



# Bisimilar control affine systems<sup>☆</sup>

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## Abstract

The notion of bisimulation plays a very important role in theoretical computer science where it provides several notions of equivalence between models of computation. These equivalences are in turn used to simplify verification and synthesis for these models as well as to enable compositional reasoning. In systems theory, a similar notion is also of interest in order to develop modular verification and design tools for purely continuous or hybrid control systems. In this paper, we introduce two notions of bisimulation for nonlinear systems. We present differential geometric characterizations of these notions and show that bisimilar systems of different dimensions are obtained by factoring out certain invariant distributions. Furthermore, we also show that all bisimilar systems of different dimension are of this form.

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## 1. Introduction

In theoretical computer science, the notion of bisimulation inspired the definition of various notions of equivalence between models of computation. Each of these equivalences identifies classes of systems with similar properties, so that proving a property for a certain system can be done on a smaller equivalent system, thereby simplifying the proof process.

Similar notions are also important in the context of hybrid systems, where the inherent complexity of the hybrid model renders its analysis or design very difficult. Motivated by this, we were naturally led to understand the continuous counterpart of this notion. Previous steps towards this objective have been given in [16], where linear control systems are embedded in the class of transition systems for which the notion of bisimulation was originally introduced by Park [20] and also Milner [12]. It is shown in [16] that different embeddings give rise to semantically different notions of bisimulation being characterized by different conditions. For nonlinear systems no such attempt has appeared in the literature so far, except for [6] where the notion of bisimulation is presented in a sufficiently abstract categorical context to unify discrete and continuous interpretations. Compared to the work in [6], this paper seeks not to unify, but to

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characterize the notion by easily checkable (algebraic) conditions.

A characterization of bisimulation for nonlinear systems is important for several reasons that go beyond its application in hybrid systems. In the series of papers [18,19,21], a methodology has been introduced to compute abstractions of linear and nonlinear control systems. These abstractions are clearly important for verification problems, but also for hierarchical synthesis. For example, in [17] hierarchical stabilization of linear systems is discussed in the framework of abstractions while in [22] abstractions are used to hierarchically design trajectories for classes of nonlinear systems. The ability to perform hierarchical synthesis depends on finding low-level trajectories that implement or refine trajectories of the abstracted model. A sufficient condition is given by bisimilarity, and this fact constitutes another reason to provide algebraic tests for its characterization.

The notion of bisimulation is also very interesting from a system theoretic point of view as it provides an equivalence relation on the class of control systems. This can be regarded as another tool in the quest of classifying nonlinear control systems. Furthermore, this equivalence relation has the important property of rendering as equivalent, control systems of possibly different dimensions. This contrasts with other known equivalences such as diffeomorphisms [11], or feedback transformations [3,8,10]. Furthermore, the notion of bisimulation also has interesting connections with other well-known notions in systems theory such as controlled invariance [7,9,13] and symmetries for nonlinear control systems [5,14].

In this paper, we introduce two local notions of bisimulation for nonlinear control system systems based on the original definition in [12]. We then focus on control affine systems and relations between them defined by submersions, providing algebraic characterizations for these notions. These characterizations turn out to be related with the notion of  $\Phi$ -related control systems introduced in [18]. We then show that by factoring out certain invariant distributions one obtains bisimilar systems and that all bisimilar systems are obtained in this way. The distinguishing power of the two introduced notions is also discussed by showing that, locally, they are equivalent up to a feedback transformation. This is achieved by relating the introduced notions of bisimulation with controlled invariance.

## 2. Geometrical preliminaries

Let  $M$  be a differentiable manifold and  $T_x M$  its tangent space at  $x \in M$ . In this paper, we will consider that all the manifolds are smooth, that is  $C^\infty$ , and that all related mathematical objects are also smooth. The tangent bundle of  $M$  is denoted by  $TM = \bigcup_{x \in M} T_x M$  and  $\pi_M$  is the canonical projection map  $\pi_M : TM \rightarrow M$  taking a tangent vector  $X(x) \in T_x M \subset TM$  to the base point  $x \in M$ . Given manifolds  $M$  and  $N$  and given a map  $\phi : M \rightarrow N$ , we denote by  $T_x \phi : T_x M \rightarrow T_{\phi(x)} N$  the induced tangent map which maps tangent vectors  $X$  at  $T_x M$  to tangent vectors  $T_x \phi \cdot X$  at  $T_{\phi(x)} N$ . If  $\phi$  is such that  $T_x \phi$  is surjective at  $x \in M$  then we say that  $\phi$  is a submersion at  $x$ . When  $\phi$  is a submersion at every  $x \in M$  we simply say that it is a submersion. If furthermore, the submanifolds  $\phi^{-1}(y) = \{x \in M : \phi(x) = y\} \subset M$  are connected, we say that  $\phi$  has connected fibers. When  $\phi$  has an inverse which is also smooth we call  $\phi$  a diffeomorphism. Given a map  $f : M \rightarrow N$  and a set  $L \subseteq M$  we employ the notation  $f|_L$  to denote the restriction of  $f$  to  $L$ .

A fiber bundle is a tuple  $(B, M, \pi_B, \mathcal{F}, \{O_i\}_{i \in I})$ , where  $B$ ,  $M$  and  $\mathcal{F}$  are manifolds called the *total space*, the *base space* and *standard fiber* respectively. The map  $\pi_B : B \rightarrow M$  is a surjective submersion and  $\{O_i\}_{i \in I}$  is an open cover of  $M$  such that for every  $i \in I$  there exists a diffeomorphism  $\Psi_i : \pi_B^{-1}(O_i) \rightarrow O_i \times \mathcal{F}$  satisfying  $\pi_{O_i} \circ \Psi_i = \pi_B$ , where  $\pi_{O_i}$  is the projection from  $O_i \times \mathcal{F}$  to  $O_i$ . The submanifold  $\pi_B^{-1}(x)$  is called the fiber at  $x \in M$  and is diffeomorphic to  $\mathcal{F}$ . Since a fiber bundle is locally a product, we can always find local coordinates, which we shall call trivializing coordinates, of the form  $(x, b)$ , where  $x$  are coordinates for the base space and  $b$  are coordinates for the local representative of the standard fiber.

**Definition 2.1** (Control system). A control system  $\Sigma_M = (M \times V, F_M)$  consists of smooth manifolds  $M$  called the state space,  $V$  called the input space and a smooth map  $F_M : M \times V \rightarrow TM$  that assigns a vector  $X \in T_x M$  to each pair  $(x, v) \in M \times V$ .

Although the previous definition captures the usual notion of control systems, in certain situations it is more natural to model available inputs as being dependent on the state. This dependence can be captured by replacing the product  $M \times V$  by a fiber bundle. In this

situation, we define a control system as  $\Sigma_M = (U_M, F_M)$  consisting of a fiber bundle  $\pi_{U_M} : U_M \rightarrow M$  called the control bundle and a map  $F_M : U_M \rightarrow TM$  making the following diagram commutative:

$$\begin{array}{ccc}
 U_M & \xrightarrow{F_M} & TM \\
 \pi_{U_M} \downarrow & & \swarrow \pi_M \\
 M & & 
 \end{array} \quad (2.1)$$

that is,  $\pi_M \circ F_M = \pi_{U_M}$ , where  $\pi_M : TM \rightarrow M$  is the tangent bundle projection.

In trivializing coordinates  $(x, v)$ , the map  $F_M : U_M \rightarrow TM$  reduces to the familiar expression  $\dot{x} = f(x, v)$  with  $v \in \pi_{U_M}^{-1}(x)$ . In the special case where the control bundle is trivial, that is,  $U_M = M \times V$  we recover Definition 2.1.

Having defined control systems the concept of trajectories or solutions of a control system is naturally expressed as follows:

**Definition 2.2** (Trajectories of control systems). A smooth curve  $c : I \rightarrow M$ ,  $0 \in I = ]\tau_1, \tau_2[ \subseteq \mathbb{R}$  is called a trajectory of control system  $\Sigma_M = (U_M, F_M)$ , if there exists a curve<sup>1</sup>  $c^V : I \rightarrow V$  such that

$$\frac{d}{dt} c(t) = F(c(t), c^V(t)) \quad \forall t \in I. \quad (2.2)$$

When we need to consider a fiber bundle  $U_M$  instead of the product  $M \times V$ , we replace  $c^V$  by  $c^U : I \rightarrow U_M$  and require commutativity of the following diagrams:

$$\begin{array}{ccc}
 & U_M & \\
 c^U \nearrow & \downarrow \pi_{U_M} & \\
 I & \xrightarrow{c} & M
 \end{array}
 \quad
 \begin{array}{ccc}
 & U_M & \\
 c^U \nearrow & \downarrow F_M & \\
 I & \xrightarrow{Tc} & TM
 \end{array} \quad (2.3)$$

where we have identified  $I$  with  $TI$ . These commutative diagrams are equivalent to the equalities  $\pi_{U_M} \circ$

$c^U = c$  and  $Tc = F_M(c^U)$  which express in the language of fiber bundles equality (2.2). A (left) action of a Lie group  $G$  on a manifold  $M$  is a map  $\theta : G \times M \rightarrow M$  such that  $\theta(e, x) = x$  and  $\theta(g_1 g_2, x) = \theta(g_1, \theta(g_2, x))$ , where  $e$  is the group identity and  $g_1, g_2 \in G$  (see [2]). Given a point  $x \in M$  we can define the orbit of  $\theta$  through  $x$  to be the following subset of  $M$ :

$$\{x' \in M : x' = \theta(x, g) \text{ for some } g \in G\}$$

An action is said to be free when  $\theta(g, x) = x \Rightarrow g = e$  and proper when the map  $\bar{\theta}(g, x) = (x, \theta(g, x))$  is proper. When  $\theta : G \times M \rightarrow M$  is a free and proper action, then  $M/G$ , the space of orbits of  $\theta$  is a smooth manifold and the projection  $\pi : M \rightarrow M/G$  taking each point in  $M$  to its orbit is a smooth surjective submersion [2]. Furthermore by fixing any  $g \in G$  we obtain  $\theta(g, -) = \theta_g : M \rightarrow M$  a diffeomorphism of  $M$ .

### 3. Bisimulation relations

The notion of bisimulation is originally credited to [20,12], and since then many authors have made important contributions to its development. In the context of continuous control systems, bisimulations have been discussed for the first time in [16] for linear control systems. We start by recalling the concept of transition system and bisimulation as presented in [12].

**Definition 3.1** (Transition systems). A transition system is a tuple  $T = (S, L, \rightarrow)$  consisting of

- a set of states  $S$ ,
- a set of labels  $L$ ,
- a transition relation  $\rightarrow \subseteq S \times L \times S$ .

We use the graphical representation  $q_1 \xrightarrow{l} q_2$  to denote  $(q_1, l, q_2) \in \rightarrow$ . Intuitively, one can regard a transition system as a nondeterministic control system. Given a state  $s \in S$ , one interprets the set of labels  $l \in L$  such that  $s \xrightarrow{l} s'$  for some  $s' \in S$ , as the set of control inputs available at state  $s$ . Choosing one of those inputs will make the transition system evolve to the new state or states  $s'$  satisfying  $(s, l, s') \in \rightarrow$ . The nondeterminism is captured by the fact that different triples  $(s, l, s')$  and  $(s, l, s'')$  may belong to  $\rightarrow$ . This is the analogy that we shall make use to provide a

<sup>1</sup> The curve  $c^V : I \rightarrow V$  is assumed to belong to a class of functions for which  $c(t)$  is uniquely defined for sufficiently small  $I$ .

continuous counterpart of the notion of bisimulation that we now recall.

**Definition 3.2** (Bisimulation relation). Let  $T_1 = (S_1, L, \rightarrow_1)$  and  $T_2 = (S_2, L, \rightarrow_2)$  be transition systems. A relation  $R \subseteq S_1 \times S_2$  is said to be a bisimulation relation between  $T_1$  and  $T_2$  if  $(s_1, s_2) \in R$  implies for all  $l \in L$ :

- if  $s_1 \xrightarrow{l} s'_1$  then there exists a  $s'_2 \in S_2$  such that  $s_2 \xrightarrow{l} s'_2$  and  $(s'_1, s'_2) \in R$ ,
- if  $s_2 \xrightarrow{l} s'_2$  then there exists a  $s'_1 \in S_1$  such that  $s_1 \xrightarrow{l} s'_1$  and  $(s'_1, s'_2) \in R$ .

To import this notion into the continuous context we face the difficulty of not being able to express the continuous dynamics in terms of the “atomic” jumps  $s_1 \xrightarrow{l} s'_1$ . We shall, therefore, replace the atomic jumps with *any* evolution, that is, we will ask a control system to match the evolution of another control system for *every* instant of time. Furthermore, as trajectories must be obtained by using the same input trajectory, the input space cannot depend on the state space. We shall, therefore assume, that the control bundle is a product  $U_M = M \times V$ , being  $V$  the input space.

Naturally, this leads to the following local notion of bisimulation for control systems:

**Definition 3.3** (Local bisimulation of control systems). Let  $\Sigma_M = (U_M, F_M)$  and  $\Sigma_N = (U_N, F_N)$  be control systems such that  $U_M = M \times V$  and  $U_N = N \times V$ . A relation  $R \subseteq M \times N$  is said to be a local bisimulation relation between  $\Sigma_M$  and  $\Sigma_N$  if  $(x, y) \in R$  implies

1. for any state trajectory  $c_M : I \rightarrow M$  of  $\Sigma_M$  with  $c_M(0) = x$  determined by input trajectory  $c^V : I \rightarrow V$  there exists a set  $J \subseteq I$ ,  $0 \in J$  and a state trajectory  $c_N : J \rightarrow N$  of  $\Sigma_N$  with  $c_N(0) = y$  determined by input trajectory  $c^V|_J : J \rightarrow V$  such that  $(c_M(t), c_N(t)) \in R$  for every  $t \in J$ .
2. for any state trajectory  $c_N : I \rightarrow N$  of  $\Sigma_N$  with  $c_N(0) = y$  determined by input trajectory  $c^V : I \rightarrow V$  there exists a set  $J \subseteq I$ ,  $0 \in J$  and a state trajectory  $c_M : J \rightarrow M$  of  $\Sigma_M$  with  $c_M(0) = x$  determined by input trajectory  $c^V|_J : J \rightarrow V$  such that  $(c_M(t), c_N(t)) \in R$  for every  $t \in J$ .

As we shall see soon, this notion of bisimulation will be quite restrictive. This will motivate more relaxed notions of bisimulation, and in particular, we shall consider an input abstract version. This new notion relaxes the requirement that both systems have the same input trajectories and furthermore can be easily expressed without the assumption of trivial control bundles.

**Definition 3.4** (Input abstract local bisimulation of control systems). Let  $\Sigma_M = (U_M, F_M)$  and  $\Sigma_N = (U_N, F_N)$  be control systems. A relation  $R \subseteq M \times N$  is said to be an input abstract local bisimulation relation between  $\Sigma_M$  and  $\Sigma_N$  if  $(x, y) \in R$  implies:

1. for any state trajectory  $c_M : I \rightarrow M$  of  $\Sigma_M$  with  $c_M(0) = x$  there exists a set  $J \subseteq I$ ,  $0 \in J$  and a state trajectory  $c_N : J \rightarrow N$  of  $\Sigma_N$  with  $c_N(0) = y$  such that  $(c_M(t), c_N(t)) \in R$  for every  $t \in J$ .
2. for any state trajectory  $c_N : I \rightarrow N$  of  $\Sigma_N$  with  $c_N(0) = y$  there exists a set  $J \subseteq I$ ,  $0 \in J$  and a state trajectory  $c_M : J \rightarrow M$  of  $\Sigma_M$  with  $c_M(0) = x$  such that  $(c_M(t), c_N(t)) \in R$  for every  $t \in J$ .

We shall say that two control systems are (input abstract) locally bisimilar when there exists a (input abstract) local bisimulation between them. A global notion of bisimulation can also be defined by requiring  $J = I$ . This implies that trajectories  $c_M$  and  $c_N$  have necessarily the same time domain in addition to satisfy the conditions specified in Definition 3.4.

The above introduced notions of bisimulation are also important from a systems perspective since they allow a new type of classification of control systems. Indeed, it is not difficult to show that the notion of bisimulation defines an equivalence relation in the class of control systems.

**Proposition 3.5.** *Local bisimulation and input abstract local bisimulation are equivalence relations on the class of control systems.*

These equivalence relations have the important characteristic of rendering equivalent systems of possibly different dimensions. It therefore makes sense to consider as representative of each equivalence class, the system of smallest dimension, leading to notions of minimality.

#### 4. A characterization of bisimulation

For presentation purposes, all proofs of the main results in this section can be found at Appendix A. We start by making some assumptions that will allow to provide simple characterizations of locally bisimilar control systems:

1. The control systems are assumed to be control affine, that is, there are local (trivializing) coordinates  $(x, v)$  where the system map  $F_M$  takes the form  $F_M(x, v) = f_M(x) + \sum_{i=1}^k g_M^i(x)v_i$ . In this case, we shall denote by  $\mathcal{D}_M$  the affine distribution defined by  $\mathcal{D}_M(x) = f_M(x) + \text{span}\{g_M^1(x), g_M^2(x), \dots, g_M^k(x)\}$ .
2. The relation  $R \subseteq M \times N$  is induced by a smooth map  $r: M \rightarrow N$ , that is  $(x, y) \in R$  iff  $r(x) = y$  where  $r$  is a submersion, that is,  $T_x r$  is surjective at every  $x \in M$ .

The first assumption is not very restrictive since the results obtained for affine control systems can be lifted to fully nonlinear control systems by making use of the notion of extended control system [15]. The second assumption is more restrictive but its justified by the fact that in [19] an algorithm has been presented for the computation of quotients of control systems based on such a quotient map. It is therefore of extreme importance to be able to determine when such quotients are in fact locally bisimilar to the original one with respect to the quotient map.

A characterization of local bisimulation can now be given as follows:

**Theorem 4.1.** *Let  $\Sigma_M = (U_M, F_M)$  and  $\Sigma_N = (U_N, F_N)$  be two control affine systems such that  $U_M = M \times V$ ,  $U_N = N \times V$ , and let  $r: M \rightarrow N$  be a submersion. Then,  $\Sigma_M$  is locally bisimilar to  $\Sigma_N$  via  $r$  iff for every  $x \in M$ :*

$$T_x r \cdot F_M(x, v) = F_N(r(x), v) \quad \forall v \in V.$$

The above characterization shows how restrictive the notion of local bisimulation is since vector fields  $f_M$  and  $g_M^i$  have to be  $r$ -related to vector fields  $f_N$  and  $g_N^i$ , respectively, that is

$$T_x r \cdot f_M(x) = f_N \circ r(x), \quad T_x r \cdot g_M^i(x) = g_N^i \circ r(x) \\ \forall x \in M, i \in \{1, 2, \dots, k\}.$$

Relaxing this condition was the motivating factor behind the notion of input abstract local bisimulation whose characterization is now presented.

**Theorem 4.2.** *Let  $\Sigma_M$  and  $\Sigma_N$  be two control affine systems and  $r: M \rightarrow N$  a submersion. Then,  $\Sigma_M$  is input abstract locally bisimilar to  $\Sigma_N$  via  $r$  iff for every  $x \in M$ :*

$$T_x r(\mathcal{D}_M(x)) = \mathcal{D}_N \circ r(x).$$

We note that since the relation  $R \subseteq M \times N$  is defined by a map  $r: M \rightarrow N$ , the first condition on the definition of input abstract local bisimulation can be rephrased as: *for every trajectory  $c_M$  of  $\Sigma_M$ ,  $r(c_M)$  must be a trajectory of  $\Sigma_N$* . This was the basic definition of abstraction introduced in [18], so that it is natural that the characterization of (input abstract) local bisimulation is a stronger version of the concept of  $r$ -related control systems, which is the algebraic characterization of abstractions. It is also interesting to note that the characterization of input abstract local bisimilarity, given in Theorem 4.2 distinguishes these systems from general abstractions at the level of the structure of the control bundle as discussed in [21]. In fact, when a control system is locally bisimilar to its abstraction, no new inputs will appear on the abstraction, a phenomenon that does not occur for general abstracted systems [18]. Several obstructions exist to the extension of local bisimulation to a global version. Possible finite explosion times may prevent existence trajectory  $c_M(t)$  for every  $t \in I$ . Since  $I$  may be arbitrarily long one needs to assume that solutions of  $\Sigma_M$  are defined for all positive times which is difficult to characterize for general control affine systems. In addition, existence of “holes” in  $M$  may also prevent trajectory  $c_M$  from satisfying  $r(c_M(t)) = c_N(t)$  for every  $t \in I$ . Removing this obstruction requires different topological conditions on the manifold  $M$  and/or the map  $r$  that we shall not pursue in this paper.

We now clarify how different can (input abstract) locally bisimilar control systems be if they have different dimensions. For this we will assume that  $\dim(M) > \dim(N)$ , and recall the notions of invariant and controlled invariant distributions:

**Definition 4.3** (Invariant and controlled invariant distributions). Let  $\Sigma_M$  be a control affine system and let  $\mathcal{E}$  be a regular and involutive distribution on  $M$ .

Distribution  $\mathcal{E}$  is said to be invariant for  $\Sigma_M$  when

$$[\mathcal{D}_M, \mathcal{E}] \subseteq \mathcal{E}.$$

Distribution  $\mathcal{E}$  is said locally controlled invariant if there exist a local feedback transformation around each  $x \in M$ , such that  $\mathcal{E}$  is invariant for the feedback transformed system.

Locally controlled invariant distributions also admit the following characterization:

**Theorem 4.4** (Adapted from Grasse [4]). *Let  $\Sigma_M$  be a control affine system and  $\mathcal{E}$  a regular and involutive distribution on  $M$ . The distribution  $\mathcal{E}$  is locally controlled invariant for  $\Sigma_M$  iff:*

$$[\mathcal{D}_M, \mathcal{E}] \subseteq \mathcal{E} + \Delta_M,$$

where  $\Delta_M = \text{span}\{g_M^1, g_M^2, \dots, g_M^k\}$ .

Equipped with the notions of invariant and controlled invariant distributions we can now understand the relationship between (input abstract) locally bisimilar systems of different dimensions.

**Theorem 4.5.** *Let  $\Sigma_M = (U_M, F_M)$  and  $\Sigma_N = (U_N, F_N)$  be control affine systems such that  $U_M = M \times V$ ,  $U_N = N \times V$ ,  $\dim(M) > \dim(N)$  and let  $r: M \rightarrow N$  be a surjective submersion with connected fibers. Then,  $\Sigma_M$  is locally bisimilar to  $\Sigma_N$  via  $r$  iff  $\ker(\text{Tr})$  is invariant for  $\Sigma_M$  and  $\Sigma_N$  is defined by the affine distribution:*

$$\mathcal{D}_N \circ r(x) = \bigcup_{x' \in r^{-1} \circ r(x)} T_{x'} r(\mathcal{D}_M(x')).$$

For input abstract local bisimulation we recover local controlled invariance:

**Theorem 4.6.** *Let  $\Sigma_M$  and  $\Sigma_N$  be control affine systems such that  $\dim(M) > \dim(N)$  and let  $r: M \rightarrow N$  be a surjective submersion with connected fibers. Then,  $\Sigma_M$  is input abstract locally bisimilar to  $\Sigma_N$  via  $r$  iff  $\ker(\text{Tr})$  is locally controlled invariant for  $\Sigma_M$  and  $\Sigma_N$  is defined by the affine distribution*

$$\mathcal{D}_N \circ r(x) = \bigcup_{x' \in r^{-1} \circ r(x)} T_{x'} r(\mathcal{D}_M(x')).$$

The previous characterization of (input abstract) local bisimulation shows that although dimension is not constant on the equivalence classes of this equivalence, two control systems  $\Sigma_M$  and  $\Sigma_N$  of different dimensions are in the same equivalence class if and only if it is possible to obtain one from the other by factoring out (controlled) invariant distributions. As an immediate corollary of the previous results we have that factoring out symmetries also produces locally bisimilar systems.

**Corollary 4.7.** *Let  $\Sigma_M$  be an affine control system and  $\theta: G \times M \rightarrow M$  be a free and proper action of a Lie group  $G$  such that for every  $X \in \mathcal{D}_M$  we have  $\theta_g^* X = X$  for every  $g \in G$ . Then  $\Sigma_M$  is locally bisimilar via  $\pi$  to  $\Sigma_M/G$  defined by*

$$\mathcal{D}_N \circ \pi(x) = \bigcup_{x' \in \pi^{-1} \circ \pi(x)} T_{x'} \pi(\mathcal{D}_M(x')).$$

For input abstract locally bisimilar systems it is still the case that factoring out symmetries implies input abstract local bisimilarity but we allow a larger class of symmetries (partial symmetries in the context of [14]).

**Corollary 4.8.** *Let  $\Sigma_M$  be an affine control system and  $\theta: G \times M \rightarrow M$  be a free and proper action of a Lie group  $G$  such that for every  $X \in \mathcal{D}_M$  we have  $\theta_g^* X \in \mathcal{D}_M$  for every  $g \in G$ . Then  $\Sigma_M$  is input abstract locally bisimilar via  $\pi$  to  $\Sigma_M/G$  defined by*

$$\mathcal{D}_N \circ \pi(x) = \bigcup_{x' \in \pi^{-1} \circ \pi(x)} T_{x'} \pi(\mathcal{D}_M(x')).$$

We have not explicitly discussed the quotient system  $\Sigma_M/G$  control bundle geometry. We defer the reader to Ref. [21], where these issues are addressed for general quotients and to [5,14] where symmetries are modeled by group actions acting on the control bundle as well.

It is clear that the equivalence relation defined by local bisimulation is strictly finer (in the sense that it distinguishes more control systems) than the equivalence relation defined by input abstract local bisimulation. However, locally, every two input abstract locally bisimilar control systems are locally bisimilar up to a feedback transformation. This fact is a

simple consequence of Theorem 4.4. This proves the following result:

**Proposition 4.9.** *Let  $\Sigma_M$  and  $\Sigma_N$  be affine control systems input abstract locally bisimilar via  $r : M \rightarrow N$ . Then, for every  $x \in M$  there exists a local feedback transformation for  $\Sigma_M$  rendering it locally bisimilar to  $\Sigma_N$  via  $r$ .*

Note that the previous result does not assert that  $\Sigma_M$  is locally bisimilar to  $\Sigma_N$  since the feedback transformation is not a local bisimulation relation.

## 5. Conclusions

Motivated by notions of equivalence in computer science and hybrid systems, we have introduced the notion of (input abstract) local bisimulation for nonlinear control systems. A differential geometric characterization was given for the introduced notions which can be seen as a strengthening of the notion of  $\Phi$ -related control systems of [18]. Although this notion constitutes an equivalence relation on the class of control systems which does not require the dimension of the systems to be an invariant, it was shown that locally bisimilar systems of different dimensions must be related in a special way. In fact, one of the systems must be obtained from the other by factoring out (controlled) invariant distributions.

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## Appendix A. Proofs

**Proof of Theorem 4.2 (Necessity).** Since the relation  $R$  is defined by a map  $r : M \rightarrow N$ , the first condition of local bisimilarity implies that for every trajectory  $c_M(t)$  of  $\Sigma_M$ ,  $r(c_M(t))$  must be a trajectory of  $\Sigma_N$  for  $t \in J$ . This, in turn, implies

$$T_x r(\mathcal{D}_M(x)) \subseteq \mathcal{D}_N \circ r(x) \quad (\text{A.1})$$

as was shown in [18]. The second condition for local bisimilarity implies that for every trajectory  $c_N$  of  $\Sigma_N$  such that  $y = c_N(0)$  belongs to the range of  $r$ , there exists a trajectory  $c_M$  of  $\Sigma_M$  for every  $x \in r^{-1}(c_N(0))$  satisfying  $c_M(0) = x$  and  $r(c_M) = c_N$ . Let  $X = d/dt|_{t=0} c_M(t)$  and  $Y = d/dt|_{t=0} c_N(t)$ . The equality  $r(c_M) = c_N$  implies  $T_x r \cdot X = Y$ , and as  $c_N$  is any trajectory originating at  $y$ ,  $Y$  is any vector belonging to  $\mathcal{D}_N(y)$ . As such, we must have

$$\mathcal{D}_N(y) = \mathcal{D}_N \circ r(x) \subseteq T_x r(\mathcal{D}_M(x)) \quad (\text{A.2})$$

for every  $x \in M$ . Combining (A.1) with (A.2) gives

$$T_x r(\mathcal{D}_M(x)) = \mathcal{D}_N \circ r(x) \quad (\text{A.3})$$

for every  $x \in M$ .

(Sufficiency). Condition  $T_x r(\mathcal{D}_M(x)) = \mathcal{D}_N \circ r(x)$  implies  $T_x r(\mathcal{D}_M(x)) \subseteq \mathcal{D}_N \circ r(x)$  from which follows that, for every trajectory  $c_M$  of  $\Sigma_M$ ,  $r(c_M)$  is a trajectory of  $\Sigma_N$ , as proved in [18]. This shows that the first condition of input abstract local bisimulation is satisfied.

The second condition requires more work. Let  $c_N$  be a trajectory of  $\Sigma_N$  such that  $c_N(0)$  belongs to the range of  $r$ . Consider the graph  $\Gamma \subset I \times N$  of  $c_N$  which is a submanifold of  $\bar{N} = I \times N$  as  $c_N$  is a smooth map. On  $\Gamma$  we define the following vector field  $Y : \Gamma \rightarrow T\Gamma$ ,  $Y(t, c_N(t)) = (\partial/\partial t, T_t c_N \cdot \partial/\partial t)$ . Note that the integral curve of this vector field is precisely the graph of  $c_N$ . Consider now the manifold  $\bar{M} = I \times M$  with local coordinates  $(t, x)$ , where  $x$  are coordinates for  $M$  and  $t$  coordinates for  $I \subseteq \mathbb{R}$ . On this manifold, we introduce a new control system defined by the affine distribution  $\overline{\mathcal{D}_M} = \{\bar{X} \in T\bar{M} : \bar{X} = X\partial/\partial x + \partial/\partial t, \text{ for } X \in \mathcal{D}_M\}$ . We will now restrict this control system so as to project on  $Y$ . To do so, we consider the map  $\bar{r} = id_I \times r : \bar{M} \rightarrow \bar{N}$  which being a submersion, since  $r$  is a submersion, allows to define the submanifold  $\Pi = (\bar{r})^{-1}(\Gamma)$  of  $\bar{M}$ . The set  $\Pi$  is a submanifold since  $\bar{r}$  is transversal to the submanifold  $\Gamma$  [1]. We now restrict  $\overline{\mathcal{D}_M}$  to obtain a new affine distribution on  $\Pi$ :

$$\widehat{\mathcal{D}_M}(p) = \{\bar{X} \in \overline{\mathcal{D}_M}(p) \cap T_p \Pi :$$

$$T_p \bar{r} \cdot \bar{X}(p) = Y \circ \bar{r}(p)\}$$

<sup>2</sup> We are abusing language and identifying the points  $p \in \Pi$  and  $i(p) = p \in \bar{M}$ , where  $i : \Pi \rightarrow \bar{M}$  is the natural inclusion map.

It is not difficult to see that  $\widehat{\mathcal{D}}_M$  is also an affine distribution and is nontrivial since  $T_x r(\mathcal{D}_M(x)) = \mathcal{D}_N \circ r(x)$ . It then follows by definition of  $\widehat{\mathcal{D}}_M$  that  $T\bar{r}(\widehat{\mathcal{D}}_M) = Y \circ \bar{r}$ . However, as shown in [18], this condition implies that every trajectory of the control system defined by  $\widehat{\mathcal{D}}_M$  is mapped by  $\bar{r}$  to a trajectory of the trivial control system defined by  $Y$ . Furthermore, as<sup>3</sup>  $\widehat{\mathcal{D}}_M \subseteq \overline{\mathcal{D}}_M$  we also know that trajectories of  $\widehat{\mathcal{D}}_M$  are also trajectories of  $\overline{\mathcal{D}}_M$ . Finally, by realizing that trajectories of the control system defined by  $\overline{\mathcal{D}}_M$  are simply the graph of (some) trajectories of  $\Sigma_M$ , we conclude that for every  $x \in r^{-1}(y)$  there exist a trajectory  $c_M$  of  $\Sigma_M$  satisfying  $c_M(0) = x$  and  $r(c_M(t)) = c_N(t)$  for every  $t \in J$  and sufficiently small  $J \subseteq I$ .  $\square$

**Proof of Theorem 4.1 (Necessity).** As locally bisimilar control systems are input abstract locally bisimilar we only have to show the equality  $T_x r \cdot F_M(x, v) = F_N(r(x), v)$ . Let  $c_N$  be an arbitrary trajectory of  $\Sigma_N$  with initial condition  $c_N(0) = y$  belonging to the range of  $r$  and  $c^V$  the corresponding input trajectory. Since  $\Sigma_M$  is locally bisimilar to  $\Sigma_N$ , for any  $x \in M$  such that  $r(x) = y$ , there is a trajectory  $c_M$  of  $\Sigma_M$  satisfying  $r \circ c_M(t) = c_N(t)$  for  $t \in J$  and  $c_M(0) = x$ . Time differentiation at  $t = 0$  of  $r \circ c_M(t) = c_N(t)$  gives

$$\begin{aligned} T_x r \cdot F_M(x, c^V(0)) &= F_N(y, c^V(0)) \\ &= F_N(r(x), c^V(0)). \end{aligned} \quad (\text{A.4})$$

As  $y$  is an arbitrary point in the range of  $r$ , the above equality must hold for every  $x \in M$ . Furthermore,  $c_N$  is also an arbitrary trajectory of  $\Sigma_N$  so that  $c^V(0)$  can be any value in  $V$  and this implies that (5.4) also holds for any  $c^V(0) \in V$  which concludes the necessity part of the proof.

(Sufficiency). We need to show that for any trajectory  $c_N$  of  $\Sigma_N$  generated by input trajectory  $c^V$  and such that  $c_N(0)$  belongs to the range of  $r$  there exist trajectories  $c_M$  of  $\Sigma_M$  generated by the same input trajectory  $c^V$  for every  $x \in r^{-1}(c_N(0))$  such that  $c_M(0) = x$  and  $r(c_M) = c_N$ . Existence of  $c_M$  defined by  $c^V$  for  $t \in J \subseteq I$  is an immediate consequence of considering input curves for which solutions are uniquely locally defined. The proof now reduces to

<sup>3</sup> This inclusion is to be understood as  $T_{p_i}(\widehat{\mathcal{D}}_M(p)) \subseteq \overline{\mathcal{D}}_M \circ i(p)$  for the natural inclusion  $i: \Pi \rightarrow \bar{M}$ .

show equality  $r \circ c_M(t) = c_N(t)$ . Since  $c_N$  is a trajectory of  $\Sigma_N$  corresponding to input trajectory  $c^V$ , we have by definition of trajectory

$$\frac{d}{dt} c_N(t) = F_N(c_N(t), c^V(t)). \quad (\text{A.5})$$

Similarly, trajectory  $r \circ c_M(t)$  satisfies

$$\begin{aligned} \frac{d}{dt} r \circ c_M(t) &= T_{c_M(t)} r \cdot F_M(c_M(t), c^V(t)) \\ &= F_N(r(c_M(t)), c^V(t)), \end{aligned}$$

which shows that  $r \circ c_M(t)$  is also a trajectory of  $\Sigma_N$  for the same input trajectory  $c^V$ . Furthermore, as  $c_N(0) = y = r(x) = r(c_M(0))$  we conclude the equality  $c_N(t) = r \circ c_M(t)$  from uniqueness of control systems trajectories. The same argument also shows that for any trajectory  $c_M$  of  $\Sigma_M$  with corresponding input trajectory  $c^V$ , there exists a trajectory  $c_N$  of  $\Sigma_N$  for the same input trajectory  $c^V$  which satisfies  $r \circ c_M(t) = c_N(t)$  for  $t \in J$ .  $\square$

The proof of Theorem 4.6 requires the following lemma:

**Lemma A.1.** *Let  $\phi: M \rightarrow N$  be a surjective submersion with connected fibers and  $\mathcal{D} = X + \Delta$  an affine distribution on  $M$ . Distribution  $\mathcal{D}$  satisfies  $T_x \phi(\mathcal{D}(x)) = T_{x'}(\mathcal{D}(x'))$  for any  $x, x' \in M$  such that  $\phi(x) = \phi(x')$  iff  $[\mathcal{D}, \ker(T\phi)] \subseteq \Delta + \ker(T\phi)$ .*

**Proof.** We reduce the proof of this lemma to the equivalence  $TK_t(\mathcal{E}) = \mathcal{E} \circ K_t$  iff  $[\mathcal{E}, \ker(T\phi)] \subseteq \Delta_{\mathcal{E}}$  shown in [19] for an affine distribution  $\mathcal{E} = Y + \Delta_{\mathcal{E}}$  on  $M$  and the flow  $K_t$  of any vector field  $K \in \ker(T\phi)$ .

Assume that  $T_x \phi(\mathcal{D}(x)) = T_{x'}(\mathcal{D}(x'))$  holds for any  $x, x' \in M$  such that  $\phi(x) = \phi(x')$  and define  $\mathcal{E}$  as  $\mathcal{E} = \mathcal{D} + \ker(T\phi) = X + \Delta_{\mathcal{E}}$ ,  $\Delta_{\mathcal{E}} = \Delta + \ker(T\phi)$ . In particular we will assume that  $x' \in M$  is such that there exists a flow  $K_t$  of a vector field  $K \in \ker(T\phi)$  satisfying  $K_t(x) = x'$ . Note that  $\phi(x) = \phi(x') = \phi \circ K_t(x)$  implies that:

$$T_x \phi(x) = T_{x'} \phi \circ T_x K_t \quad (\text{A.6})$$

and this allows to rewrite  $T_x \phi(\mathcal{D}(x)) = T_{x'} \phi(\mathcal{D}(x'))$  as

$$T_x \phi(\mathcal{D}(x)) = T_{x'} \phi(\mathcal{D}(x'))$$

$$\Leftrightarrow T_x \phi(\mathcal{D}(x) + \ker(T_x \phi)) = T_{x'} \phi(\mathcal{D} \circ K_t(x))$$

$$\Leftrightarrow T_{x'}\phi \circ T_x K_t(\mathcal{D}(x) + \ker(T_x\phi)) = T_{x'}\phi(\mathcal{D} \circ K_t(x))$$

$$\Leftrightarrow T_x K_t(\mathcal{E}) = \mathcal{D} \circ K_t(x) + \ker(T_{x'}\phi) = \mathcal{E} \circ K_t(x).$$

From Proposition 5.1 in [19] now follows:

$$[\mathcal{E}, \ker(T\phi)] \subseteq \Delta_{\mathcal{E}}.$$

This inclusion can be written as

$$\begin{aligned} [\mathcal{D} + \ker(T\phi), \ker(T\phi)] &\subseteq \Delta + \ker(T\phi) \\ \Rightarrow [\mathcal{D}, \ker(T\phi)] + [\ker(T\phi), \ker(T\phi)] & \\ \subseteq \Delta + \ker(T\phi) & \\ \Rightarrow [\mathcal{D}, \ker(T\phi)] &\subseteq \Delta + \ker(T\phi) \\ -[\ker(T\phi), \ker(T\phi)] & \\ \Rightarrow [\mathcal{D}, \ker(T\phi)] &\subseteq \Delta + \ker(T\phi) \\ -\ker(T\phi) = \Delta + \ker(T\phi) & \end{aligned}$$

which shows the desired inclusion. The converse is similarly proved since by connectedness of the fibers of  $\phi$  any two points  $x, x' \in M$  such that  $\phi(x) = \phi(x')$  can be connected by local flows of vector fields in  $\ker(T\phi)$ .  $\square$

**Proof of Theorem 4.6.** Assume that  $\Sigma_M$  and  $\Sigma_N$  are input abstract locally bisimilar via  $r$ . Then by Theorem 4.2 we have

$$T_x r(\mathcal{D}_M) = \mathcal{D}_N \circ r(x) \quad (\text{A.7})$$

for every  $x \in M$ . This means that for every  $x' \in r^{-1} \circ r(x)$  the equality

$$T_x r(\mathcal{D}_M(x)) = T_{x'} r(\mathcal{D}_M(x')) \quad (\text{A.8})$$

holds and by Lemma A.1 we have  $[\mathcal{D}_M, \ker(Tr)] \subseteq \ker(Tr) + \Delta_M$ . Furthermore, by Theorem 4.4 we have controlled invariance since  $r$  being a submersion ensures that  $\ker(Tr)$  is regular and involutive. In addition, we also have

$$\begin{aligned} \bigcup_{x' \in r^{-1} \circ r(x)} T_{x'} r(\mathcal{D}_M(x')) &= T_x r(\mathcal{D}_M(x)) \\ &= \mathcal{D}_N \circ r(x) \end{aligned} \quad (\text{A.9})$$

where the first equality holds by (A.8) and the second equality holds by Theorem 4.2.

Assume now that  $\ker(Tr)$  is controlled invariant for  $\Sigma_M$ , then the equality  $T_x(\mathcal{D}_M(x)) = T_{x'} r(\mathcal{D}_M(x'))$  holds for every  $x' \in r^{-1} \circ r(x)$  in virtue of Theorem 4.4 and Lemma 5.1. Therefore, since  $\mathcal{D}_N$  is defined by

$$\mathcal{D}_N \circ r(x) = \bigcup_{x' \in r^{-1} \circ r(x)} T_{x'} r(\mathcal{D}_M(x')) \quad (\text{A.10})$$

it follows that  $T_x r(\mathcal{D}_M(x)) = \mathcal{D}_N \circ r(x)$  for every  $x \in M$  and the result now follows from Theorem 4.2.  $\square$

**Proof of Theorem 4.5.** The proof is similar to the proof of Theorem 4.6, except we now use the fact that  $\ker(Tr)$  is invariant for  $\Sigma_M$  iff every vector field in  $\mathcal{D}_M$  is  $r$ -related to its projection on  $N$ .  $\square$

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