 1: Proof of NCA

Definition. Connectivity pattern**:** Given a matrix A, and a set $Z_o \subset Z^2$ we say that A is characterized by the connectivity pattern imposed by Z_0 if and only if: $a_{ij} \equiv o \quad for$

$i, j \in Z_o$

Theorem. NCA: Given a matrix E (N x M) if the following conditions are satisfied:

1. There exists a matrix A (N x L) with a connectivity pattern defined by Z_0 and a matrix P (L x M), such that

$$E = A\,P. \qquad \textbf{[A1]}$$

2. A has full column rank and the reduced matrix Ar_j (see definition in text) has rank L-1 for all j.
3. P has full row rank.

Then, for any matrices $\overline{A}$ (N x L), $\overline{P}$ (L x M), such that $\overline{A}$ has full column rank and has the same connectivity pattern Z_0 as A, and $E = \overline{A}\overline{P}$ there exists a diagonal nonsingular matrix X (L x L), such that

$$\overline{A} = AX^{-1} \qquad \textbf{[A2]}$$

$$\overline{P} = XP \qquad \textbf{[A3]}$$

Thus, the decomposition of E into A and P is unique up to a scaling matrix X.

Proof. Given $\overline{A}$, $\overline{P}$ such that

$$E = AP = \overline{A}\overline{P} \qquad \textbf{[A4]}$$

Because $\overline{A}$ has full column rank we can write:

$$\overline{A}^T AP = \overline{A}^T \overline{A}\overline{P} \qquad \textbf{[A5]}$$

$$\overline{P} = \left(\overline{A}^T \overline{A}\right)^{-1} \overline{A}^T AP = XP \qquad \textbf{[A6]}$$

From **A4** we obtain:

$$AP = \overline{A}XP$$

$$\left(A - \bar{A}X\right)P = 0 \qquad \text{[A7]}$$

Because P has full row rank, the above equation implies that

$$A = \bar{A}X \qquad \text{[A8]}$$

We can write **A8** as:

$$a_{ij} = \sum_{l=1}^{L} \bar{a}_{il} x_{lj} \qquad \text{[A9]}$$

In particular, the connectivity pattern Z_0 imposes the following constraints:

$$a_{ij} = \sum_{l=1}^{L} \bar{a}_{il} x_{lj} \equiv 0, \qquad for \quad i, j \in Z_0 \qquad \text{[A10]}$$

In the set of equations (**A10**) the diagonal terms of X do not appear because, if $a_{ij} = 0$ we necessarily have that:

$$\bar{a}_{ij} x_{jj} = 0 x_{jj} \qquad \text{[A11]}$$

Thus, Eq. **A10** can be written by the following matrix form:

$$\overline{A_u} X_r = 0 \qquad \text{[A12]}$$

where $\overline{A_u}$ is the derived from $\bar{A}$ as defined as

$$\bar{A}_u \equiv \begin{bmatrix} \bar{A}r_1 & 0 & & 0 \\ 0 & \bar{A}r_2 & & \\ : & & : & 0 \\ 0 & & & \bar{A}r_L \end{bmatrix} \qquad \text{[A13]}$$

and X_r is the column vector of dimension [L(L-1) x 1] obtained by stacking each column of X after deleting the diagonal elements. Because A and $\bar{A}$ have the same connectivity pattern, the rank of $\overline{A_u}$ is the same as the rank of A_u, which is equal to L(L-1). Note that there are [L(L-1)] unknown variables in X_r. Because $\overline{A_u}$ has rank L(L-1), the unique solution of the linear system of equations defined by **A12** is X_r =0. Thus,

$$x_{ij} = 0 \qquad for \quad i \neq j$$

$$x_{ij} = arbitrary \qquad for \quad i = j$$

Therefore, X is a diagonal matrix. Because A is full column rank, a zero diagonal entry would violate **A8**. This completes the proof.