On the Stability of Zeno Equilibria*

Aaron D. Ames[1], Paulo Tabuada[2], and Shankar Sastry[1]

[1] Department of Electrical Engineering and Computer Sciences,
University of California at Berkeley, Berkeley, CA 94720
{adames, sastry}@eecs.berkeley.edu
[2] Department of Electrical Engineering,
University of Notre Dame, Notre Dame, IN 46556
ptabuada@nd.edu

Abstract. Zeno behaviors are one of the (perhaps unintended) features of many hybrid models of physical systems. They have no counterpart in traditional dynamical systems or automata theory and yet they have remained relatively unexplored over the years. In this paper we address the stability properties of a class of Zeno equilibria, and we introduce a necessary paradigm shift in the study of hybrid stability. Motivated by the peculiarities of Zeno equilibria, we consider a form of asymptotic stability that is global in the continuous state, but local in the discrete state. We provide sufficient conditions for stability of these equilibria, resulting in sufficient conditions for the existence of Zeno behavior.

1 Introduction

Hybrid models have been used successfully during the past decade to describe systems exhibiting both discrete and continuous dynamics, while they have simultaneously allowed complex models of continuous systems to be simplified. We are interested in the rich dynamical behavior of hybrid models of physical systems. These hybrid models admit a kind of equilibria that is not found in continuous dynamical systems or in automata theory: *Zeno equilibria*. Zeno equilibria are collections of points which are invariant under the discrete component of the hybrid dynamics, and which can be stable is many cases of interest.

Mechanical systems undergoing impacts are naturally modeled as hybrid systems (cf. [1] and [2]). The convergent behavior of these systems is often of interest—even if this convergence is not to "classical" notions of equilibrium points. This motivates the study of Zeno equilibria because even if the convergence is not classical, it still is important. For example, simulating trajectories of these systems is an important component in their analysis, yet this may not be possible due to the relationship between Zeno equilibria and Zeno behavior.

An equally important reason to address the stability of Zeno equilibria is to be able to assess the existence of Zeno trajectories. This behavior is infamous in the hybrid system community for its ability to halt simulations. The only way to

* This research is supported by the National Science Foundation (award numbers CCR-0225610 and CSR-EHS-0509313).

J. Hespanha and A. Tiwari (Eds.): HSCC 2006, LNCS 3927, pp. 34–48, 2006.

prevent this undesirable outcome is to give *a priori* conditions on the existence of Zeno behavior. This has motivated a profuse study of Zeno hybrid systems (see [1, 3, 4, 5, 6, 7, 8] to name a few) but a concrete notion of convergence (in the sense of stability) has not yet been introduced. As a result, there is a noticeable lack of sufficient conditions for the existence of Zeno behavior. We refer the reader to [3, 7, 8] for a more thorough introduction to Zeno behavior.

Our investigations into the stability of Zeno equilibria are made possible through a categorical framework for hybrid systems (as first introduced in [9] and later utilized in [10]). This theory allows "non-hybrid" objects to be generalized to a hybrid setting. Specifically, let T be a category, i.e., a collection of mathematical objects that share a certain property together with morphisms between these objects. A *hybrid object* over this category is a special type of small category H, termed an *H-category*, together with a functor (either covariant or contravariant) $\boldsymbol{S} : \mathsf{H} \to \mathsf{T}$. Morphisms between objects of T are generalized to a hybrid setting through the use of natural transformations.

The main contribution of this paper is sufficient conditions for the stability of Zeno equilibria. As a byproduct, we are able to give sufficient conditions for the existence of Zeno behavior. The categorical approach to hybrid systems allows us to decompose the study of stability into two manageable steps. The first step consists of identifying a sufficiently rich, yet sufficiently simple, class of hybrid systems embodying the desired stability properties: *first quadrant hybrid systems.* The second step is to understand the stability of general hybrid systems by understanding the relationships between these systems and first quadrant hybrid systems described by morphisms (in the category of hybrid systems).

2 Classical Stability: A Categorical Approach

In this section we revisit classical stability theory under a categorical light. The new perspective afforded by category theory is more than a simple exercise in abstract nonsense because it motivates the development of an analogous stability theory for hybrid systems and hybrid equilibria to be presented in Sections 4 and 5. We shall work on Dyn, the category of dynamical systems, which has as objects pairs (M, X), where M is a smooth manifold[1] and $X : M \to TM$ is a smooth vector field. The morphisms are smooth maps $f : N \to M$ making the following diagram commutative:

$$\begin{array}{ccc} TN & \xrightarrow{Tf} & TM \\ Y \uparrow & & \uparrow X \\ N & \xrightarrow{f} & M \end{array} \tag{1}$$

The subcategory $\mathsf{Interval}(\mathsf{Dyn})$ of Dyn will play an especially important role in the theory developed in this paper. This subcategory is the full subcategory of Dyn

[1] We assume that M is a Riemannian manifold, and so has a metric $d(x, y) = \|x - y\|$. Alternatively, we could assume that M is a subset of $\mathbb{R}^n$.

defined by objects[2] $(I, \frac{d}{dt})$ with I a subset of $\mathbb{R}$ of the form $[t,t']$, $(t,t']$, $[t,t')$, (t,t') and $\{t\}$, where $[t,t']$ is a manifold with boundary (and so is $(t,t']$ and $[t,t')$) and $\{t\}$ is a zero-dimensional manifold consisting of the single point t (which is trivially a smooth manifold). The following observation shows the relevance of $\mathsf{Interval}(\mathsf{Dyn})$. A morphism $c : (I, d/dt) \to (M, X)$ is a smooth map $c : I \to M$ making diagram (1) commutative and thus satisfying:

$$\dot{c}(t) = Tc \cdot \frac{d}{dt} = X \circ c(t).$$

We can therefore identify a morphism $c : (I, d/dt) \to (M, X)$ with a trajectory of (M, X). Furthermore, the existence of a morphism $f : (N, Y) \to (M, X)$ implies that for every trajectory $c : (I, d/dt) \to (N, Y)$, the composite $f \circ c : (I, d/dt) \to (M, X)$ is a trajectory of (M, X). In other words, a morphism $f : (N, Y) \to (M, X)$ carries trajectories of (N, Y) into trajectories of (M, X).

Remarkably, stability also can be described through the existence of certain morphisms. Let us first recall the definition of globally asymptotically stable equilibria.

Definition 1. *Let (M, X) be an object of* Dyn. *An equilibrium point $x^* \in M$ of X is said to be globally asymptotically stable when for any morphism $c : ([t, \infty), \frac{d}{dt}) \to (M, X)$, for any $t_1 > t$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ satisfying:*

1. $\|c(t_1) - x^*\| < \delta \quad \Rightarrow \quad \|c(t_2) - x^*\| < \varepsilon \qquad \forall t_2 \geq t_1 \geq t,$
2. $\lim_{\tau \to \infty} c(\tau) = x^*$.

Consider now the full subcategory of Dyn denoted by GasDyn and defined by objects $(\mathbb{R}_0^+, -\alpha)$ where α is a class $\mathcal{K}_\infty$ function. Lyapunov's second method can then be described as follows:

Theorem 1. *Let (M, X) be an object of* Dyn. *An equilibrium point $x^* \in M$ of X is globally asymptotically stable if there exists a morphism:*

$$(M, X) \xrightarrow{v} (\mathbb{R}_0^+, -\alpha) \in \mathsf{GasDyn}$$

in Dyn *satisfying:*

1. *$v(x) = 0$ implies $x = x^*$,*
2. *$v : M \to \mathbb{R}_0^+$ is a proper (radially unbounded) function.*

The previous result suggests that the study of stability properties can be carried out in two steps. In the first step we identify a suitable subcategory having the desired stability properties. In the case of global asymptotic stability, this subcategory is GasDyn; for local stability we could consider the full subcategory defined by objects of the form $(\mathbb{R}_0^+, -\alpha)$ with α a non-negative definite function.

[2] We do not consider more general objects of the form $(J, g(t)d/dt)$ with $g > 0$ since each such object is isomorphic to $(I, d/dt)$.

The chosen category corresponds in some sense to the simplest possible objects having the desired stability properties. In the second step we show that existence of a morphism from a general object (M, X) to an object in the chosen subcategory implies that the desired stability properties also hold in (M, X). This is precisely the approach we will develop in Sections 4 and 5 for the study of Zeno equilibria.

3 Categorical Hybrid Systems

This section is devoted to the study of first quadrant hybrid systems, categorical hybrid systems, and their interplay. We begin by defining a very simple class of hybrid systems; these systems are easy to understand and analyze, but lack generality. We then proceed to define general hybrid systems through the framework of hybrid category theory; these systems are general but difficult to analyze. The advantage of introducing these two concepts is that not only can they be related through explicit constructions, but also through the more general framework of morphisms in the category of hybrid systems. This relationship will be important in understanding the stability of general hybrid systems.

First Quadrant Systems. In order to understand the stability of general hybrid systems, we must consider a class of hybrid systems analogous to the objects of GasDyn; these are termed *first quadrant hybrid systems*. It is not surprising that these would be chosen as the "canonical" hybrid systems with which to understand the stability of Zeno equilibria as they already have been used to derive sufficient conditions for the existence of Zeno behavior in [3].

A *first quadrant hybrid system* is a tuple:

$$\mathscr{H}_{\mathbf{FQ}} = (\Gamma, D, G, R, F),$$

where

- $\Gamma = (Q, E)$ is an oriented cycle, with
$$Q = \{1, \dots, k\}, \qquad E = \{e_1 = (1,2), e_2 = (2,3), \dots, e_k = (k,1)\}.$$
- $D = \{D_i\}_{i \in Q}$, where for all $i \in Q$,
$$D_i = (\mathbb{R}_0^+)^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 \geq 0\}.$$
- $G = \{G_e\}_{e \in E}$, where for all $e \in E$
$$G_e = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \geq 0\}.$$
- $R = \{R_e\}_{e \in E}$, where $R_e : G_e \to (\mathbb{R}_0^+)^2$ and for all $e \in E$ there exists a function $r_e : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with
$$R_e(0, x_2) = (r_e(x_2), 0).$$
- $F = \{f_i\}_{i \in Q}$, where f_i is a Lipshitz vector field on $(\mathbb{R}_0^+)^2$.

Before discussing the stability properties of first quadrant hybrid systems, we need to relate them to more general hybrid systems. This is accomplished by introducing a categorical framework for hybrid systems. As outlined in the introduction, a hybrid object over a category is a pair $\boldsymbol{S} : \mathsf{H} \to \mathsf{T}$. Since we allow $\boldsymbol{S}$ to be any functor, the main component of the definition is the small category H which must be an H-category; the special form of this small category directly reflects its ability to describe "hybrid objects." Therefore, in order to define hybrid objects over a category, we must invest a rather sizable amount of effort in understanding the definition and structure of H-categories.

H-Cateogries. We start by defining a specific type of small category termed an *oriented H-category* and denoted by H. This is a small category (cf. [11]) in which every diagram has the form:[3]

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \quad \cdots \quad \bullet \longleftarrow \bullet \longrightarrow \bullet$$

That is, an H-category has as its basic atomic unit a diagram of the form: $\bullet \longleftarrow \bullet \longrightarrow \bullet$, and any other diagram in this category must be obtainable by gluing such atomic units along the target of a morphism (and not the source). In addition, we require the existence of an orientation on H. Before defining such an orientation, some additional definitions are needed. Denote by $\mathsf{Ob}(\mathsf{H})$ the objects of H, denote by $\mathsf{Mor}(\mathsf{H})$ the morphisms of H, and by $\mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H})$ the set of non-identity morphisms of H. For a morphism $\alpha : a \to b$ in H, its domain (or source) is denoted by $\mathsf{dom}(\alpha) = a$ and its codomain (or target) is denoted by $\mathsf{cod}(\alpha) = b$. For H-categories, there are two sets of objects that are of particular interest; these are subsets of the set $\mathsf{Ob}(\mathsf{H})$. The first of these is called the *edge set of* H, is denoted by $\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$, and is defined to be

$$\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) = \{a \in \mathsf{Ob}(\mathsf{H}) : \ a = \mathsf{dom}(\alpha) = \mathsf{dom}(\beta), \\ \alpha, \beta \in \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H}),\ \alpha \neq \beta\}.$$

The symbol $\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\,\cdot\,)$ is used because every object $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$ sits in a diagram of the form:

$$\mathsf{cod}(\alpha_a) = b \xleftarrow{\ \alpha_a\ } \mathsf{dom}(\alpha_a) = a = \mathsf{dom}(\beta_a) \xrightarrow{\ \beta_a\ } c = \mathsf{cod}(\beta_a)$$

called a *bac-diagram*. Note that giving all diagrams of this form (of which there is one for each $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$) gives all the objects in H, i.e., every object of H is the target of α_a or β_a, or their source, for some $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$. More specifically, we can define the *vertex set of* H by

$$\mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}) = \left(\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})\right)^c$$

where $\left(\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})\right)^c$ is the complement of $\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$ in the set $\mathsf{Ob}(\mathsf{H})$.

[3] Where $\bullet$ denotes an arbitrary object in H and $\longrightarrow$ denotes an arbitrary morphism.

Oriented H-Categories. We can orient an H-category by picking a specific labeling of its morphisms. Specifically, we define an *orientation* of an H-category H as a pair of maps (α, β) between sets:

$$\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H})$$

such that for every $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$, there is a bac-diagram in H:

$$b \xleftarrow{\alpha_a} a \xrightarrow{\beta_a} c. \tag{2}$$

We can form the category of oriented H-categories: Hcat. A morphism between two oriented H-categories, H and H' (with orientations (α, β) and (α', β'), respectively), is a functor $\boldsymbol{F} : \mathsf{H} \to \mathsf{H}'$ such that the following diagrams

$$\begin{array}{ccc} \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) & \xrightarrow{\boldsymbol{F}} & \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}') \\ \alpha \downarrow & & \downarrow \alpha' \\ \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H}) & \xrightarrow{\boldsymbol{F}} & \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H}') \end{array} \qquad \begin{array}{ccc} \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) & \xrightarrow{\boldsymbol{F}} & \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}') \\ \beta \downarrow & & \downarrow \beta' \\ \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H}) & \xrightarrow{\boldsymbol{F}} & \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H}') \end{array} \tag{3}$$

commute. This requirement implies that, if $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$ with corresponding bac-diagram (2) in H, there is a corresponding bac-diagram:

$$\boldsymbol{F}(b) \xleftarrow{\boldsymbol{F}(\alpha_a) = \alpha'_{\boldsymbol{F}(a)}} \boldsymbol{F}(a) \xrightarrow{\boldsymbol{F}(\beta_a) = \beta'_{\boldsymbol{F}(a)}} \boldsymbol{F}(c)$$

where $\boldsymbol{F}(a) \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}')$.

From Graphs to H-Categories. To every oriented H-category, we can associate (a possibly infinite) oriented graph, and vice versa. That is, we have functors (see [12] for the explicit construction of these functors):

$$\begin{array}{ll} \boldsymbol{\Gamma} : \mathsf{Grph} \longrightarrow \mathsf{Hcat} & \qquad \boldsymbol{H} : \mathsf{Hcat} \longrightarrow \mathsf{Grph} \\ \Gamma \mapsto \boldsymbol{\Gamma}(\Gamma) := \mathsf{H}_\Gamma & \qquad \mathsf{H} \mapsto \boldsymbol{H}(\mathsf{H}) := \Gamma_{\mathsf{H}} \end{array}$$

where Grph is the category of oriented graphs. The functor $\boldsymbol{\Gamma}$ is, roughly speaking, defined on every edge $e_i \in E$ by:

$$\boldsymbol{\Gamma}\left(i \xrightarrow{e_i} j \right) = i \xleftarrow{\alpha_{e_i}} e_i \xrightarrow{\beta_{e_i}} j.$$

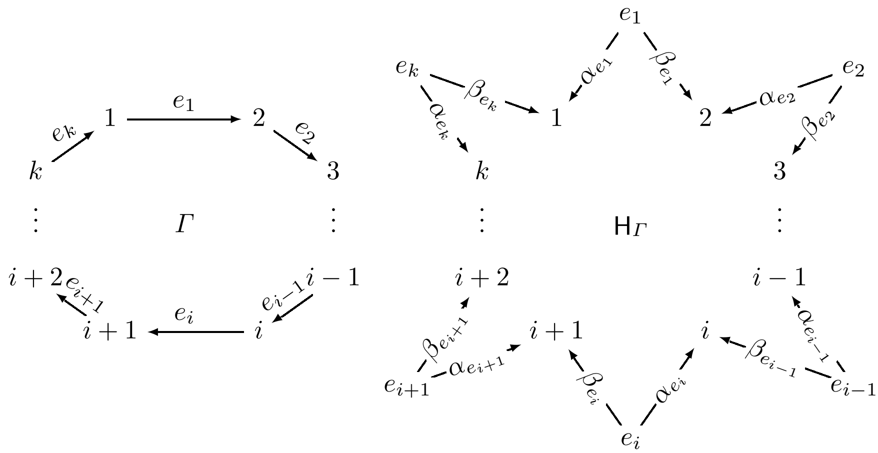
The relationship between oriented H-categories and oriented graphs is made more precise in the following theorem (again, see [12]).

Theorem 2. *There is an isomorphism of categories,*

$$\mathsf{Grph} \cong \mathsf{Hcat}, \tag{4}$$

where this isomorphism is given by the functor $\boldsymbol{H} : \mathsf{Hcat} \to \mathsf{Grph}$ *with inverse* $\boldsymbol{\Gamma} : \mathsf{Grph} \to \mathsf{Hcat}$.

Example 1. For the hybrid system $\mathscr{H}_{\mathbf{FQ}}$, and by utilizing (4), we can associate to the graph Γ a H-category H_Γ. Both Γ and the corresponding H-category are given in the following diagrams:



We now have the necessary framework in which to introduce hybrid objects over a category.

Definition 2. *Let* T *be a category. Then a hybrid object over* T *is a pair* $(\mathsf{H}, \boldsymbol{S})$*, where* H *is an H-category and*

$$\boldsymbol{S} : \mathsf{H} \to \mathsf{T}$$

is a functor (either covariant or contravariant). Given two hybrid objects, $(\mathsf{H}, \boldsymbol{S})$ *and* $(\mathsf{H}', \boldsymbol{S}')$*, a morphism between these objects is a functor and a natural transformation* $(\boldsymbol{F}, \boldsymbol{f}) : (\mathsf{H}, \boldsymbol{S}) \to (\mathsf{H}', \boldsymbol{S}')$ *where* $\boldsymbol{F} : \mathsf{H} \to \mathsf{H}'$ *is a morphism in* Hcat *and* $\boldsymbol{f} : \boldsymbol{S} \stackrel{\cdot}{\longrightarrow} \boldsymbol{S}' \circ \boldsymbol{F}$.

Hybrid Manifolds. An important example of a hybrid object is a hybrid manifold, defined to be a functor $\boldsymbol{M} : \mathsf{H}_{\boldsymbol{M}} \to \mathsf{Man}$ where $\mathsf{H}_{\boldsymbol{M}}$ is an H-category and Man is the category of smooth manifolds; in this paper, we assume that for every diagram (2), there is the following diagram:

$$\boldsymbol{M}_b \xleftarrow{\boldsymbol{M}_{\alpha_a} = i_{\alpha_a}} \boldsymbol{M}_a \xrightarrow{\boldsymbol{M}_{\beta_a}} \boldsymbol{M}_c$$

in Man, where $\boldsymbol{M}_a \subseteq \boldsymbol{M}_b$ and i_{α_a} is the natural inclusion. If $(\mathsf{H}_{\boldsymbol{N}}, \boldsymbol{N})$ is another hybrid manifold, a morphism of hybrid manifolds is a pair $(\boldsymbol{F}, \boldsymbol{f}) : (\mathsf{H}_{\boldsymbol{N}}, \boldsymbol{N}) \to (\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M})$ where $\boldsymbol{F} : \mathsf{H}_{\boldsymbol{N}} \to \mathsf{H}_{\boldsymbol{M}}$ is a morphism in Hcat and $\boldsymbol{f}$ is a natural transformation: $\boldsymbol{f} : \boldsymbol{N} \stackrel{\cdot}{\longrightarrow} \boldsymbol{M} \circ \boldsymbol{F}$.

Example 2. For $\mathscr{H}_{\mathbf{FQ}}$, the "hybrid manifold" portion of this hybrid system corresponds to the tuple (Γ, D, G, R). To make this explicit, the hybrid manifold associated to $\mathscr{H}_{\mathbf{FQ}}$ is given by the pair $(\mathsf{H}_\Gamma, \boldsymbol{M}^{\mathscr{H}_{\mathbf{FQ}}})$ where $\boldsymbol{M}^{\mathscr{H}_{\mathbf{FQ}}}$ is the functor defined on each bac-diagram in H_Γ to be

$$\boldsymbol{M}^{\mathscr{H}_{\mathbf{FQ}}}\left(i \xleftarrow{\alpha_{e_i}} e_i \xrightarrow{\beta_{e_i}} i+1 \right) = D_i \xleftarrow{\;i\;} G_{e_i} \xrightarrow{R_{e_i}} D_{i+1}.$$

Hybrid Systems. A hybrid system is a tuple $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$, where $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M})$ is a hybrid manifold and $\boldsymbol{X} = \{\boldsymbol{X}_b\}_{b \in \mathsf{Ob}_{(\rightarrow \cdot \leftarrow)}(H)}$ with $\boldsymbol{X}_b : \boldsymbol{M}_b \to T\boldsymbol{M}_b$ a Lipschitz vector field on $\boldsymbol{M}_b$. With this formulation of hybrid systems (it can be verified that this definition is consistent with the standard one), we can form the category of hybrid systems, HySys. The objects are hybrid systems and the morphisms are pairs $(\boldsymbol{F}, \boldsymbol{f}) : (\mathsf{H}_{\boldsymbol{N}}, \boldsymbol{N}, \boldsymbol{Y}) \to (\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$, where $(\boldsymbol{F}, \boldsymbol{f})$ is a morphism from the hybrid manifold $(\mathsf{H}_{\boldsymbol{N}}, \boldsymbol{N})$ to the hybrid manifold $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M})$ such that there is a commuting diagram for all $b \in \mathsf{Ob}_{(\rightarrow \cdot \leftarrow)}(\mathsf{H}_{\boldsymbol{N}})$

$$\begin{array}{ccc} T\boldsymbol{N}_b & \xrightarrow{T\boldsymbol{f}_b} & T\boldsymbol{M}_{\boldsymbol{F}(b)} \\ {\scriptstyle \boldsymbol{Y}_b}\uparrow & & \uparrow{\scriptstyle \boldsymbol{X}_{\boldsymbol{F}(b)}} \\ \boldsymbol{N}_b & \xrightarrow{\boldsymbol{f}_b} & \boldsymbol{M}_{\boldsymbol{F}(b)} \end{array}$$

That is, for all $b \in \mathsf{Ob}_{(\rightarrow \cdot \leftarrow)}(\mathsf{H}_{\boldsymbol{N}})$, $\boldsymbol{f}_b : (\boldsymbol{N}_b, \boldsymbol{Y}_b) \to (\boldsymbol{M}_{\boldsymbol{F}(b)}, \boldsymbol{X}_{\boldsymbol{F}(b)})$ is a morphism in Dyn.

Morphisms of hybrid systems are composed by composing the associated functor and natural transformation, respectively.

Example 3. The categorical hybrid system associated to $\mathscr{H}_{\mathbf{FQ}}$ is given by

$$(\mathsf{H}_{\Gamma}, \boldsymbol{M}^{\mathscr{H}_{\mathbf{FQ}}}, \boldsymbol{X}^{\mathscr{H}_{\mathbf{FQ}}}),$$

where $(\mathsf{H}_{\Gamma}, \boldsymbol{M}^{\mathscr{H}_{\mathbf{FQ}}})$ is the hybrid manifold defined in the previous example and $\boldsymbol{X}^{\mathscr{H}_{\mathbf{FQ}}} = \{f_i\}_{i \in \mathsf{Ob}_{(\rightarrow \cdot \leftarrow)}(\mathsf{H}_{\Gamma}) = Q}$.

Hybrid Intervals. As with the continuous case discussed in Section 2, we need to introduce a notion of intervals for hybrid systems. Let $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ be a finite or infinite indexing set, from which we can associate a graph $\Gamma_\Lambda = (Q_\Lambda, E_\Lambda)$, where $Q_\Lambda = \Lambda$ and E_Λ is the set of pairs $\eta_{j+1} = (j, j+1)$ such that $j, j+1 \in \Lambda$. From this graph we obtain an H-category $\mathsf{H}_{\Gamma_\Lambda}$ via (4); this implies that every bac-diagram in this H-category must have the form:

$$j - 1 \xleftarrow{\alpha_j} \eta_j = (j-1, j) \xrightarrow{\beta_j} j, \tag{5}$$

and so we denote by 0 the object of $\mathsf{H}_{\Gamma_\Lambda}$ corresponding to the vertex $0 \in Q_\Lambda = \Lambda$. Define $\mathsf{Interval}(\mathsf{Hcat})$ to be the subcategory of Hcat consisting of all H-categories obtained from graphs of this form. A hybrid interval now can be defined as a pair:

$$\boldsymbol{I} : \mathsf{H}_{\boldsymbol{I}} \to \mathsf{Interval}(\mathsf{Dyn}),$$

where $\mathsf{H}_{\boldsymbol{I}}$ is an object of $\mathsf{Interval}(\mathsf{Hcat})$, and we assume that for every bac-diagram in $\mathsf{H}_{\boldsymbol{I}}$, there exist (switching times) $\tau_{j-1}, \tau_j, \tau_{j+1} \in \mathbb{R} \cup \{\infty\}$, with $\tau_{j-1} \le \tau_j \le \tau_{j+1}$ such that:

$$\boldsymbol{I}_{j-1} = [\tau_{j-1}, \tau_j] \xleftarrow{\boldsymbol{I}_{\alpha_j} = \iota} \boldsymbol{I}_{\eta_j} = \{\tau_j\} \xrightarrow{\boldsymbol{I}_{\beta_j} = \iota} \boldsymbol{I}_j = [\tau_j, \tau_{j+1}] \text{ or } [\tau_j, \tau_{j+1}).$$

We also suppose that $\boldsymbol{I}_0 = [0, \tau_1]$ or $[0, \tau_1)$.

Trajectories of Hybrid Systems. The importance of hybrid intervals is that, like classical intervals, they can be used to define trajectories of hybrid systems (which correspond to the classical notion of an execution for a hybrid system). The interval category of HySys, denoted by $\mathsf{Interval}(\mathsf{HySys})$, is the full subcategory of HySys with objects consisting of hybrid systems of the form $(\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt})$ where $(\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I})$ is a hybrid interval and $\boldsymbol{d/dt}_j = d/dt$ for all $j \in \Lambda = \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}_{\boldsymbol{I}})$.

Definition 3. *A trajectory of a hybrid system* $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ *is a morphism* $(\boldsymbol{C}, \boldsymbol{c})$ *in* HySys*:*

$$(\boldsymbol{C}, \boldsymbol{c}) : (\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt}) \to (\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X}),$$

where $(\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt})$ *is an object of* $\mathsf{Interval}(\mathsf{HySys})$*. In particular, this implies that* $\dot{\boldsymbol{c}}_j(t) = \boldsymbol{X}_{\boldsymbol{C}(j)}(\boldsymbol{c}_j(t))$ *for every object* $j \in \Lambda = \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}_{\boldsymbol{I}})$*.*

Note that the functor $\boldsymbol{C}$ corresponds to the "discrete" portion of the trajectory, while the natural transformation $\boldsymbol{c}$ corresponds to the "continuous" portion. The discrete initial condition is given by $\boldsymbol{C}(0)$ and the continuous initial condition is given by $\boldsymbol{c}_0(0) \in \boldsymbol{M}_{\boldsymbol{C}(0)}$, i.e., the initial condition to the trajectory is $(\boldsymbol{c}_0(0), \boldsymbol{C}(0))$.

Example 4. To better understand the categorical formulation of trajectories, we enumerate the consequences of Definition 3 for first quadrant hybrid systems. Let

$$(\boldsymbol{C}, \boldsymbol{c}) : (\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt}) \to (\mathsf{H}_{\Gamma}, \boldsymbol{M}^{\mathscr{H}_{\mathbf{FQ}}}, \boldsymbol{X}^{\mathscr{H}_{\mathbf{FQ}}})$$

be a trajectory of the hybrid system $(\mathsf{H}_{\Gamma}, \boldsymbol{M}^{\mathscr{H}_{\mathbf{FQ}}}, \boldsymbol{X}^{\mathscr{H}_{\mathbf{FQ}}})$. Since $\boldsymbol{c}$ is a natural transformation, we have a commuting diagram:

$$\begin{array}{ccccc}
\boldsymbol{I}_{j-1} = [\tau_{j-1}, \tau_j] & \xleftarrow{\boldsymbol{I}_{\alpha_j} = \iota} & \boldsymbol{I}_{\eta_j} = \{\tau_j\} & \xrightarrow{\boldsymbol{I}_{\beta_j} = \iota} & \boldsymbol{I}_j = [\tau_j, \tau_{j+1}] \text{ or } [\tau_j, \tau_{j+1}) \\
\downarrow \boldsymbol{c}_{j-1} & & \downarrow \boldsymbol{c}_{\eta_j} & & \downarrow \boldsymbol{c}_j \\
D_{\boldsymbol{C}(j-1)} & \xleftarrow{\iota} & G_{\boldsymbol{C}(\eta_j)} & \xrightarrow{R_{\boldsymbol{C}(\beta_j)}} & D_{\boldsymbol{C}(j)}
\end{array}$$

This in turn implies that a trajectory must satisfy the following conditions:

$$\boldsymbol{c}_{j-1}(\tau_j) \in G_{\boldsymbol{C}(\eta_j)}, \qquad R_{\boldsymbol{C}(\beta_j)}(\boldsymbol{c}_{j-1}(\tau_j)) = \boldsymbol{c}_j(\tau_j),$$

which are the standard requirements on a trajectory.

We end this section by noting that, as with the continuous case, if $(\boldsymbol{F}, \boldsymbol{f})$: $(\mathsf{H}_{\boldsymbol{N}}, \boldsymbol{N}, \boldsymbol{Y}) \to (\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ is a morphism of hybrid systems, and $(\boldsymbol{C}, \boldsymbol{c})$: $(\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt}) \to (\mathsf{H}_{\boldsymbol{N}}, \boldsymbol{N}, \boldsymbol{Y})$ is a trajectory of $(\mathsf{H}_{\boldsymbol{N}}, \boldsymbol{N}, \boldsymbol{Y})$, then

$$(\boldsymbol{F} \circ \boldsymbol{C}, \boldsymbol{f} \bullet \boldsymbol{c}) : (\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt}) \to (\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$$

is a trajectory of $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$.

4 Stability of Zeno Equilibria

The purpose of this section is to study the stability of a type of equilibria that is unique to hybrid systems: Zeno equilibria. The uniqueness of these equilibria necessitates a paradigm shift in the current notions of stability, i.e., we must introduce a type of stability that is both local and global in nature and, therefore, has no direct analogue in continuous and discrete systems. The main result of this section is sufficient conditions for the stability of Zeno equilibria in first quadrant hybrid systems.

It is important to note that we do not claim that Zeno equilibria are the most general form of equilibria corresponding to Zeno behavior. We do claim that the type of Zeno equilibria considered are general enough to cover a wide range of interesting (and somewhat peculiar) behavior, while being specific enough to allow for analysis.

Definition 4. *Let* $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ *be a hybrid system. A Zeno equilibria* $(\mathsf{H}^{\Gamma}_{\boldsymbol{M}}, \boldsymbol{z})$ *is a H-subcategory* $\mathsf{H}^{\Gamma}_{\boldsymbol{M}}$ *of* $\mathsf{H}_{\boldsymbol{M}}$ *obtained from a cycle* Γ *together with a set* $\boldsymbol{z} = \{\boldsymbol{z}_a\}_{a \in \mathsf{Ob}(\mathsf{H}^{\Gamma}_{\boldsymbol{M}})}$ *such that*

- $\boldsymbol{z}_a \in \boldsymbol{M}_a$ *for all* $a \in \mathsf{Ob}(\mathsf{H}^{\Gamma}_{\boldsymbol{M}})$,
- $\boldsymbol{z}_b = \boldsymbol{M}_{\gamma}(\boldsymbol{z}_a)$ *for all* $\gamma : a \to b$ *in* $\mathsf{H}^{\Gamma}_{\boldsymbol{M}}$,
- $\boldsymbol{X}_a(\boldsymbol{z}_a) \neq 0$ *for all* $a \in \mathsf{Ob}(\mathsf{H}^{\Gamma}_{\boldsymbol{M}})$.

Another Interpretation of Zeno Equilibria. There is a more categorical definition of a Zeno equilibria. Starting with the one point set $*$, we obtain a hybrid manifold $(\mathsf{H}^{\Gamma}_{\boldsymbol{M}}, \triangle(*))$ where $\triangle(*) : \mathsf{H}^{\Gamma}_{\boldsymbol{M}} \to \mathsf{Man}$ with $\triangle$ the diagonal functor. Denoting by $\mathbf{Inc} : \mathsf{H}^{\Gamma}_{\boldsymbol{M}} \to \mathsf{H}_{\boldsymbol{M}}$ the inclusion functor, a Zeno equilibria is a morphism of hybrid manifolds:

$$(\mathbf{Inc}, \boldsymbol{z}) : (\mathsf{H}^{\Gamma}_{\boldsymbol{M}}, \triangle(*)) \to (\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M})$$

such that $\boldsymbol{X}_a(\boldsymbol{z}_a) \neq 0$; in this case (and by slight abuse of notation) $\boldsymbol{z}_a(*) := \boldsymbol{z}_a$.

Example 5. For the hybrid system $\mathscr{H}_{\mathbf{FQ}}$, and since we are assuming the underlying graph to be a cycle, the conditions expressed in Definition 4 imply that a set $\boldsymbol{z} = \{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k\}$ is a Zeno equilibria if for all $i = 1, \ldots, k$, $\boldsymbol{z}_i \in G_{e_i}$ and

$$R_{e_{i-1}} \circ \cdots \circ R_{e_1} \circ R_{e_k} \circ \cdots \circ R_{e_i}(\boldsymbol{z}_i) = \boldsymbol{z}_i. \tag{6}$$

Because of the special structure of $\mathscr{H}_{\mathbf{FQ}}$, (6) holds iff $\boldsymbol{z}_i = 0$ for all i. That is, the only Zeno equilibria of $\mathscr{H}_{\mathbf{FQ}}$ is the singleton set $\boldsymbol{z} = \{0\}$.

Induced Hybrid Subsystems. Let $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ be a hybrid system, $\mathsf{H}^{\Gamma}_{\boldsymbol{M}}$ be an H-subcategory of $\mathsf{H}_{\boldsymbol{M}}$, and $\mathbf{Inc} : \mathsf{H}^{\Gamma}_{\boldsymbol{M}} \to \mathsf{H}_{\boldsymbol{M}}$ be the inclusion functor. In this case, there is a hybrid subsystem $(\mathsf{H}^{\Gamma}_{\boldsymbol{M}}, \boldsymbol{M}^{\Gamma}, \boldsymbol{X}^{\Gamma})$ of $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ corresponding to this inclusion, i.e., there is an inclusion in HySys:

$$(\mathbf{Inc}, \mathbf{id}) : (\mathsf{H}^{\Gamma}_{\boldsymbol{M}}, \boldsymbol{M}^{\Gamma}, \boldsymbol{X}^{\Gamma}) \hookrightarrow (\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$$

where $\mathbf{id}$ is the identity natural transformation.

Definition 5. *A Zeno equilibria* $(\mathsf{H}_{\boldsymbol{M}}^{\Gamma}, \boldsymbol{z})$ *of* $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ *is globally asymptotically stable relative to* $(\mathsf{H}_{\boldsymbol{M}}^{\Gamma}, \boldsymbol{M}^{\Gamma}, \boldsymbol{X}^{\Gamma})$ *if the inclusion* $\mathbf{Inc} : \mathsf{H}_{\boldsymbol{M}}^{\Gamma} \to \mathsf{H}_{\boldsymbol{M}}$ *satisfies: for all* $b \in \mathsf{Ob}_{(\to\cdot\leftarrow)}(\mathsf{H}_{\boldsymbol{M}}^{\Gamma})$,

$$\mathsf{cod}(\alpha_{a_1}) = \mathbf{Inc}(b) = \mathsf{cod}(\alpha_{a_2}) \quad \Rightarrow \quad a_1 = a_2,$$

and for every trajectory:

$$(\boldsymbol{C}, \boldsymbol{c}) : (\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt}) \to (\mathsf{H}_{\boldsymbol{M}}^{\Gamma}, \boldsymbol{M}^{\Gamma}, \boldsymbol{X}^{\Gamma}),$$

with $\Lambda = \mathbb{N}$, *and for any* $\varepsilon_{\boldsymbol{C}(j)}$ *there exists* $\delta_{\boldsymbol{C}(i)}$ *such that:*

1. *If* $\|\boldsymbol{c}_i(\tau_i) - \boldsymbol{z}_{\boldsymbol{C}(i)}\| < \delta_{\boldsymbol{C}(i)}$ *for* $i = 0, 1, \ldots, k \in Q$ *then*
$$\|\boldsymbol{c}_j(t) - \boldsymbol{z}_{\boldsymbol{C}(j)}\| < \varepsilon_{\boldsymbol{C}(j)}$$
with $j \in \Lambda$ *and* $t \in \boldsymbol{I}_j = [\tau_j, \tau_{j+1}]$.
2. *For all* $a \in \mathsf{Ob}_{(\to\cdot\leftarrow)}(\mathsf{H}_{\boldsymbol{M}}^{\Gamma})$
$$\lim_{\substack{j\to\infty \\ \boldsymbol{C}(j)=a}} \boldsymbol{c}_j(\tau_j) = \boldsymbol{z}_a, \qquad \lim_{\substack{j\to\infty \\ \boldsymbol{C}(j)=a}} \boldsymbol{c}_j(\tau_{j+1}) = \boldsymbol{z}_a.$$

We say that a Zeno equilibria $(\mathsf{H}_{\boldsymbol{M}}^{\Gamma}, \boldsymbol{z})$ *of* $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ *is globally asymptotically stable if it is globally asymptotically stable relative to* $(\mathsf{H}_{\boldsymbol{M}}^{\Gamma}, \boldsymbol{M}^{\Gamma}, \boldsymbol{X}^{\Gamma})$ *and* $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X}) = (\mathsf{H}_{\boldsymbol{M}}^{\Gamma}, \boldsymbol{M}^{\Gamma}, \boldsymbol{X}^{\Gamma})$.

The definition of relative global asymptotic stability implicitly makes some very subtle points. The first is that this type of stability is both local and global in nature—hence the use of the words “global” and “relative” in the definition. While for traditional dynamical systems this would seem contradictory, the complexity of hybrid systems requires us to view stability in a much different light, i.e., we must expand the paradigm for stability.

To better explain the mixed global and local nature of relatively globally asymptotically stable Zeno equilibria, we note that the term “global” is used because the hybrid subsystem $(\mathsf{H}_{\boldsymbol{M}}^{\Gamma}, \boldsymbol{M}^{\Gamma}, \boldsymbol{X}^{\Gamma})$ is globally stable to the Zeno equilibria; this also motivates the use of the word “relative” as $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ is stable relative to a hybrid subsystem. Finally, the local nature of this form of stability is in the discrete portion of the hybrid system, rather than the continuous one. That is, the H-subcategory $\mathsf{H}_{\boldsymbol{M}}^{\Gamma}$ can be thought of as a neighborhood inside the H-category (see Fig. 1, where the H-categories $\mathsf{H}_{\boldsymbol{M}}^{\Gamma}$ and $\mathsf{H}_{\boldsymbol{M}}$ are represented by graphs in order to make their orientations explicit). The condition on the inclusion functor given in the definition is a condition that all edges (or morphisms) are pointing into the neighborhood.

Definition 6. *A trajectory of a hybrid system* $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$*:*

$$(\boldsymbol{C}, \boldsymbol{c}) : (\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt}) \to (\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$$

is Zeno if $\Lambda = \mathsf{Ob}_{(\to\cdot\leftarrow)}(\mathsf{H}_{\boldsymbol{I}}) = \mathbb{N}$ *and*

$$\lim_{j\to\infty} \tau_j = \tau_\infty$$

for a finite τ_∞.

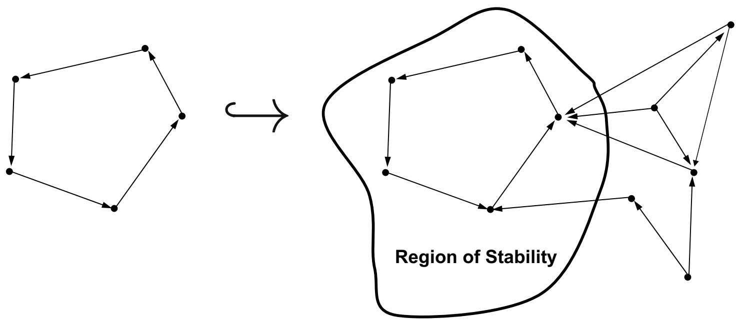


Fig. 1. A graphical representation of the "local" nature of relatively globally asymptotically stable Zeno equilibria

Zeno equilibria are intimately related to Zeno behavior for first quadrant hybrid systems.

Proposition 1. *If a first quadrant hybrid system* $\mathscr{H}_{\mathbf{FQ}}$ *is globally asymptotically stable at the Zeno equilibria* $\boldsymbol{z} = \{0\}$*, then every trajectory with* $\Lambda = \mathbb{N}$ *is Zeno.*

Conditions for the Stability of $\mathscr{H}_{\mathbf{FQ}}$. In order to give conditions on the stability of Zeno equilibria, it is necessary to give conditions on both the continuous and discrete portions of the hybrid system. That is, the conditions on stability will relate to three aspects of the behavior of the hybrid system: the continuous portion, the existence of events and the discrete portion.

Continuous conditions: For all $i \in Q$,

(**I**) $f_i(x) \neq 0$ for all $x \in (\mathbb{R}_0^+)^2$.

(**II**) There exists a function $v_i : (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+$ of class $\mathcal{K}_\infty$ along each ray emanating from the origin in D_i and $dv_i(x)f_i(x) \leq 0$ for all $x \in (\mathbb{R}_0^+)^2$.

Event conditions: For all $i \in Q$,

(**III**) $(f_i(x_1, 0))_2 \geq 0$.

Now consider the map ψ_i defined by requiring that:

$$\psi_i(x) = y \quad \text{if} \quad (0, y) = v_i^{-1}(v_i(x, 0)) \cap \{x_1 = 0 \ \text{ and } \ x_2 \geq 0\}$$

which is well-defined by condition (**II**). Using ψ_i we introduce the function $P_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by:

$$P_i(x) = r_{e_{i-1}} \circ \psi_{i-1} \circ \cdots \circ r_{e_1} \circ \psi_{e_1} \circ r_{e_k} \circ \psi_{e_k} \circ \cdots \circ r_{e_1} \circ \psi_1(x).$$

The map P_i can be thought of as both a Poincaré map or a discrete Lyapunov function depending on the perspective taken. The final conditions are given by:

Discrete conditions: For all $i \in Q$ and $e \in E$,

(**IV**) r_e is order preserving.

(**V**) There exists a class $\mathcal{K}_\infty$ function α such that $P_i(x) - x \leq -\alpha(x)$.

Theorem 3. *A first quadrant hybrid system $\mathscr{H}_{\mathbf{FQ}}$ is globally asymptotically stable at the Zeno equilibria $\boldsymbol{z} = \{0\}$ if conditions $(\mathbf{I}) - (\mathbf{V})$ hold.*

Corollary 1. *If $\mathscr{H}_{\mathbf{FQ}}$ is a first quadrant hybrid system satisfying conditions $(\mathbf{I}) - (\mathbf{V})$, then there exist trajectories with $\Lambda = \mathbb{N}$ and every such trajectory is Zeno.*

Note that the condition that $\Lambda = \mathbb{N}$ in Proposition 1 and Corollary 1 is due to the fact that there always are trajectories with finite indexing set Λ, e.g., any trajectory with $\Lambda = \mathbb{N}$ has "sub-trajectories" with finite indexing sets. These trajectories are trivially non-Zeno, so we necessarily rule them out.

5 Hybrid Stability: A Categorical Approach

Building upon the results of the previous section, we are able to derive sufficient conditions for the stability of general hybrid systems. Mirroring the continuous case, we simply find a morphism to the "simplest stable object," i.e., a first quadrant hybrid system.

Theorem 4. *A Zeno equilibria $(\mathsf{H}_\Gamma, \boldsymbol{z})$ of $(\mathsf{H}_{\boldsymbol{M}}, \boldsymbol{M}, \boldsymbol{X})$ is globally asymptotically stable relative to $(\mathsf{H}_\Gamma, \boldsymbol{M}^\Gamma, \boldsymbol{X}^\Gamma)$ if there exists a morphism of hybrid systems:*

$$(\mathsf{H}_\Gamma, \boldsymbol{M}^\Gamma, \boldsymbol{X}^\Gamma) \xrightarrow{(\boldsymbol{V}, \boldsymbol{v})} (\mathsf{H}^{\mathbf{SFQ}}, \boldsymbol{M}^{\mathbf{SFQ}}, \boldsymbol{X}^{\mathbf{SFQ}})$$

where $(\mathsf{H}^{\mathbf{SFQ}}, \boldsymbol{M}^{\mathbf{SFQ}}, \boldsymbol{X}^{\mathbf{SFQ}})$ is the object of HySys *corresponding to a stable first quadrant hybrid system, and for all $a \in \mathsf{Ob}(\mathsf{H}_\Gamma)$ the following holds:*

1. *$\boldsymbol{v}_a(x) = 0$ implies $x = \boldsymbol{z}_a$,*
2. *$\boldsymbol{v}_a$ is a proper (radially unbounded) function.*

Furthermore, there exist trajectories

$$(\mathsf{H}_{\boldsymbol{I}}, \boldsymbol{I}, \boldsymbol{d/dt}) \xrightarrow{(\boldsymbol{C}, \boldsymbol{c})} (\mathsf{H}_\Gamma, \boldsymbol{M}^\Gamma, \boldsymbol{X}^\Gamma)$$

with $\Lambda = \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}_{\boldsymbol{I}}) = \mathbb{N}$ and every such trajectory is Zeno.

Example 6. The bouncing ball is the classical example of a hybrid system that is Zeno (cf. [6]). Although it is possible to show that the bouncing ball is Zeno by explicitly solving for the vector fields, we will demonstrate that it is Zeno by applying our results on the stability of Zeno equilibria. In order to do so, we can view the classical model of a bouncing ball as a first quadrant hybrid system by adding an additional discrete mode; we then will apply Theorem 3.

The classical hybrid model for the bouncing ball has $(\{q\}, \{e = (q, q)\})$ as its graph. The domain is given by the set of positive positions:

$$D_q = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$$

and the guard is given by the ground together with the condition that the velocity is not positive:

$$G_e = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \le 0\}.$$

The equations of motion for the bouncing ball are given by the Hamiltonian

$$H(x_1, x_2) = \frac{1}{2}x_2^2 + mgx_1,$$

where x_1 is the position of the ball and x_2 is its velocity; here we have assumed that the mass of the ball is $m = 1$ for simplicity. This can be used (see [2]) to define both the vector field on D_q and the reset map $R_e : G_e \to D_q$:

$$f_q(x_1, x_2) = \begin{pmatrix} x_2 \\ -g \end{pmatrix}, \qquad R_e(x_1, x_2) = \begin{pmatrix} x_1 \\ -ex_2 \end{pmatrix},$$

where $0 \le e \le 1$ is the coefficient of restitution.

The bouncing ball can be viewed as a first quadrant hybrid system $\mathscr{H}_B = (\Gamma, D, G, R, F)$ by dividing the original domain into two components, and changing the vector fields accordingly. We first define

$$\Gamma = (\{1, 2\}, \{e_1 = (1, 2), e_2 = (2, 1)\}).$$

Since it is a first quadrant hybrid system, the domains and guards are given as in Section 3. The domain D_1 is obtained from the top half of the original domain for the bouncing ball by reflecting it around the line $x_1 = x_2$. The domain D_2 is obtained from the bottom half of the original domain by reflecting it around the line $x_2 = 0$. This implies that the reset maps are given by:

$$R_{e_1}(x_1, x_2) = (x_2, x_1), \qquad R_{e_2}(x_1, x_2) = (ex_2, x_1).$$

Finally, the transformed vector fields are given by

$$f_1(x_1, x_2) = \begin{pmatrix} -g \\ x_1 \end{pmatrix}, \qquad f_2(x_1, x_2) = \begin{pmatrix} -x_2 \\ g \end{pmatrix}$$

as pictured in Fig. 2.

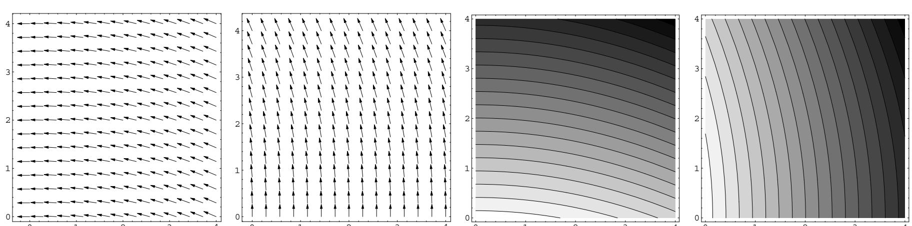


Fig. 2. (Left) Vector fields for the modified bouncing ball hybrid system. (Right) Level sets of the Lyapunov functions on each domain.

To verify that $\mathscr{H}_B$ is globally asymptotically stable at the Zeno point $\boldsymbol{z} = \{0\}$, and hence Zeno by Proposition 1, we need only show that conditions (**I**)−(**V**) are satisfied. It is easy to see that conditions (**I**) and (**III**) are satisfied. Since $r_{e_1}(x) = x$ and $r_{e_2}(x) = ex$, condition (**IV**) holds. We use the original Hamiltonian, suitably transformed, for the Lyapunov type functions given in (**II**), i.e., we pick:

$$v_1(x_1, x_2) = \frac{1}{2}x_1^2 + gx_2, \qquad v_2(x_1, x_2) = \frac{1}{2}x_2^2 + gx_1.$$

It is easy to see that these functions meet the specifications given in (**II**); some of the level sets of these functions can be seen in Fig. 2. Note that the level sets on one domain increase, but this is compensated for by the decreasing level sets on the other domain. Finally, condition (**V**) is satisfied when $e < 1$ since $P_1(x) = P_2(x) = ex$.

References

1. Ames, A.D., Zheng, H., Gregg, R.D., Sastry, S.: Is there life after Zeno? Taking executions past the breaking (Zeno) point. (Submitted to the 2006 American Control Conference)
2. Brogliato, B.: Nonsmooth Mechanics. Springer-Verlag (1999)
3. Ames, A.D., Abate, A., Sastry, S.: Sufficient conditions for the existence of Zeno behavior. 44th IEEE Conference on Decision and Control and European Control Conference ECC (2005)
4. Branicky, M.S.: Stability of hybrid systems: State of the art. In: Proceedings of the 36th IEEE Conference on Decision and Control, San Diego, CA (1997)
5. Heymann, M., Lin, F., Meyer, G., Resmerita, S.: Analysis of Zeno behaviors in hybrid systems. In: Proceedings of the 41st IEEE Conference on Decision and Control, Las Vagas, NV (2002)
6. Johansson, K.H., Lygeros, J., Sastry, S., Egerstedt, M.: Simulation of Zeno hybrid automata. In: Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, AZ (1999)
7. Zhang, J., Johansson, K.H., Lygeros, J., Sastry, S.: Zeno hybrid systems. Int. J. Robust and Nonlinear Control **11**(2) (2001) 435–451
8. Zheng, H., Lee, E.A., Ames, A.D.: Beyond Zeno: Get on with it! (To appear in Hybrid Systems: Computation and Control, 2006)
9. Ames, A.D., Sastry, S.: A homology theory for hybrid systems: Hybrid homology. In Morari, M., Thiele, L., eds.: Hybrid Systems: Computation and Control. Volume 3414 of Lecture Notes in Computer Science., Springer-Verlag (2005) 86–102
10. Ames, A.D., Sangiovanni-Vincentelli, A., Sastry., S.: Homogenous semantic preserving deployments of heterogenous networks of embedded systems. In: Workshop on Networked Embedded Sensing and Control, Notre Dame, IN (2005)
11. Lane, S.M.: Categories for the Working Mathematician. second edn. Volume 5 of Graduate Texts in Mathematics. Springer (1998)
12. Ames, A.D., Tabuada, P., Sastry, S.: H-categories and graphs. (Technical Note)