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Fundamental Study

Bisimulation relations for dynamical, control, and hybrid systems

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Abstract

The fundamental notion of bisimulation equivalence for concurrent processes, has escaped the world of continuous, and subsequently, hybrid systems. Inspired by the categorical framework of Joyal, Nielsen and Winskel, we develop novel notions of bisimulation equivalence for dynamical systems as well as control systems. We prove that these notions can be captured by the abstract notion of bisimulation as developed by Joyal, Nielsen and Winskel. This is the first unified notion of system equivalence that transcends discrete and continuous systems. Furthermore, this enables the development of a novel and natural notion of bisimulation for hybrid systems, which is the final goal of this paper.

Keywords: Bisimulation; Open maps; Dynamical systems; Control systems; Hybrid systems

1. Introduction

Embedded computing devices have fostered the paradigm of digital programs interacting with an analog world. Examples include portable accessories such as mobile phones

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and PDAs; medical equipment such as defibrillators, dialysis machines and MRIs among many other systems. These embedded computing devices interact with the continuous environment reacting to external stimuli while regulating the behavior of several continuous processes. Hybrid systems have recently emerged as a mathematical model for embedded computing devices interacting with the continuous environment, see for example [2,3,24] for an introduction to hybrid systems. The interaction between discrete and continuous components creates enormous difficulties in the analysis and design of this class of complex engineered systems. In particular, a major challenge in the research area of hybrid systems is how to define notions of equivalence enabling the development of compositional analysis and design techniques.

Bisimulation is a notion of system equivalence that has become one of the primary tools in the analysis of concurrent processes. When two concurrent systems are bisimilar, known properties are readily transferred from one system to the other. For purely discrete systems these problems are now reasonably well understood and for every notion of concurrency or process algebra there has been a different notion of bisimulation and frequently several competing notions. In [12], Joyal, Nielsen and Winskel proposed the notion of *span of open maps* in an attempt to understand the various equivalence notions for concurrency in an abstract categorical setting. They also showed that this abstract definition of bisimilarity captures the strong bisimulation relation of Milner [19]. Subsequently in [7] it was shown that abstract bisimilarity can also capture Hennessy's testing equivalences [9], Milner and Sangiorgi's barbed bisimulation [20] and Larsen and Skou's probabilistic bisimulation [16]. More recently, in [4], a bisimulation relation for Markov processes on Polish spaces was formulated in this categorical framework, extending the work of Larsen and Skou. Other attempts to formulate the notion of bisimulation in categorical language, include the coalgebraic approach of [11,23]. We will further discuss these methods in Section 7 where we compare our approach to those in the literature.

Despite the plethora of bisimulation notions in concurrency, the notion of bisimulation has escaped the world of continuous and dynamical systems, as noted in [29,28]. Furthermore, the lack of bisimulation notions for continuous systems has impeded developing bisimulation equivalence for hybrid systems. Inspired by the abstract framework in [12], in this paper we transcend from the discrete to the continuous world and develop novel notions of bisimulation equivalence for dynamical systems, control systems, and subsequently hybrid systems.

Despite the existence of traditional notions of equivalence in dynamical systems and control theory [13], the notion of bisimulation offers two novelties even in the more traditional setting of continuous systems. Dynamical systems are deterministic systems for which bisimulation equivalence is equivalent to trajectory equivalence. For control systems, however, one can think of the control input as producing nondeterministic system behavior, and therefore bisimulation equivalence is a finer notion of equivalence for nondeterministic dynamical systems than trajectory equivalence. Furthermore, system equivalence by bisimulation relation is a notion of equivalence that does not require control systems to be of minimal dimension or even of the same dimension.

There has been very recent work by the second and the third authors, characterizing the notion of bisimulation for dynamical and control systems in a functional setting, that is, the bisimulation relation is a functional relation [21,27]. In [8], we have extended this notion to relational setting and further have shown that this equivalence relation is captured by the abstract bisimulation relation of [12]. In this paper, we also develop novel and natural notions of bisimulation for hybrid systems, and show that this notion is also captured in the framework of [12]. In addition to providing novel notions of system equivalence for dynamical and control systems, unifying the notion of bisimulation across discrete and continuous domains, our results also extend the applicability of the categorical framework to the domain of hybrid dynamical systems. This completes our program of unifying bisimulation notions for discrete, continuous, and hybrid systems.

Our choice to work with path objects and path categories à la Joyal, Nielsen and Winskel is due to the fact that in this approach, the flow of the system is made explicit and the notion of abstract bisimulation has the idea of paths and trajectories built into the definition through the **P**-open maps. We have found this approach very beneficial in trying to formulate a notion of bisimulation for dynamical and especially for hybrid systems where it provided us with an idea as to what the abstract notion of time should be for a hybrid system. The approach of **P**-open maps generalizes from the context of labeled transition systems, where they were first introduced, to that of dynamical, control and hybrid systems.

The rest of the paper is organized as follows: In Section 2, we briefly review the abstract formulation of the notion of bisimilarity as developed in [12]. Section 3 provides the main application of this method in concurrency theory and recalls that the abstract bisimilarity captures Milner's strong bisimulation relation. Section 4 reviews our recently developed notions of bisimulation for dynamical systems and Section 5 does the same for control systems. The main results of the paper are contained in Section 6 where we introduce and discuss bisimulation relations for hybrid systems. Section 7 briefly reviews the coalgebraic approach to bisimulation and discusses the reasons for our choice of working within the framework of [12]. We also review some other categorical approaches to the modeling of hybrid systems and compare those to our models. Finally in Section 8 we conclude our study while presenting some future research directions. Given that the sections on dynamical, control and hybrid systems use definitions and facts from differential geometry, we have included an appendix that reviews as much of this background material as we need to develop our work.

2. Bisimulation and open maps

The notion of bisimilarity, as defined in [19], has turned out to be one of the most fundamental notions of operational equivalences in the field of process algebras. This has inspired a great amount of research on various notions of bisimulation for a variety of concurrency models. In order to unify most of these notions, Joyal et al. gave in [12] an abstract formulation of bisimulation in a category theoretical setting.

The approach of [12] introduces a category of models where the objects are the systems in question, and the morphisms are simulations. More precisely, it consists of the following components:

- *Model category*: The category **M** of *models*, with objects the systems being studied, and morphisms $f : X \to Y$ in **M**, that should be thought of as a simulation of system *X* in system *Y*.

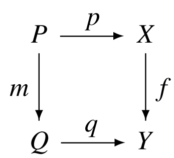
- *Path category*: The category **P**, a subcategory of **M**, of *path objects*, with morphisms expressing path extensions.

The path category will serve as an abstract notion of time. Since the path category **P** is a subcategory of the category **M** of models, time is thus modeled as a (possibly trivial) system within the same category **M** of models. This allows the unification of notions of time across discrete and continuous domains.

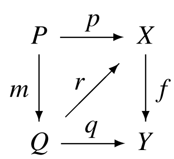
Definition 1. A *path* or *trajectory* in an object X of **M** is a morphism $p : P \to X$ in **M** where P is an object in **P**.

Let $f : X \to Y$ be a morphism in **M**, and $p : P \to X$ be a path in X, then clearly $f \circ p : P \to Y$ is a path in Y. Note that a path is a morphism in **M** and so is the map f and hence $f \circ p$ is a map in **M**. This is the sense in which Y *simulates* X; any path (trajectory) p in X is matched by the path $f \circ p$ in Y.

The abstract notion of bisimulation in [12] demands a slightly stronger version of simulation as follows: Let $m : P \to Q$ be a morphism in **P** and let the diagram



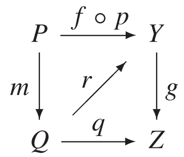
commute in **M**, i.e., the path $f \circ p$ in Y can be extended via m to a path q in Y. Then we require that there exist $r : Q \to X$ such that in the diagram



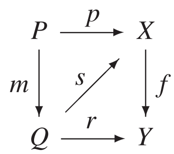
both triangles commute. Note that this means that the path p can be extended via m to a path r in X which matches q. In this case, we say that $f : X \to Y$ is **P**-*open*. It can be shown that **P**-open maps form a subcategory of **M**.

Proposition 2. *Let* **M** *be a category and* **P** *be the subcategory of path objects. Then,* **P**-*open maps in* **M** *form a subcategory of* **M**.

Proof. Let X be an object in **M**, we first show that $id_X : X \to X$ is a **P**-open map. Let $p : P \to X$ and $q : Q \to X$ and $m : P \to Q$, where P and Q are path objects in **P**. Assume also that $id_X p = qm$. Then let $r = q : Q \to X$: $id_X r = id_X q = q$ and $qm = p$. Now suppose, $f : X \to Y$ and $g : Y \to Z$ are **P**-open maps, let $p : P \to X$ and $q : Q \to Z$, and $m : P \to Q$. Also assume that $(gf)p = qm$. As $g : Y \to Z$ is a **P**-open map, there exists an $r : Q \to Y$ such that the triangles in the following diagram commute:

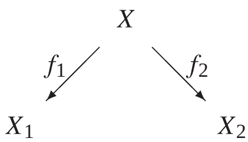


and as $f : X \to Y$ is **P**-open, there exists a map $s : Q \to X$ making the triangles in the following diagram commute:

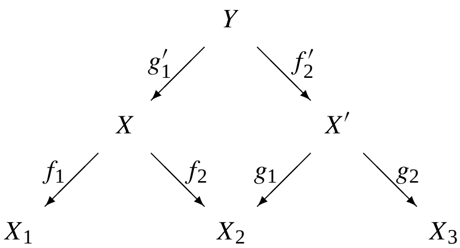


Now $(gf)s = g(fs) = gr = q$, using the second and the first diagrams for the last two equalities, respectively. Also $sm = p$ from the second diagram above. □

The definition of **P**-open maps leads to the notion of **P**-bisimilarity. We say that objects X_1 and X_2 of **M** are **P**-*bisimilar*, denoted $X_1 \sim_{\mathbf{P}} X_2$ iff there is a span (X, f_1, f_2) of **P**-open maps as shown below:



The relation of **P**-bisimilarity between objects is clearly reflexive (identities are **P**-open) and symmetric. It is also transitive *provided the model category* **M** *has pullbacks*, due to the fact that pullbacks of **P**-open morphisms are **P**-open (see [12] for a proof). Indeed suppose $X_1 \sim_{\mathbf{P}} X_2$ and $X_2 \sim_{\mathbf{P}} X_3$, then $X_1 \sim_{\mathbf{P}} X_3$ as can be seen from the following diagram.



Note that given X_1 and X_2 in **M**, if there exists a **P**-open morphism $f : X_1 \to X_2$, or a **P**-open morphism $g : X_2 \to X_1$, then X_1 and X_2 are **P**-bisimilar. The spans are (X_1, id_{X_1}, f) and (X_2, g, id_{X_2}), respectively.

Not all model categories that we consider have pullbacks of all morphisms. In particular the category of smooth manifolds and smooth mappings does not have pullbacks of *all* morphisms. We discuss the solution to this problem in the sections below.

3. Labelled transition systems

We briefly recall the definitions and results in [12] for labeled transition systems. We will also refer to these definitions and results later, when we discuss hybrid dynamical systems.

Definition 3. A *labeled transition system* $T = (S, i, L, \rightarrow)$ consists of the following:
- A set S of states with a distinguished state $i \in S$ called the *initial state*. Note that we do not require S be finite.
- A set L of labels.
- A ternary relation $\rightarrow \subseteq S \times L \times S$.

The model category $\mathbf{T}$, of transition systems has labeled transition systems as objects and a morphism $f : T_1 \rightarrow T_2$ with $T_1 = (S_1, i_1, L_1, \rightarrow_1)$ and $T_2 = (S_2, i_2, L_2, \rightarrow_2)$ is given by $f = (\sigma, \lambda)$ where $\sigma : S_1 \rightarrow S_2$ with $\sigma(i_1) = i_2$ and $\lambda : L_1 \rightarrow L_2$ is a partial function such that
(1) $(s, a, s') \in \rightarrow_1$ and $\lambda(a)$ defined, implies $(\sigma(s), \lambda(a), \sigma(s')) \in \rightarrow_2$ and
(2) $(s, a, s') \in \rightarrow_1$ and $\lambda(a)$ undefined, implies $\sigma(s) = \sigma(s')$.
In order to discuss the usual bisimilarity of transition systems we need to restrict our model category to the subcategory $\mathbf{T}_L$ of transition systems with the same label set L and morphisms of the form $f = (\sigma, id_L)$ which preserve all the labels. The category $\mathbf{T}_L$ has both binary products and pullbacks [12].

Definition 4. Given transition systems $T_1 = (S_1, i_1, L, \rightarrow_1)$ and $T_2 = (S_2, i_2, L, \rightarrow_2)$ in $\mathbf{T}_L$ we define their product $T = (S, i, L, \rightarrow)$ as follows:
- $S = S_1 \times S_2$ with projections $\rho_1 : S \rightarrow S_1$ and $\rho_2 : S \rightarrow S_2$,
- $i = (i_1, i_2)$,
- $((s_1, s_2), a, (s_1', s_2') \in \rightarrow$ iff $(s_1, a, s_1') \in \rightarrow_1$ and $(s_2, a, s_2') \in \rightarrow_2$.

It is straightforward to show that $(T, (\rho_1, id_L), (\rho_2, id_L))$ is a product in the category $\mathbf{T}_L$.

Definition 5. Given $f_1 = (\sigma_1, id_L) : T_1 \rightarrow U$ and $f_2 = (\sigma_2, id_L) : T_2 \rightarrow U$ morphisms in $\mathbf{T}_L$ with $T_1 = (S_1, i_1, L, \rightarrow_1)$ and $T_2 = (S_2, i_2, L, \rightarrow_2)$. We define the pullback of f_1 and f_2 as (T, f_1', f_2') with $f_1' : T \rightarrow T_2$, $f_2' : T \rightarrow T_1$ as follows:
- $T = (S, i, L, \rightarrow)$ where,
  - $S = \{(s_1, s_2) \mid \sigma_1(s_1) = \sigma_2(s_2)\} \subseteq S_1 \times S_2$,
  - $i = (i_1, i_2)$,
  - $((s_1, s_2), a, (s_1', s_2')) \in \rightarrow$ iff $(s_1, a, s_1') \in \rightarrow_1$ and $(s_2, a, s_2') \in \rightarrow_2$
- $f_1' = (\rho_2, id_L)$ where $\rho_2 : S \rightarrow S_2$ is the projection map.
- $f_2' = (\rho_1, id_L)$ where $\rho_1 : S \rightarrow S_1$ is the projection map.

We define the path category $\mathbf{Bran}_L$ as the full subcategory of $\mathbf{T}_L$ of all synchronization trees with a single finite branch (possibly empty). Now a path in a transition system T in $\mathbf{T}_L$ is a morphism $p : P \rightarrow T$ in $\mathbf{T}_L$, with P an object in $\mathbf{Bran}_L$. Clearly this simply means that we look at the traces of the transition system. The $\mathbf{Bran}_L$-open maps in $\mathbf{T}_L$ are characterized as follows:

Proposition 6. *The* $\mathbf{Bran}_L$*-open morphisms of* $\mathbf{T}_L$ *are morphisms* $(\sigma, id_L) : T \to T'$ *with* $T, T' \in \mathbf{T}_L$ *such that*:

If $\sigma(s) \xrightarrow{a} s'$ *in* T', *then there exists* $u \in S$, $s \xrightarrow{a} u$ *in* T *and* $\sigma(u) = s'$.

We now recall the strong notion of bisimulation introduced in [19]. Let T_1 and T_2 be two transition systems in $\mathbf{T}_L$, as in Definition 5 above.

Definition 7. A binary relation $\mathcal{R} \subseteq S_1 \times S_2$ is a *strong bisimulation* if $(s, t) \in \mathcal{R}$ implies, for all $a \in L$:

(1) Whenever $s \stackrel{a}{\to}_1 s'$ then, there is t', $t \stackrel{a}{\to}_2 t'$ and $(s', t') \in \mathcal{R}$,

(2) Whenever $t \stackrel{a}{\to}_2 t'$ then, there is s', $s \stackrel{a}{\to}_1 s'$ and $(s', t') \in \mathcal{R}$.

Transition systems T_1 and T_2 are called strongly bisimilar, written $T_1 \sim T_2$, if $(i_1, i_2) \in \mathcal{R}$ for some strong bisimulation relation $\mathcal{R}$. The following theorem, proven in [12], shows that the abstract notion of $\mathbf{Bran}_L$-bisimilarity coincides with the traditional notion of strong bisimulation.

Theorem 8 (*Joyal et al. [12]*). *Two transition systems* (*hence synchronization trees*) *over the same labeling set L, are* $\mathbf{Bran}_L$*-bisimilar iff they are* strongly bisimilar *in the sense of Milner* [19].

In the next sections, we consider the notion of **P**-bisimilarity in the categories of dynamical, control, and hybrid systems.

4. Dynamical systems

The material in this and the subsequent sections require some background knowledge on differential geometry that we have included in the Appendix for the convenience of the reader.

We begin with a motivating example. Suppose we would like to describe the evolution of the temperature inside a car in a cold winter day when we need the heating system turned on. If we denote by *x* the temperature inside the car and by *y* the temperature outside, it is natural to assume that, since $x > y$, the interior of the car will cool down until reaching the outside temperature. Such decrease is described by the derivative $\frac{\mathrm{d}}{\mathrm{d}t}x(t)$ of temperature $x(t)$ which can be described by

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = c(y - x(t)), \tag{1}$$

where *c* is a positive coefficient describing how well the car is thermally isolated from the outside. This decrease can, however, be balanced by the car heating system. If heat is produced at rate *u* we can modify (1) to account for the produced heat resulting in the differential equation:

$$\mathrm{d}x(t)/\mathrm{d}t = c(y - x(t)) + u. \tag{2}$$

This is an example of a dynamical system $X : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ with $X(x) = (x, c(y - x) + u)$. Given a value for the temperature $x(0)$ inside the car at time $t = 0$, Eq. (2) completely defines the value of the temperature $x(t)$ for all future times $t \in \mathbb{R}$.

A dynamical system or vector field on a manifold M is a smooth section of the tangent bundle on M, that is, a smooth map $X : M \to TM$ such that $\pi_M X = id_M$ where $\pi_M : TM \to M$ is the canonical projection of the tangent bundle onto the manifold M.

We proceed to define the model category **Dyn** of dynamical systems. The objects in **Dyn** are dynamical systems $X : M \to TM$ where M is a smooth manifold. A morphism in **Dyn** from object $X : M \to TM$ to object $Y : N \to TN$ is a smooth map $f : M \to N$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \mathrm{X}\downarrow & & \downarrow\mathrm{Y} \\ TM & \xrightarrow{Tf} & TN \end{array}$$

commutes. Thus related systems are said to be *f-related* [14]. The identity morphisms and composition are induced by those in the category **Man** of smooth manifolds and smooth mappings.

We proceed to define the path category **P** as the full subcategory of **Dyn** with objects $P : I \to TI$, where $P(t) = (t, 1)$ and I is an open interval of $\mathbb{R}$ containing the origin. Note that I is a manifold since it is an open set and it is also parallelizable (trivializable), that is, $TI \cong I \times \mathbb{R}$. Observe that P represents the differential equation $\mathrm{d}x(t)/\mathrm{d}t = 1$ modeling a clock running on the interval I at unit rate. Note that any other choice $P' : I \to TI$ with $P'(t) = (t, c)$, $0 \neq c \in \mathbb{R}$, for path object is isomorphic to $P : I' \to TI'$ via $f : P' \to P$ with $f(t) = tc$. Here $I' = \{t/c \mid t \in I\}$.

Definition 9. A *path* or *trajectory* in a dynamical system $X : M \to TM$ is a morphism $c : P \to X$ in **Dyn**, where P is an object in **P**. More explicitly, a path c is a map $c : I \to M$ such that the following diagram commutes.

$$\begin{array}{ccc} I & \xrightarrow{c} & M \\ P\downarrow & & \downarrow X \\ TI & \xrightarrow{Tc} & TM \end{array}$$

This means that a path in X is a smooth map $c : I \to M$ for some open interval I such that $c'(t) = X(c(t))$ for all $t \in I$. Thus, a path in X is just an integral curve in M. Observe that given a path c in X, and $f : X \to Y$, $f \circ c$ is a path in Y. This is the sense of *Y simulating or over-approximating X*.

The next issue to understand is the meaning of path extension. Suppose $P : I \to TI$ and $Q : J \to TJ$ are objects in **P** with I, J open intervals in $\mathbb{R}$ containing the origin, and $m : P \to Q$. Then, m is a smooth map from I to J, such that $m'(t) = 1$ or $m(t) = t - t_0$ for some $t_0 \in \mathbb{R}$ and for all $t \in I$.

We now introduce the following notation: let $\phi_X(x_1, x_2, t)$ denote the predicate that is true iff system X evolves from state x_1 to state x_2 in time $|t|$. Hence, $\phi_X(x_1, x_2, t)$ is true iff there is an open interval I in $\mathbb{R}$ containing the origin and an integral curve $c : I \to M$ such that $c(0) = x_1$ and $c(t) = x_2$. The following important result will be central to the characterization of **P**-open maps in **Dyn**.

Theorem 10 (*Boothby [5]*). *Let X be a smooth vector field on a manifold M and suppose $p \in M$. Then there is a uniquely determined open interval of $\mathbb{R}$, $I(p) = (\alpha(p), \beta(p))$ containing $t = 0$ and having the properties*:

(1) *there exists a smooth integral curve $F(t)$ defined on $I(p)$ and such that $F(0) = p$*;
(2) *given any other integral curve $G(t)$ with $G(0) = p$, then the interval of definition of G is contained in $I(p)$ and $F(t) = G(t)$ on this interval.*

The characterization of **P**-open maps is given by the following proposition.

Proposition 11. *Given the dynamical systems X on M and Y on N, $f : X \to Y$ is* **P**-*open if and only if*

For any state x_1 of X ($x_1 \in M$) and $t \in \mathbb{R}$, if $\phi_Y(f(x_1), y_2, t)$, then there exists $x_2 \in M$ such that $\phi_X(x_1, x_2, t)$ where $y_2 = f(x_2)$.

Proof. Suppose $f : X \to Y$ is a **P**-open map and $\phi_Y(f(x_1), y_2, t)$. Then there exists a path $d_1 : J_1 \to N$ such that $d_1(0) = f(x_1)$ and $d_1(t) = y_2$. Then, by the existence and uniqueness theorem for vector fields there exists a path $d : J \to N$ with J maximal such that $d(0) = f(x_1)$ and thus $J_1 \subseteq J$ and $d_1(t) = d(t)$ for all $t \in J_1$. On the other hand, there is a path $c : I \to M$ with $c(0) = x_1$ for some open interval I of $\mathbb{R}$. Thus $fc(0) = f(x_1)$. By maximality, $I \subseteq J$ and $fc(t) = d(t)$ for all $t \in I$. Thus the following diagram (with i the inclusion map) commutes:

$$
\begin{array}{ccc}
I & \xrightarrow{c} & M \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\
J & \xrightarrow{d} & N
\end{array}
$$

The **P**-openness of *f*, then implies that there exists $r : J \to M$ such that $ri = c$ and $fr = d$. Hence we have $ri(0) = c(0) = x_1$ and $fr(t) = d(t) = y_2$. Let $x_2 = r(t)$, then clearly we have established $\phi_X(x_1, x_2, t)$.

Conversely, suppose that the condition of Proposition 11 holds and given $P, Q, m : P \to Q$, with $p : P \to X$ and $q : Q \to Y$, the equation $fp = qm$ holds. Note that as was observed earlier with $P : I \to TI$ and $Q : J \to TJ$, $m(t) = t - t_0$ for some $t_0 \in \mathbb{R}$. Consider the point $p(0) \in M$, by Theorem 10 there exists an integral curve $\tilde{r} : \tilde{I} \to M$ with $\tilde{I}$ maximal such that $\tilde{r}(0) = p(0)$. We will show that for every $t \in J$, $t + t_0 \in \tilde{I}$. Suppose there exists a $t \in J$ such that $t + t_0 \notin \tilde{I}$. Note that q is a **Dyn**-morphism, so we have $\phi_Y(q(-t_0), q(t), t_0 + t)$, but $\phi_Y(q(-t_0), q(t), t_0 + t) = \phi_Y(q(m(0)), q(t), t_0 + t) = \phi_Y(f(p(0)), q(t), t_0 + t)$ where the latter equality follows from assumption. Hence, there exists a point $x \in M$ such that $\phi_X(p(0), x, t_0 + t)$ with $f(x) = q(t)$. Hence, there exists an integral curve $c : I_c \to M$ with $c(0) = p(0)$ and $c(t + t_0) = x$, and $t + t_0 \in I_c \setminus \tilde{I}$

contradicting the maximality of $\tilde{I}$. Now define r by $r(t) = \tilde{r}(t + t_0)$ for all $t \in J$. Clearly r is a **Dyn**-morphism and is well defined. Now, $rm(0) = r(-t_0) = \tilde{r}(0) = p(0)$ and hence $rm = p$. On the other hand, $fr(-t_0) = f\tilde{r}(0) = fp(0) = qm(0) = q(-t_0)$ and hence $fr = q$.

Intuitively, this condition simply requires that $p(t)$ be extendible on both sides if necessary to a solution $r(t)$ of X that matches the solution q of Y, i.e., $f(r(t)) = q(t)$ for all $t \in J$. □

In the special case where vector fields are *complete*, that is solutions exist for all time (i.e., for all $t \in \mathbb{R}$), the previous proposition takes the following form.

Proposition 12. *Let X and Y be complete vector fields on manifolds M and N respectively. Then any $f : X \to Y$ is* **P**-*open.*

Proof. Note that for complete vector fields any integral curve is defined on the whole of $\mathbb{R}$. Suppose $p : P \to X$ and $q : Q \to Y$ are paths and that $fp = qm$. Recall that $m : P \to Q$ is given by $m(t) = t - t_0$ for some $t_0 \in \mathbb{R}$. Consider the point $p(0) \in M$, then by Theorem 10 and completeness of X, there exists an integral curve $d : \mathbb{R} \to M$ such that $d(0) = p(0)$, define $r : J \to M$ by $r(t) = d(t + t_0)$ for all $t \in J$. Clearly r is a **Dyn**-morphism. Now, $fr(-t_0) = fd(0) = fp(0) = qm(0) = q(-t_0)$ and hence $fr = q$. Similarly, $rm(0) = r(-t_0) = d(0) = p(0)$ and hence $rm = p$. □

Recall that by the general definition in Section 2, two objects X_1 and X_2 in the model category are **P**-bisimilar if there is a span of **P**-open maps, that is, an object X with **P**-open maps $f_1 : X \to X_1$ and $f_2 : X \to X_2$. The **P**-bisimulation relation has to be an equivalence relation and for that purpose one requires the existence of pullbacks in the underlying model category, to ensure transitivity. However, as it is well known in differential geometry [1,14], in the category **Man** of smooth manifolds and smooth mappings, arbitrary pullbacks do not exist. Structure needs to be imposed on the maps in order to guarantee that pullbacks exist.

Definition 13. Given smooth manifolds M and N, a smooth map $f : M \to N$ and $x \in M$, let $T_x f : T_x M \to T_{f(x)} N$ be the differential of f. We say that:
(i) f is an *immersion* at x if and only if the map $T_x f$ is injective.
(ii) f is a *submersion* at x if and only if the map $T_x f$ is surjective.

Definition 14. Let M, N be smooth manifolds and $f : M \to N$ be a smooth mapping and P be a submanifold of N. The map f is *transversal* on P iff for each $x \in M$ such that $f(x)$ lies in P, the composite

$$T_x(M) \xrightarrow{T_x f} T_{f(x)}(N) \to T_{f(x)}(N)/T_{f(x)}(P)$$

is surjective.

In particular, if for every $x \in M$, $T_x f$ is surjective, that is, if f is a submersion on M, then the composite in the definition above will be surjective and hence every submersion

$f : M \to N$ is transversal on every submanifold *P* of *N*. The importance of transversality is that one can prove submanifold property, that is, given $f : M \to N$ a smooth transversal map on a submanifold *P* of *N*, $f^{-1}(P)$ is a smooth submanifold of *M*.

Definition 15. Given smooth maps $f : M \to P$ and $g : N \to P$, we say that *f* and *g* are transversal if $f \times g : M \times N \to P \times P$ is transversal on the diagonal submanifold Δ_P of $P \times P$.

Proposition 16 (*Abraham et al. [1]*). *Let M and N be smooth manifolds and* $f : M \to N$ *a smooth map, then graph*(f) *is a smooth submanifold of* $M \times N$.

Proposition 17. *The category* **Man** *has transversal pullbacks.*

Proof. Suppose M, N, P are smooth manifolds and $f_1 : M \to P$ and $f_2 : N \to P$ are smooth transversal maps. Form the fiber product of *M* and *N* on *P*, denoted $M \times_P N = \{(x, y) \in M \times N \mid f_1(x) = f_2(y)\}$. As f_1 and f_2 are transversal, $(f_1 \times f_2)^{-1}\Delta_P = M \times_P N$ is a submanifold of $M \times N$, the smooth structure is induced by that of $M \times N$, for more details see [14]. The rest of the proof consists of checking the universal property of the pullback which follows from the set theoretical construction. □

Obviously transversality is a sufficient condition and hence there are other pullbacks in the category **Man**. In view of this proposition we have the following result.

Proposition 18. *Pullbacks of submersions exists in* **Man**. *Moreover, the pullback of any submersion is a submersion.*

Proof. First note that the transversality condition for a given $f_1 : M \to P$ and $f_2 : N \to P$ is equivalent to the following condition: for any $p \in P$ such that $p = f_1(x) = f_2(y)$ for some $x \in M$ and $y \in N$, $im(T_x f_1) + im(T_y f_2) = T_p P$ [14]. In other words, the tangent spaces on the left together must span the whole of $T_p P$. Now given that f_1 and f_2 are submersions, we conclude that $im(T_x f_1) = im(T_y f_2) = T_p P$ and hence transversality follows. To prove the second statement, recall that the pullback morphisms are projections restricted to $M \times_P N$, let $g_1 : M \times_P N \to N$ be the pullback of f_1 (see the diagram below), $T g_1 : T(M \times_P N) \cong TM \times_{TP} TN \to TN$. Given any $(x, y) \in M \times_P N$, $T_{(x,y)} g_1 : T_x M \times_{T_{f_1(x)} P} T_y N \to T_y N$ is surjective as f_1 is a submersion. Hence g_1 is a submersion.

$$\begin{array}{ccc} M \times_P N & \xrightarrow{g_1} & N \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle f_2} \\ M & \xrightarrow{f_1} & P \end{array}$$

□

After all these preliminary results in the category **Man** of manifolds, we can finally get to our desired goal in the category of dynamical systems.

Proposition 19. *The category* **Dyn** *has binary products and transversal pullbacks.*

Proof. Given the dynamical systems $X : M \to TM$ and $Y : N \to TN$, define $X \times Y : M \times N \to TM \times TN \cong T(M \times N)$ by $(X \times Y)(x, y) = (X(x), Y(y))$. The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are morphisms in **Dyn** as can be easily seen from the definition.

Let X, Y and Z be dynamical systems on the manifolds M, N, P respectively and $f_1 : X \to Z$ and $f_2 : Y \to Z$. By assumption the maps $f_1 : M \to P$ and $f_2 : N \to P$ are transversal, so $M \times_P N$ is a smooth submanifold of $M \times N$. We define the dynamical system $W : M \times_P N \to T(M \times_P N) \cong TM \times_{TP} TN$, denoted $X \times_P Y$ by $W = X \times Y|_{M \times_P N}$. For this definition to be well-defined one has to ensure that for every point $(x, y) \in M \times_P N$, $(X \times Y)(x, y) \in TM \times_{TP} TN$, in other words one has to show that the vector field $X \times Y$ is tangent to the submanifold $M \times_P N$. We proceed by proving the equivalent statement: for any $(x, y) \in M \times_P N$ the flow of (x, y) along $X \times Y$ at any time t (for which the flow is defined), denoted $Fl_t^{X \times Y}(x, y)$ is in $M \times_P N$.

$$
\begin{aligned}
&(Z \circ f_1)(x) = (Z \circ f_2)(y), && \text{as } (x, y) \in M \times_P N,\\
&T_x f_1 X(x) = T_y f_2 Y(y), && \text{as } f_1, f_2 \text{ are } \mathbf{Dyn}\text{-morphisms},\\
&(\mathcal{L}_X f_1)|_x = (\mathcal{L}_Y f_2)|_y, && \text{Lie derivative},\\
&f_1(Fl_t^X(x)) = f_2(Fl_t^Y(y)), && \text{by integration},\\
&Fl_t^{X \times Y}(x, y) \in M \times_P N, && \text{by definition}.
\end{aligned}
$$

The fact that $M \times_P N$ is a pullback in the category **Man** implies that W is a pullback in **Dyn**. □

In this case, as we have seen above, we can only guarantee the transversal pullbacks. Hence we modify the definition for **P**-bisimulation to ensure that it becomes an equivalence relation. That is, we require that there be a span of **P**-*open surjective submersions.*

Definition 20. We say that two dynamical systems X_1 and X_2 are **P**-bisimilar, denoted $X_1 \sim_{\mathbf{P}} X_2$, if there exists a span $(Z, f_1 : Z \to X_1, f_2 : Z \to X_2)$ of **P**-*open surjective submersions.*

Note that if there exists a **P**-open surjective submersion $f : X \to Y$, then $X \sim_{\mathbf{P}} Y$ with the span (X, id_X, f).

Proposition 21. *The relation of* **P**-*bisimilarity is an equivalence relation on the class of all dynamical systems.*

Proof. Reflexivity follows from the fact that id_X is a **P**-open surjective submersion for any dynamical system X. Symmetry is trivial. For transitivity, suppose that $X_1 \sim_{\mathbf{P}} X_2$ and $X_2 \sim_{\mathbf{P}} X_3$. Then, there are the spans $(Z_1, f_1 : Z_1 \to X_1, f_2 : Z_1 \to X_2)$ and $(Z_2 : g_1 : Z_2 \to X_2, g_2 : Z_2 \to X_3)$. The pullback of f_2 and g_1 exists as these are submersions, denote these pullbacks by f_2' and g_1', respectively. We also know that f_2' and g_1' are **P**-open surjective submersions, as pullback preserves all these properties. Moreover, the composition of **P**-open surjective submersions is a **P**-open surjective submersion. Thus we have the span of **P**-open surjective submersions $(Z, f_1g_1' : Z \to X_1, g_2f_2' : Z \to X_3)$ where Z is the vertex of the pullback square. $\square$

We proceed with a definition of bisimulation for dynamical systems, for this we need a notion of a well-behaved relation. We will show that bisimulation and **P**-bisimulation coincide. The following definition which seems to be new, is inspired by a relevant definition for equivalence relations on manifolds [1,25].

Definition 22. Let M and N be smooth manifolds and $\mathcal{R}$ be a relation from M to N, that is to say, $\mathcal{R} \subseteq M \times N$. We say that $\mathcal{R}$ is *regular* iff

- $\mathcal{R}$ is a smooth submanifold of $M \times N$,
- the projection maps $\pi_1 : \mathcal{R} \to M$ and $\pi_2 : \mathcal{R} \to N$ are surjective submersions.

Proposition 23. *Let M, N and P be smooth manifolds and $\mathcal{R} \subseteq M \times N$ and $\mathcal{S} \subseteq N \times P$ be regular relations. Then $\mathcal{S} \circ \mathcal{R} \subseteq M \times P$ is a regular relation.*

Proof. As $\mathcal{R}$ and $\mathcal{S}$ are regular relations the following pullback exists

$$\begin{array}{ccc} \mathcal{R} \times_N \mathcal{S} & \xrightarrow{f_2} & \mathcal{S} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \mathcal{R} & \xrightarrow{\pi_2} & N \end{array}$$

Note that $\mathcal{R} \times_N \mathcal{S} = \{(r, s) \mid \pi_1(s) = \pi_2(r)\} = \{(x, y, y', z) \mid y = y'\}$. Now consider $\mathcal{R} \times_N \mathcal{S} \xrightarrow{\pi_1 \times \pi_2} M \times P$, then $\mathcal{S} \circ \mathcal{R} = (\pi_1 \times \pi_2)(\mathcal{R} \times_N \mathcal{S})$. However, $\pi_1 \times \pi_2$ is a submersion and hence an open map. Thus $\mathcal{S} \circ \mathcal{R}$ is an open subset of $M \times P$ and so a smooth submanifold of $M \times P$. Furthermore, $\pi_1 : \mathcal{S} \circ \mathcal{R} \to M$ is given by $\mathcal{R} \times_N \mathcal{S} \xrightarrow{f_1} \mathcal{R} \xrightarrow{\pi_1} M$ which is a surjective submersion. Similarly for $\pi_2 : \mathcal{S} \circ \mathcal{R} \to P$. $\square$

Definition 24. Given two dynamical systems X on M and Y on N, we say that a relation $\mathcal{R} \subseteq M \times N$ is a *bisimulation* relation iff

(1) $\mathcal{R}$ is a regular relation,
(2) for all $(x, y) \in M \times N$, $(x, y) \in \mathcal{R}$ implies for all $t \in \mathbb{R}$,
  - if $\phi_X(x, x', t)$, there exists $y' \in N$ such that $\phi_Y(y, y', t)$ and $(x', y') \in \mathcal{R}$,
  - if $\phi_Y(y, y', t)$, there exists $x' \in M$ such that $\phi_X(x, x', t)$ and $(x', y') \in \mathcal{R}$.

We say that two dynamical systems X and Y on manifolds M and N, respectively are *bisimilar* if there exists a bisimulation relation $\mathcal{R} \subseteq M \times N$.

Theorem 25. *Given dynamical systems X and Y on manifolds M and N respectively, X and Y are bisimilar iff they are* **P***-bisimilar*.

Proof. Suppose that $X \sim_{\mathbf{P}} Y$ and $(Z, f : Z \to X, g : Z \to Y)$ is the span where $Z : P \to TP$. Note that *graph*$(f) \subseteq P \times M$ and *graph*$(g) \subseteq P \times N$ are regular relations. Consider the converse relation $\overline{graph(f)}$ and let $\mathcal{R} = graph(g) \circ \overline{graph(f)}$. It can be shown that $\overline{graph(f)}$ is regular. Also, note that by the proposition above, $\mathcal{R}$ is regular. Let $(x, y) \in \mathcal{R}$ and $\phi_X(x, x', t)$, then there exists a $z \in P$ such that $(x, z) \in \overline{graph(f)}$ and $(z, y) \in graph(g)$, so $x = f(z)$. As f is a **P**-open map, then there exist $z' \in P$ such that $\phi_Z(z, z', t)$ and $f(z') = x'$, i.e. $(z', x') \in graph(f)$. Let $y' = g(z')$, then $\phi_Y(g(z), g(z'), t) = \phi_Y(y, y', t)$ and $(x', y') \in \mathcal{R}$. Similarly, the other bisimilarity condition is satisfied.

Conversely, suppose that X and Y are bisimilar and $\mathcal{R}$ is the bisimulation relation. As $\mathcal{R}$ is regular, it is a smooth manifold. Consider the dynamical system $Z : \mathcal{R} \to T\mathcal{R}$ defined by $Z = (X \times Y)|_{\mathcal{R}}$. Note that as in Proposition 19 for Z to be well defined, one has to show that $X \times Y$ is tangent to the submanifold $\mathcal{R}$. We prove: for any point $(x, y) \in \mathcal{R}$, $Fl_t^{X\times Y}(x, y) = (Fl_t^X(x), Fl_t^Y(y)) \in \mathcal{R}$. Let $Fl_t^X(x) = x'$, then $\phi_X(x, x', t)$ and as $\mathcal{R}$ is a bisimulation relation, there exists y' such that $\phi_Y(y, y', t)$ and $(x', y') \in \mathcal{R}$, where $y' = Fl_t^Y(y)$. Also $\pi_1 : \mathcal{R} \to M$ is a surjective submersion, as $\mathcal{R}$ is regular. We need to show that π_1 is **P**-open. Let $\phi_X(\pi_1(x, y), x', t) = \phi_X(x, x', t)$, then there exists y' such that $\phi_Y(y, y', t)$ and $(x', y') \in \mathcal{R}$, so $\phi_Z((x, y), (x', y'), t)$ and $\pi_1(x', y') = x'$, so π_1 is **P**-open. Similarly for π_2 and hence $(Z, \pi_1 : Z \to X, \pi_2 : Z \to Y)$ is a span of **P**-open surjective submersions and hence $X \sim_{\mathbf{P}} Y$. $\square$

The above theorem shows that the abstract notion of **P**-bisimilarity coincides with the expected and natural notion of bisimulation for dynamical systems.

The following gives an example of two bisimilar dynamical systems.

Example 26. Consider the vector field X on $M = \mathbb{R}^2$ defined by $\dot{x} = Ax$, where

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}.$$

Since M is a Euclidean space we can make the identification $TM = \mathbb{R}^2 \times \mathbb{R}^2$ and X as a map from M to TM is then described by $X(x) = (x, Ax)$. Also consider the vector field Y on $N = \mathbb{R}$ defined by $\dot{y} = 5y$. The linear map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1 + x_2$ is a **Dyn**-morphism from X to Y, indeed:

$$TfX(x) = [1\ \ 1] \begin{bmatrix} x_1 + 3x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = 5x_1 + 5x_2 = 5(x_1 + x_2) = 5y = Y(f(x)).$$

As linear vector fields are known to be complete [5] we have by Proposition 12 that f is P-open. Note that f is a surjective submersion. It then follows that X and Y are bisimilar by the span $(X, id : X \to X, f : X \to Y)$.

We now turn our attention to control systems.

5. Control systems

In this section we extend the treatment in the previous section to control systems. The extensions are in many cases straightforward and hence we have omitted the proofs of some propositions and theorems. On the other hand, we give enough details on product and pullback constructions.

Before we proceed with the mathematical definitions, we shall motivate the idea of a control system. Recall the example of a dynamical system in Section 4 where we modeled the temperature change in a car. Assume now that we are inside the car and that we can change the rate at which heat is generated by the car's heating system. Having the possibility of changing the value of u leads us to regard u, not as a constant, but as an input allowing to alter the temperature evolution. Eq. (2), that we repeat here for convenience:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = c(x(t) - y) + u, \tag{3}$$

now defines a control system $X : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ with $X(x, u) = (x, c(y - x) + u)$. In this case, a value for the temperature at time $t = 0$ does not uniquely define its future values since by changing u over time we can alter the temperature evolution. When the heating system is automatic we do not need to play directly with the value of u and only have to specify a desired value for the temperature. An embedded system will then measure the temperature inside and outside the car and automatically adjust the value of u in order to reach the specified temperature as quickly as possible.

We define the model category **Con** as follows. Objects of **Con** are control systems over smooth manifolds, a control system X over a manifold M is given by a pair (U_M, X_M) where $X_M : M \times U_M \to TM$ is a smooth map such that $\pi_M X_M = \pi_1$ with π_M the canonical tangent bundle projection and $\pi_1 : M \times U_M \to M$, the first projection map. Here U_M is a smooth manifold called the *input space*. A morphism in **Con** from a control system $X = (U_M, X_M)$ to $Y = (U_N, Y_N)$ is given by a pair (ϕ_1, ϕ_2) of smooth maps with $\phi_1 : M \times U_M \to N \times U_N$ and $\phi_2 : M \to N$, such that

$$\begin{array}{ccc} M \times U_M & \xrightarrow{\phi_1} & N \times U_N \\ {\scriptstyle X_M}\downarrow & & \downarrow{\scriptstyle Y_N} \\ TM & \xrightarrow{T\phi_2} & TN \end{array} \qquad \begin{array}{ccc} M \times U_M & \xrightarrow{\phi_1} & N \times U_N \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow{\phi_2} & N \end{array}$$

both commute. Thus related control systems are said to be (ϕ_1, ϕ_2)-related [22]. Note that since π_1 is a surjective map, ϕ_2 is uniquely determined given ϕ_1. The identity morphism $id_X : X \to X$ for an object X in **Con** is given by $id_X = (id_{M\times U_M}, id_M)$. Given $f : X \to Y$ and $g : Y \to Z$, the composite $gf : X \to Z$ is given by $gf = (g_1 f_1, g_2 f_2)$.

The path category **P** is defined as the full subcategory of **Con** with objects, control systems (U_I, P_I) where U_I is the singleton space with trivial topology and thus $I \times U_I \cong I$ and I is an open interval of $\mathbb{R}$ containing the origin. Hence, $P_I : I \to TI$ which we define as $P(t) = (t, 1)$ for all $t \in I$. Thus (U_I, P_I) is a well-defined control system.

Definition 27. A *path* in a control system $X = (U_M, X_M)$ is then a morphism $c = (c_1, c_2) : (U_I, P_I) \to (U_M, X_M)$ in **Con** with $c_1 : I \to M \times U_M$ and $c_2 : I \to M$ such that the diagrams

$$\begin{array}{ccc} I & \xrightarrow{c_1} & M \times U_M \\ {\scriptstyle P_I}\downarrow & & \downarrow{\scriptstyle X_M} \\ TI & \xrightarrow{Tc_2} & TM \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{c_1} & M \times U_M \\ {\scriptstyle id_I}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ I & \xrightarrow{c_2} & M \end{array}$$

commute.

This means that a path in X is a pair of smooth maps $c_1 : I \to M \times U_M$ and $c_2 : I \to M$ for some open interval I with $0 \in I$ such that $c_2'(t) = X(c_2(t), u(t))$ for all $t \in I$, where $u(t) = \pi_2 c_1(t)$. Let (I, P_I) and (J, Q_J) be two path objects in **P** and $m = (m_1, m_2) : P \to Q$ be a path extension. Then from the diagram on the right above we get that $m_1 = m_2 : I \to J$ and then the diagram on the left coincides with the condition we had for dynamical systems. Thus a path extension $m = (m_1, m_2)$ is of the form $m_1 = m_2 : I \to J$, $m_1(t) = t - t_0$ for some $t_0 \in \mathbb{R}$ and for all $t \in I$.

Definition 28. Given control systems $X = (U_M, X_M)$, $Y = (U_N, Y_N)$ and $Z = (U_P, Z_P)$, $f = (f_1, f_2) : X \to Z$ and $g = (g_1, g_2) : Y \to Z$ are said to be *transversal* if $f_2 \times g_2 : M \times N \to P \times P$ is transversal on Δ_P and $f_1 \times g_1 : (M \times U_M) \times (N \times U_N) \to (P \times U_P) \times (P \times U_P)$ is transversal on $\Delta_{P \times U_P}$.

Proposition 29. *The category* **Con** *has binary products and transversal pullbacks.*

Proof. Let $X = (U_M, X_M)$ and $Y = (U_N, Y_N)$ be control systems on manifolds M and N, respectively. Their product $X \times Y = (U_M \times U_N, (X \times Y)_{M \times N})$ is given by

$(X \times Y)_{M \times N} := (M \times N) \times (U_M \times U_N) \xrightarrow{\cong} (M \times U_M) \times (N \times U_N) \xrightarrow{X_M \times Y_N} TM \times TN \xrightarrow{\cong} T(M \times N)$.

Suppose now that $f = (f_1, f_2) : X \to Z$ and $g = (g_1, g_2) : Y \to Z$ where $Z = (U_P, Z_P)$ is a control system on a smooth manifold P. The pullback of f and g is given by (Q, f', g') where Q is a control system on the manifold $M \times_P N$ with input space $U_M \times_P U_N := (\pi_2 \times \pi_2)((f_1 \times g_1)^{-1} \Delta_{P \times U_P}))$ which is a submanifold of $U_M \times U_N$ due to transversality of f_1 and g_1 and the fact that $\pi_2 \times \pi_2$ is an open map. The dynamics $X_M \times_P Y_N$ is defined by restricting $X_M \times Y_N$ to $(M \times_P N) \times (U_M \times_P U_N)$, see the proof of Proposition 19. $\square$

We introduce the following notation: let $\phi_X(x_1, x_2, t)$ denote the predicate that is true iff the control system $X = (U_M, X_M)$ evolves from state x_1 to state x_2 in time t, under some input in U_M. Hence, $\phi_X(x_1, x_2, t)$ is true iff there is an open interval I of $\mathbb{R}$ containing the origin, a morphism $c = (c_1, c_2) : (U_I, P_I) \to X$ such that $c_2(0) = x_1$ and $c_2(t) = x_2$. The input driving the system is given by $\pi_2 c_1 : I \to U_M$. Similarly to the case of dynamical systems, we characterize the **P**-open maps as follows.

Proposition 30. *Given the control systems* $X = (U_M, X_M)$ *and* $Y = (U_N, Y_N)$, $f = (f_1, f_2) : X \to Y$ *is* **P**-*open iff*

For any state x_1 of X ($x_1 \in M$) and $t \in \mathbb{R}$, if $\phi_Y(f_2(x_1), y_2, t)$, then there exists $x_2 \in M$ such that $\phi_X(x_1, x_2, t)$ where $y_2 = f_2(x_2)$.

Definition 31. Given control systems $X = (U_M, X_M)$ and $Y = (U_N, Y_N)$, a morphism $f : X \to Y$ is said to be a *surjective submersion* if both its components f_1 and f_2 are surjective submersions.

Definition 32. We say that two control systems X_1 and X_2 are **P**-bisimilar, denoted $X_1 \sim_{\mathbf{P}} X_2$, if there exists a span $(Z, f_1 : Z \to X_1, f_2 : Z \to X_2)$ of **P**-*open surjective submersions.*

Proposition 33. *The relation of* **P**-*bisimilarity is an equivalence relation on the class of all control systems.*

We define the bisimulation relation for control systems, similarly to the case of dynamical systems.

Definition 34. Given two control systems $X = (U_M, X_M)$ and $Y = (U_N, Y_N)$, we say that a relation $\mathcal{R} \subseteq M \times N$ is a *bisimulation* relation iff

(1) $\mathcal{R}$ is a regular relation,

(2) for all $(x, y) \in M \times N$, $(x, y) \in \mathcal{R}$ implies, for all $t \in \mathbb{R}$,
- if $\phi_X(x, x', t)$, there exists $y' \in N$ such that $\phi_Y(y, y', t)$ and $(x', y') \in \mathcal{R}$,
- if $\phi_Y(y, y', t)$, there exists $x' \in M$ such that $\phi_X(x, x', t)$ and $(x', y') \in \mathcal{R}$.

We say that two control systems X and Y as above are *bisimilar* if there exists a bisimulation relation $\mathcal{R} \subseteq M \times N$.

Theorem 35. *Given control systems $X = (U_M, X_M)$ and $Y = (U_N, Y_N)$, X and Y are bisimilar if and only if they are* **P**-*bisimilar*.

The above theorem, shows that the categorical notion of bisimulation described in Section 2, also captures the natural notion of bisimulation for control systems.

6. Hybrid systems

A hybrid system is just a family of smooth dynamical systems indexed over the states of an underlying labelled transition system. The dynamical systems are glued together by the transitions of the underlying labelled transition system.

Definition 36. A *hybrid* (*dynamical*) *system* H is a tuple

$$H = (S, i, L, \to, \{X_s\}_{s \in S}, \{Inv_s\}_{s \in S}, \{G_{s,a}\}_{s=src(a), a \in L}, \{R_{s,a}\}_{s=src(a), a \in L})$$

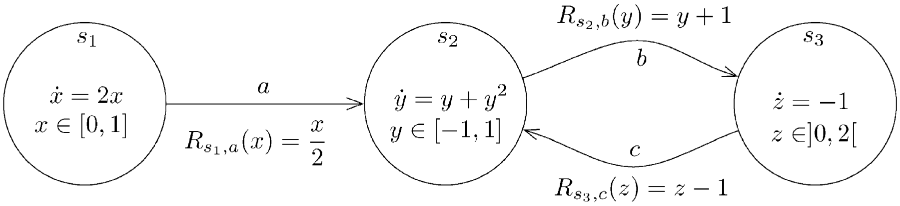


Fig. 1. Hybrid system *H*.

where:
- $(S, i, L, \rightarrow)$ is a labelled transition system,
- X_s is a smooth dynamical system $X_s : M_s \rightarrow TM_s$, for each $s \in S$, notice that we do not require that the dynamical systems be identical, nor do we require that the underlying manifolds be the same for all states $s \in S$,
- $Inv_s \subseteq M_s$, for each $s \in S$ is called *the invariant set* at state *s*, Inv_s is not required to be a submanifold,
- $G_{s,a} \subseteq Inv_s$ called the *guard* of the transition $a \in L$, for each $a \in L$, where *s* is the source of the action *a*, that is, there is $t \in S$ such that $(s, a, t) \in\rightarrow$.
- With $(s, a, t) \in\rightarrow$, $R_{s,a} : G_{s,a} \rightarrow Inv_t$ is a function, called the *reset* function.

Note that we have indexed the guard and the reset functions on a subset of $S \times L$ due to the fact that there might be two different edges with the same label *a* and different source states and these might very well have different guards and/or reset functions. On the other hand, identically labeled edges emerging from the same state will have identical guards and reset functions.

Example 37. We give an example of a hybrid system below, see Fig. 1. In this example $M_{s_i} = \mathbb{R}$ for $i = 1, 2, 3$ and guards are given by: $G_{s_1,a} = [1/2, 1]$, $G_{s_2,b} =]-1, 1[$ and $G_{s_3,c} = \{1/4\}$.

In order to simplify the notation we refer to the underlying transition system in a hybrid system *H*, by *T*. For a hybrid system as above, $T = (S, i, L, \rightarrow)$. We will also omit the index sets, as it will always be clear from the context. We assume that the underlying transition systems all have the same label set *L*, that is, *T* is an object in $\mathbf{T}_L$.

Given a hybrid system $H = (T, X_s, Inv_s, G_{s,a}, R_{s,a})$, the state space of *H* is defined by $Q = \{(s, x) \mid s \in S \text{ and } x \in Inv_s\} = \biguplus_{s \in S} Inv_s$. We next define a transition relation on a hybrid system as follows $\Rightarrow \subseteq Q \times (L \cup \{\tau_t\}_{t \in \mathbb{R}_0^+}) \times Q$. For $t \in \mathbb{R}_0^+$, $\tau_t \notin L$ are distinguished actions used to represent the continuous flow of the system. We let $(s, x) \stackrel{a}{\Rightarrow} (s', x')$ denote $((s, x), a, (s', x')) \in\Rightarrow$. Given states $(s, x), (s', x')$ in *Q*, $(s, x) \stackrel{a}{\Rightarrow} (s', x')$ iff either one of the following transitions takes place:

(1) *discrete transition* ($a \in L$): $s \stackrel{a}{\longrightarrow} s'$, i.e., *a* is a transition in *T*, and $x \in G_{s,a}$ and $x' = R_{s,a}(x)$. Note that $x \in M_s$ and $x' \in M_{s'}$ and M_s may be different from $M_{s'}$.
(2) *continuous transition* ($a = \tau_t, t \in \mathbb{R}_0^+$): $s = s'$ and $Fl_t^{X_s}(x) = x'$ and $Fl_{t'}^{X_s}(x) \in Inv_s$, for all $0 \leqslant t' \leqslant t$.

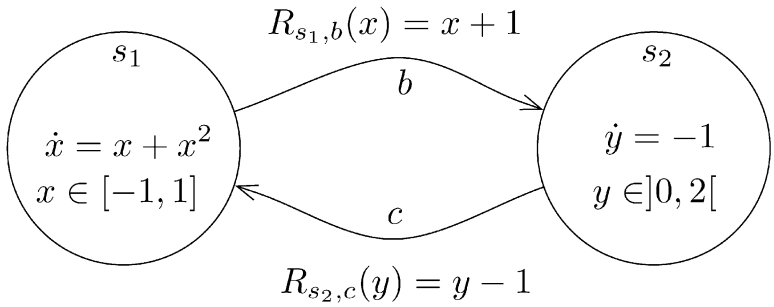


Fig. 2. Hybrid system $\tilde{H}$.

In other words, the flow in the dynamical system X_s takes x to x' while satisfying the invariant at all times in-between, and the discrete state remains the same.

Example 38. Here is an example of a trajectory that can take place in the hybrid system H of Example 37.

System starts at $(s_1, x(0) = 1/4)$ and flows continuously for $\log 2/2$ units of time reaching $(s_1, x(\log 2/2) = 1/2)$. At this point the guard is enabled and discrete transition a occurs making the system evolve from $(s_1, 1/2)$ to $(s_2, R_{s_1,a}(1/2)) = (s_2, 1/4)$. Now discrete transition b takes place and the system jumps to $(s_3, 1/4 + 1) = (s_3, 5/4)$. At this point the system flows continuously for 1 unit of time until reaching $(s_3, z(\log 2/2 + 1) = 1/4)$ and c takes the system to $(s_2, -3/4)$.

This can be neatly represented as

$$(s_1, 1/4) \overset{\tau_{\log 2/2}}{\Rightarrow} (s_1, 1/2) \overset{a}{\Rightarrow} (s_2, 1/4) \overset{b}{\Rightarrow} (s_3, 5/4) \overset{\tau_1}{\Rightarrow} (s_3, 1/4) \overset{c}{\Rightarrow} (s_2, -3/4).$$

We define the model category **Hyb** with objects, hybrid systems. A morphism f in **Hyb** from $H = (T, X, Inv, G, R)$ to $H' = (T', X', Inv', G', R')$ with $T = (S, i, L, \to)$ and $T' = (S', i', L, \to')$ is a pair $(f^1, \{f^2_s\}_{s\in S})$ where

- $f^1 : T \to T'$ is a $\mathbf{T}_L$-morphism,
- $f^2_s : X_s \to X'_{f^1(s)}$ is a **Dyn**-morphism, for all $s \in S$,
- $f^2_s(Inv_s) \subseteq Inv'_{f^1(s)}$ for all $s \in S$, and
- $f^2_s(G_{s,a}) \subseteq G'_{f^1(s),a}$ for all $a \in L$, $s = src(a)$,
- If $((s, x), a, (t, y)) \in\Rightarrow$ is a transition in H, then $(x, y) \in R_{s,a}$ implies $(f^2_s(x), f^2_t(y)) \in R'_{f^1(s),a}$.

For hybrid systems $H = (T, X, Inv, G, R)$, $H' = (T', X', Inv', G', R')$ and $H'' = (T'', X'', Inv'', G'', R'')$, the identity morphism $id : H \to H$ is defined by $id_H = (id_T, \{id_{X_s}\}_s)$. Given $f : H \to H'$ and $g : H' \to H''$, their composition $h = g \circ f$ is given by $h^1 = g^1 \circ f^1$ and $h^2_s = g^2_{f_1(s)} \circ f^2_s$ for $s \in S$. It can be easily checked that hybrid systems and their morphisms form a category.

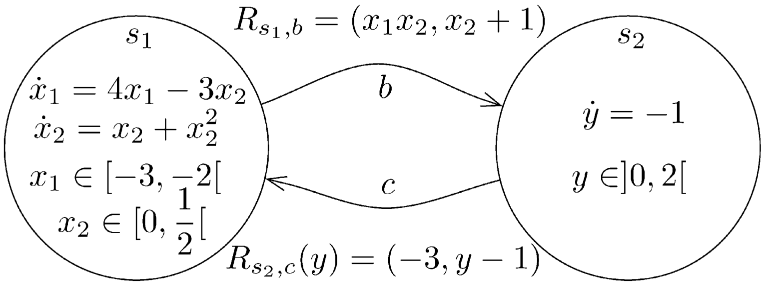


Fig. 3. Hybrid system H'.

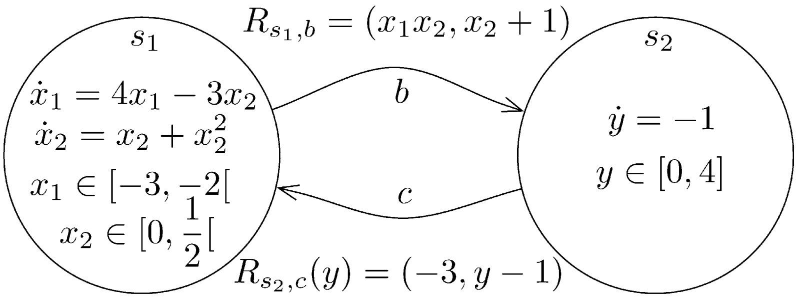


Fig. 4. Hybrid system H''.

Example 39. Consider the hybrid systems $\tilde{H}$, H' and H'' in Figs. 2, 3 and 4 respectively.

Note that on the figures we have avoided adding tilde, prime and double prime to the symbols to avoid notational complexity, instead we make such references to variables in the text. The guards in $\tilde{H}$, H' and H'' will play no role in this example, hence we leave them unspecified.

We first show that there is a morphism from H' to $\tilde{H}$. Let f^1 be defined by $f^1(s_1') = \tilde{s}_1$ and $f^1(s_2') = \tilde{s}_2$, $f^2_{s_1'}$ be defined by $f^2_{s_1'}(x_1, x_2) = x$ and finally $f^2_{s_2'}$ be the identity map, it is obvious that the conditions for $f^2_{s_2'}$ are satisfied. For $f^2_{s_1'}$ we note that:

$$Tf^2_{s_1'} \cdot \begin{bmatrix} 4x_1 - 3x_2 \\ x_2 + x_2^2 \end{bmatrix} = x_2 + x_2^2 = X_{\tilde{s}_1} \circ f^2_{s_1'},$$

which shows that $f^2_{s_1'}$ is a **Dyn**-morphism. The remaining conditions are easily checked.

Next we show that there are no morphisms from H'' to $\tilde{H}$. The $f^2_{s_2''}$ component of any morphism from H'' to $\tilde{H}$ needs to be a morphism of dynamical systems and therefore to satisfy $-\frac{\mathrm{d}}{\mathrm{d}z} f^2_{s_2''} = Tf^2_{s_2''} \cdot (-1) = -1$. This differential equation has solution $f^2_{s_2''}(z) = z + c$ for some constant $c \in \mathbb{R}$. However, for all possible choices of c we have $f^2_{s_2''}(Inv_{s_2''}) \nsubseteq Inv_{\tilde{s}_2}$ which violates the definition of morphism. Intuitively, there can be no morphism since there are trajectories in H'' that cannot be simulated by $\tilde{H}$ as their image under $f^2_{s''_2}$ would necessarily be outside $Inv_{\tilde{s}_2}$ thus contradicting the notion of trajectory that we next introduce.

We proceed to define the path category **P** as the full subcategory of **Hyb** with objects $P = (T, X, Inv, G, R)$ where $T = (S, i, L, \rightarrow)$ is a tree with a single (possibly empty) branch, and for every $s \in S$, $X_s : I_s \rightarrow TI_s$, with I_s an open interval (α_s, β_s) of $\mathbb{R}$ containing the origin, is defined by $X_s(t) = (t, 1)$. $Inv_s \subseteq I_s$, Inv_s is a closed interval of the form $[t_1, t_2]$ for some t_1, t_2, (this includes $t_1 = t_2$ possibility) that represents the duration of the continuous flow and $G_{s,a} = \{t_2\}$. Suppose $(s, a, t) \in \rightarrow$, $R_{s,a} : G_{s,a} \rightarrow Inv_t$ is the inclusion function.

Definition 40. A *path* or *trajectory* in a hybrid system H is a morphism $p : P \rightarrow H$ in **Hyb**, where P is an object in **P**.

Any path including a discrete transition will also carry the information of when this transition takes place. This in turn is captured by the choice of the appropriate path object (see the example below). The example below contains the representative cases that cover all possibilities. We content ourselves with the example as it is sufficiently self explanatory.

Example 41. Let H be a hybrid system. We will consider 3 path examples that cover all possible cases.

- Consider a path of the form

  $$(s_0, x) \stackrel{\tau_t}{\Rightarrow} (s_0, x') \stackrel{a}{\Rightarrow} (s_1, y) \stackrel{\tau_{t_1}}{\Rightarrow} (s_1, y') \stackrel{b}{\Rightarrow} (s_2, z)$$

  so in this case the system flows for duration t, starting at time 0 and then at time t the event a takes place etc. This path is represented by the path object P which has states l_0, l_1, l_2 as shown below:

  $$l_0 \xrightarrow{a} l_1 \xrightarrow{b} l_2$$

$I_{l_0} = (\alpha_0, \beta_0)$	$I_{l_1} = (\alpha_1, \beta_1)$	
with $0, t \in I_{l_0}$	with $0, t + t_1 \in I_{l_1}$	$I_{l_2} = (\alpha_2, \beta_2)$
$Inv_{l_0} = [0, t]$	$Inv_{l_1} = [t, t + t_1]$	with $0, t + t_1 \in I_{l_2}$
$G_{l_0,a} = \{t\}$	$G_{l_1,b} = \{t + t_1\}$	$Inv_{l_2} = \{t + t_1\}$
$R_{l_0,a}(t) = t$	$R_{l_1,b}(t + t_1) = t + t_1$	

  In this case we also spell out the definition of $p : P \rightarrow H$: $p^1(l_j) = s_j$, $j = 0, 1, 2$ and $p^2_{l_0}(0) = x_0$, $p^2_{l_1}(t) = x_1$ and $p^2_{l_2}(t + t_1) = x_2$, note that the p^2_s are integral curves and thus uniquely determined by these definitions.
- Next consider the path

  $$(s_0, x) \stackrel{\tau_t}{\Rightarrow} (s_0, x') \stackrel{a}{\Rightarrow} (s_1, y) \stackrel{b}{\Rightarrow} (s_2, z) \stackrel{\tau_{t_1}}{\Rightarrow} (s_2, z')$$

The path object for this path is defined as follows, the underlying tree is the same as the one above and we have:

$I_{l_0} = (\alpha_0, \beta_0)$	$I_{l_1} = (\alpha_1, \beta_1)$	
with $0, t \in I_{l_0}$	with $0, t \in I_{l_1}$	$I_{l_2} = (\alpha_2, \beta_2)$
$Inv_{l_0} = [0, t]$	$Inv_{l_1} = \{t\}$	with $0, t + t_1 \in I_{l_2}$
$G_{l_0,a} = \{t\}$	$G_{l_1,b} = \{t\}$	$Inv_{l_2} = [t, t + t_1]$
$R_{l_0,a}(t) = t$	$R_{l_1,b}(t) = t$	

- This last case follows from the one above, but we include it for the sake of clarity. Suppose we are given the path

$$(s_0, x) \overset{a}{\Rightarrow} (s_1, y) \overset{\tau_t}{\Rightarrow} (s_1, y') \overset{b}{\Rightarrow} (s_2, z)$$

The path object here too has the same underlying tree as the ones above and

$I_{l_0} = (\alpha_0, \beta_0)$	$I_{l_1} = (\alpha_1, \beta_1)$	
with $0 \in I_{l_0}$	with $0, t \in I_{l_1}$	$I_{l_2} = (\alpha_2, \beta_2)$
$Inv_{l_0} = \{0\}$	$Inv_{l_1} = [0, t]$	with $0, t \in I_{l_2}$
$G_{l_0,a} = \{0\}$	$G_{l_1,b} = \{t\}$	$Inv_{l_2} = \{t\}$
$R_{l_0,a}(0) = 0$	$R_{l_1,b}(t) = t$	

Suppose $P = (T, X, Inv, G, R)$ and $P' = (T', X', Inv', G', R')$ and $m : P \to P'$. Then, $m^1 : T \to T'$ which simply extends the tree T to T'. For any $s \in S$, m_s^2 is a smooth map from I_s to $I_{m^1(s)}$, such that $\mathrm{d}/\mathrm{d}t(m_s^2(t)) = 1$ or $m_s^2(t) = t - t_0$ for some $t_0 \in \mathbb{R}$ and for all $t \in I_s$.

We next characterize the **P**-open maps.

Proposition 42. *Let $H = (T, X_s, Inv_s, G_{s,a}, R_{s,a})$ and $H' = (T', X'_s, Inv'_s, G'_{s,a}, R'_{s,a})$ be hybrid systems with $T = (S, i, L, \to)$, $T' = (S', i', L, \to')$ and underlying state spaces Q and Q', then $f = (f^1, f_s^2) : H \to H'$ is* **P***-open iff*

(1) *for all $u \in Q$, $w \in Q'$ and $a \in L$, if $f(u) \overset{a}{\Rightarrow} w$, then there exists a $v \in Q$ such that $u \overset{a}{\Rightarrow} v$ and $f(v) = w$, and*

(2) *for all $u \in Q$, $w \in Q'$ and $t \in \mathbb{R}_0^+$, if $f(u) \overset{\tau_t}{\Rightarrow} w$, then there exists a $v \in Q$ such that $u \overset{\tau_t}{\Rightarrow} v$ and $f(v) = w$.*

Proof. Suppose $f = (f^1, f_s^2) : H \to H'$ is **P**-open and for a reachable state $u = (s, x) \in Q$ and $a \in L$, $f(u) \overset{a}{\Rightarrow} w$ in H'. Let $w = (s'', x'')$, then $f(u) = (f^1(s), f_s^2(x))$ and $f^1(s) \overset{a}{\longrightarrow} s''$ in T', $f_s^2(x) \in G_{f^1(s),a}$ and $(f_s^2(x), x'') \in R_{f^1(s),a}$. As $u = (s, x)$ is reachable in H, the state $s \in S$ is reachable from i in T, say through

$$i = s_0 \overset{a_1}{\longrightarrow} s_1 \ldots \overset{a_n}{\longrightarrow} s_n = s$$

hence there is a path object P whose underlying tree is

$$l_0 \overset{a_1}{\longrightarrow} l_1 \ldots \overset{a_n}{\longrightarrow} l_n$$

and a path $p : P \to H$ with $p^1(l_0) = s_0, \ldots, p^1(l_n) = s_n$ and appropriate p_s^2 for $s \in \{l_0, \ldots, l_n\}$. The only part of the continuous data about P relevant to the proof is the information at l_n which we will make explicit below. Suppose that a_n occurs at time t_n and consider the following cases:

Case 1: No continuous flow takes place at state s_n, hence we have, say $(s_{n-1}, x) \stackrel{a_n}{\Rightarrow} (s_n, x)$, or $(s_{n-1}, x') \stackrel{a_n}{\Rightarrow} (s_n, x)$ with $R_{s_{n-1},a_n}(x') = x$. Also $I_{l_n} = (\alpha_n, \beta_n)$ containing the origin and t_n and $Inv_{l_n} = \{t_n\}$. Define a path object P' with underlying tree

$$l'_0 \xrightarrow{a_1} l'_1 \ldots \xrightarrow{a_n} l'_n \xrightarrow{a} l'$$

The underlying continuous information is the same as in P except that we set $G_{l'_n,a} = \{t_n\}$, and $I_{l'} = (\alpha', \beta')$ containing the origin and t_n and $Inv_{l'} = \{t_n\}$. Also we define the path $q : P' \to H'$ by $q^1(l'_j) = f^1 p^1(l_j)$ for $j = 0, \ldots, n$, and $q^1(l') = s''$. And $q_s^2 = f_s^2 p_s^2$ for all $s \in \{l'_0, \ldots, l'_n\}$, and $q_{l'}^2(t_n) = x''$.

Case 2: There is a continuous flow at s_n, say we have

$$(s_{n-1}, x') \stackrel{a_n}{\Rightarrow} (s_n, \tilde{x}) \stackrel{\tau_t}{\Rightarrow} (s_n, x)$$

for some t. The path object P is as above save for $I_{l_n} = (\alpha_n, \beta_n)$ containing the origin, and $t_n + t$ and $Inv_{l_n} = [t_n, t_n + t]$. We define the path object P' as this new path object P, except for $G_{l'_n,a} = \{t_n + t\}$, and $I_{l'} = (\alpha', \beta')$ containing the origin and $t_n + t$ and $Inv_{l'} = \{t_n + t\}$. The morphism q is defined as above except that we set $q_{l'}^2(t_n + t) = x''$.

Clearly q is a path and with m the obvious embedding we have $fp = qm$. As f is **P**-open we have $r : P' \to H$, let $v = (r^1(l'), r_{l'}^2(t_n))$ in case 1 and $v = (r^1(l'), r_{l'}^2(t_n + t))$ in the second case. Clearly $u \stackrel{a}{\Rightarrow} v$ and

$$f(v) = (f^1 r^1(l'), f_{r^1(l')}^2(r_{l'}^2(t_n))) = (s'', x'')$$

in case 1 and similarly $f(v) = w$ in case 2.

Now suppose $f(u) \stackrel{\tau_{t'}}{\Rightarrow} w$, with the same notation as above, this means that $f^1(s) = s''$ and $Fl_{t'}^{X'_{f^1(s)}}(f_s^2(x)) = x''$. Again we need to distinguish two cases similar to those above: (1) There is no continuous flow at s_n. The path object P is the same as in case 1 above, we define the path object P':

$$l'_0 \xrightarrow{a_1} l'_1 \quad \ldots \xrightarrow{a_n} l'_n$$

as P except that we set $I_{l'_n} = (\alpha', \beta')$ containing 0 and $t_n + t'$, $Inv_{l'_n} = [t_n, t_n + t']$. The path q is defined as in case 1 above except that $q_{l'_n}^2(t_n + t') = x''$.

(2) There is continuous flow, say of duration t to reach (s, x), in this case P is the same as in case 2 above and we define P' as P except that $I_{l'_n} = (\alpha', \beta')$ to contain the origin and $t_n + t + t'$ and $Inv_{l'_n} = [t_n, t_n + t + t']$.

It can be easily checked that with $v = (r^1(l'_n), r_{l'_n}^2(t_n + t'))$, and $v = (r^1(l'_n), r_{l'_n}^2(t_n + t + t'))$ in cases 1 and 2 respectively, one has $u \stackrel{\tau_t}{\Rightarrow} v$ and $f(v) = w$.

Conversely, suppose that conditions (i) and (ii) of the proposition hold and that there are paths $p : P \to H$ and $q : P' \to H'$ with $m : P \to P'$ such that $fp = qm$ we need to show that *f* is **P**-open.

Note that the underlying tree of P' is either the same as or an extension of *P*, in this case we repeatedly use condition (i) above to define r^1. The argument for the definition of r^1 is the same as in [12]. We show the proof on an example, suppose *P* is given by

$$l_0 \xrightarrow{a} l_1$$

which maps to

$$s_0 \xrightarrow{a} s_1$$

in *H* under *p* and P' is given by

$$l'_0 \xrightarrow{a} l'_1 \xrightarrow{b} l'_2$$

which maps to

$$s'_0 \xrightarrow{a} s'_1 \xrightarrow{b} s'_2$$

under *q*.

Now apply condition (i) of the proposition to find s_2 such that $s_1 \xrightarrow{b} s_2$ and define $r^1(l'_j) = s_j$, $j = 0, 1, 2$.

Consider the commutative diagram

$$\begin{array}{ccc} I_{l_0} & \xrightarrow{p^2_{l_0}} & M_{s_0} \\ {\scriptstyle m^2_{l_0}}\downarrow & & \downarrow{\scriptstyle f^2_{s_0}} \\ I_{l'_0} & \xrightarrow{q^2_{l'_0}} & M'_{s'_0} \end{array}$$

and use Theorem 11 to define $r^2_{l'_0}$, similarly for $r^2_{l'_1}$. As for $r^2_{l'_2}$, suppose $Inv_{l'_2} = \{t_b\}$ where t_b is the time that *b* occurs. Then there is no continuous flow at s'_2 and we set $r^2_{l'_2}(t_b) = x_2$ where $(s_1, x_1) \overset{b}{\Rightarrow} (s_2, x_2)$. On the other hand, if time *t* elapsed at state l'_2, use (ii) above to find (s_2, x'_2) where $(s_2, x_2) \overset{\tau_t}{\Rightarrow} (s_2, x'_2)$ and set $r^2_{l'_2}(t_b) = x_2$ and $r^2_{l'_2}(t_b + t) = x'_2$.

It is not hard to see that with this definition $r : P' \to H$ is a path and that the $fr = p'$ and $rm = p$. □

Definition 43. Let H', H'' be hybrid systems with S' and S'' as the state spaces of their underlying labelled transition systems, respectively. Let $f : H' \to \tilde{H}$ and $g : H'' \to \tilde{H}$ be morphisms of hybrid systems. We say that *f* and *g* are *transversal* if for any $s' \in S'$ and $s'' \in S''$ such that $f^1(s') = g^1(s'')$ we have that the **Dyn**-morphisms $f^2_{s'} : X'_{s'} \to \tilde{X}_{f^1(s')}$ and $g^2_{s''} : X''_{s''} \to \tilde{X}_{g^1(s'')}$ are transversal (see Section 4).

Definition 44. Let H and H' be hybrid systems, and $f : H \to H'$ be a morphism of hybrid systems. Then, f is said to be a *surjective submersion* if $f_s^2 : X_s \to X'_{f^1(s)}$ is a surjective submersion, for all $s \in S$.

Proposition 45. *The category* **Hyb** *has binary products and transversal pullbacks.*

Proof. Given two hybrid systems

$$H' = (T', X', Inv', G', R')$$

and

$$H'' = (T'', X'', Inv'', G'', R'')$$

with $T' = (S', i', L, \to')$ and $T'' = (S'', i'', L, \to'')$, we define their product $H = H' \times H'' = (T, X, Inv, G, R)$ as follows:

- $T = (S, i, L, \to) = T' \times T''$. Note that this is the product in the category $\mathbf{T}_L$ of transition systems with label set L (see Section 3 above).
- For $s = (s', s'') \in S = S' \times S''$, $X_s = X'_{s'} \times X''_{s''}$, which is a product in **Dyn**.
- For $s = (s', s'') \in S$, $Inv_s = Inv'_{s'} \times Inv''_{s''}$, Cartesian product of sets.
- Finally, for $s = (s', s'') \in S$, $G_{(s',s''),a} = G'_{s',a} \times G''_{s'',a}$ and $R_{(s',s''),a} = R'_{s',a} \times R''_{s'',a}$.

Definition of projection maps is based on those for underlying transition and dynamical systems and verification of product property is routine and not included.

Let H', H'' be hybrid systems as above and $f : H' \to \tilde{H}$ and $g : H'' \to \tilde{H}$ be morphisms of hybrid systems. Now suppose f, g are transversal, we define the pullback of f and g as (H, g', f') where $H = (T, X, Inv, G, R)$ is given by

- T is the pullback in $\mathbf{T}_L$ of f^1, g^1, (see Section 3 above). Recall that, then $S = \{(s', s'') \mid f^1(s') = g^1(s'')\}$.
- For $s = (s', s'') \in S$, X_s is the pullback in **Dyn** of transversal maps $f_{s'}^2$ and $g_{s''}^2$ (see Section 4 above). Recall that $M_s = \{(x', x'') \in M'_{s'} \times M''_{s''} \mid f_{s'}^2(x') = g_{s''}^2(x'')\}$.
- For $s = (s', s'') \in S$, $Inv_s = (Inv'_{s'} \times Inv''_{s''}) \cap M_s$.
- For $s = (s', s''), t = (t', t'') \in S$ and $(x', x'') \in M_s$ and $(y', y'') \in M_t$ such that $(s', s'', x', x'') \overset{a}{\Rightarrow} (t', t'', y', y'')$ define

$$G_{(s',s''),a} = \{(x', x'') \in (G'_{s',a} \times G''_{s'',a}) \cap M_s \mid (R'_{s',a}(x'), R''_{s'',a}(x'')) \in Inv_t\}.$$

- $R_{(s',s''),a} = (R'_{s',a} \times R''_{s'',a})|_{G_{(s',s''),a}}$. Note that the range of $R_{(s',s''),a}$ is in Inv_t, for t as above. This follows from the definition of $G_{(s',s''),a}$.

Definitions of f' and g' follow using the underlying morphisms and verification of pullback property is routine and not included. □

Definition 46. We say that two hybrid systems H and H' are **P**-bisimilar if there exists a span $(\tilde{H}, f : \tilde{H} \to H, g : \tilde{H} \to H')$ of **P**-open surjective submersions.

This immediately gives us the following result.

Proposition 47. **P**-*bisimilarity is an equivalence relation on the class of all hybrid systems.*

It remains to show that the notion of **P**-bisimilarity coincides with a natural notion of bisimulation for hybrid systems, that we now define.

Definition 48. Given two hybrid systems $H = (T, X, Inv, G, R)$ and $H' = (T', X', Inv', G', R')$, with X_s and $X'_{s'}$ defined on M_s and $M'_{s'}$ respectively. Let $R^1 \subseteq S \times S'$, and for each $(s, s') \in R^1$, let $R^2_{s,s'} \subseteq M_s \times M'_{s'}$ be a regular relation.

Define $\mathcal{R} = (R^1, \{R^2_{s,s'}\}_{(s,s')\in R^1})$ to be the set

$$\{(s, x, s', x') \mid (s, s') \in R^1 \text{ and } (x, x') \in R^2_{s,s'}\}.$$

$\mathcal{R}$ is said to be a *bisimulation* relation iff for all $((s, x), (s', x')) \in Q \times Q'$, $((s, x), (s', x')) \in \mathcal{R}$ implies,

- for any $a \in L$ if $(s, x) \overset{a}{\Rightarrow} (t, y)$, then there exists t', y' such that $(s', x') \overset{a}{\Rightarrow} (t', y')$ and $((t, y), (t', y')) \in \mathcal{R}$,
- for any $t \in \mathbb{R}^+_0$ if $(s, x) \overset{\tau_t}{\Rightarrow} (t, y)$, then there exists t', y' such that $(s', x') \overset{\tau_t}{\Rightarrow} (t', y')$ and $((t, y), (t', y')) \in \mathcal{R}$
- Vice-versa.

Remark 49. Notice that $\mathcal{R}$ above is not a relation from Q to Q', as it might contain tuples (s, x, s', x') with $x \notin Inv_s$ or $x' \notin Inv'_{s'}$. However, this fact does not pose a problem in our definition, as hybrid systems always evolve inside the invariant sets.

We say that two hybrid systems H and H' are *bisimilar* if there exists a bisimulation relation $\mathcal{R}$ such that $((i, x), (i', x')) \in \mathcal{R}$ for some $x \in Inv_i$ and $x' \in Inv'_{i'}$ (recall that i, i' are the initial states of T and T', respectively).

The main theorem below shows that the intuitive definition for hybrid system bisimilarity is captured by the abstract bisimulation (**P**-bisimilarity).

Theorem 50. *Let H and H' be hybrid systems. Then H and H' are bisimilar iff they are* **P**-*bisimilar*.

Proof. Suppose H and H' are **P**-bisimilar, let the span be $f : \tilde{H} \to H$ and $g : \tilde{H} \to H'$. We define a relation $\mathcal{R} = (R^1, \{R^2_{s,s'}\}_{(s,s')\in R^1})$ as follows:

$$R^1 = graph(g^1) \circ \overline{graph(f^1)} \subseteq S \times S'.$$

For $(s, s') \in R^1$, define

$$R^2_{s,s'} = \biguplus_{\tilde{s},\, f^1(\tilde{s})=s,\, g^1(\tilde{s})=s'} graph(g^2_{\tilde{s}}) \circ \overline{graph(f^2_{\tilde{s}})}.$$

Note that $R^2_{s,s'} \subseteq M_{f^1(\tilde{s})} \times M'_{g^1(\tilde{s})} = M_s \times M'_{s'}$.

Regularity of $R^2_{s,s'}$ follows from Proposition 23 and the fact that the disjoint union of regular relations is regular.

It remains to show that $\mathcal{R}$ thus defined is a bisimulation relation, but this follows from f, g being **P**-open surjective submersions. Finally, bisimilarity of H and H' follows from the fact that f^1 and g^1 preserve initial states.

Conversely, suppose H and H' are bisimilar, let the bisimulation relation be $\mathcal{R} = (R^1, R^2_{s,s'})$, define a hybrid system $\tilde{H} = (\tilde{T}, \tilde{X}, \tilde{Inv}, \tilde{G}, \tilde{R})$ as follows:

- $\tilde{T} = (T \times T')|_{R^1}$ which means that we remove all states of $T \times T'$ not in R^1, we also remove the incident transitions on these states.
- For $\tilde{s} = (s, s') \in R^1$, define $\tilde{X}_{\tilde{s}} : R^2_{s,s'} \to TR^2_{s,s'}$ by $\tilde{X}_{\tilde{s}} = (X_s \times X'_{s'})|_{R^2_{s,s'}}$, this is well-defined by Theorem 25.
- $\tilde{Inv}_{(s,s')} = (Inv_s \times Inv'_{s'}) \cap R^2_{s,s'}$.
- $\tilde{G}_{(s,s'),a} = (G_{s,a} \times G'_{s',a}) \cap R^2_{s,s'}$, and
- $\tilde{R}_{(s,s'),a}$ is obtained from $R_{s,a} \times R'_{s',a}$ by restricting its domain to $\tilde{G}_{(s,s'),a}$. the well-definedness of $\tilde{R}$ follows from the fact that $\mathcal{R}$ is a bisimulation.

The maps $f : \tilde{H} \to H$ and $g : \tilde{H} \to H'$ are defined using the projection maps on the discrete and continuous parts and can be shown to be **P**-open surjective submersions. The proof is essentially similar to that of Theorem 25. Hence, we have a span $(\tilde{H}, f, g)$ of **P**-open surjective submersions, and H and H' are **P**-bisimilar. □

7. Related work

In this section we compare several aspects of our work with the existing ones in the literature.

7.1. Categorical approaches to modeling of hybrid systems

As much as the authors are aware the only other work that discusses categorical models of hybrid systems is the paper [18]. In this work, the authors construct an institution of hybrid systems and provide a categorical characterization of free aggregation, restriction and abstraction of such systems, thus providing a basis for compositional specification and verification of hybrid systems. However, they do not discuss bisimulations. More explicitly, they show that in the category of hybrid systems free aggregation corresponds to a product, restriction to a cartesian lifting and abstraction to a cocartesian lifting. Categorically inspired modeling of heterogeneous systems, consisting of multiple models of computation, is the primary concern of the tagged-signal model in [17], and more, recently, the trace algebraic framework in [6].

7.2. Categorical approaches to bisimulation

There has been considerable amount of research on categorical formulations of bisimulation in addition to [12]. We will be more specific on coalgebraic approach to bisimulation. See [23] for coalgebraic approaches to systems theory in general.

Coalgebraic formulation has been used successfully to model a variety of systems that include, deterministic systems, deterministic and nondeterministic labeled transition systems, supervisory control systems [15], symbolic dynamical systems, to name a few. More explicitly a labeled transition system $(S, i, L, \to)$ defined in Section 3 can be viewed as an F-system (S, α_S) with $F : Set \to Set$ a functor and $F(X) = 2^{L\times X}$ for any set X. Here $\alpha_S : S \to F(S)$ is given by $\alpha_S(s) = \{(a, s') \mid s \xrightarrow{a} s'\}$. An F-homomorphism $f : (S, \alpha_S) \to (T, \alpha_T)$ is a map $f : S \to T$ such that $F(f)\alpha_S = \alpha_T f$ which means that f both preserves and reflects the transition structure. This fact that a homomorphism reflects F-transitions makes it different from the morphisms we have in the category $T_{\mathbf{L}}$. Now suppose $F : Set \to Set$ is a functor, and (S, α_S) and (T, α_T) are F-systems, a relation $R \subseteq S \times T$ is said to be a bisimulation between S and T if there exists an F-dynamics $\alpha_R : R \to F(R)$ such that the projections from R to S and T are F-homomorphisms.

Note that in the case of dynamical systems we have a functor, the so called tangent functor $T : \mathbf{Man} \to \mathbf{Man}$, and one is tempted to view a dynamical system X on a manifold M as a coalgebra (M, X) with $X : M \to TM$. However, this is not the case on the face of it, recall that a dynamical system is $X : M \to TM$ such that $\pi_M X = id_M$ where π_M is the canonical projection. On the other hand, clearly one could work in a full subcategory of $\mathbf{coAlg}_T$ where the property above is also satisfied.

On a more essential note, our choice to work with path objects and path categories instead of coalgebraic approach was due to the fact that in coalgebraic approaches one does not have a direct way of modeling the notion of time and trajectory for the system under study. However, in path object approach the flow of the system is made explicit and the notion of abstract bisimulation has the trajectories built into the definition through the **P**-open maps. As a matter of fact, in trying to formulate a notion of bisimulation for dynamical and especially for hybrid systems we have benefited greatly from having to first define a path object. This gave as an idea as to what the abstract notion of time should be for a hybrid system. As the reader might recall, this is a tree with a single branch with bubbles on every state, representing clocks working at constant rate 1.

8. Conclusions

In this paper, we developed novel notions of system equivalence for dynamical and control systems, unified the notion of bisimulation across discrete and continuous domains, and developed bisimulation notions for hybrid dynamical systems. In all cases, we proved that this definition is captured by the abstract bisimulation framework introduced in [12].

There are several future research directions. On the one hand there is the well known connection between abstract bisimulation, and logic and game characterizations of bisimulation and presheaf semantics in the case of concurrency models [30]. This direction can be exploited for dynamical and hybrid dynamical systems and in this way one obtains specification logics for such systems. We are very keen on further exploring the relation between our models and presheaf semantics.

On the other hand we have to further investigate the use and appropriateness of the notion of bisimulation for dynamical and hybrid systems in the context of real life engineering applications. The first step in this direction is to find algebraic characterizations of bisimulation for hybrid systems or for at least a class of such systems and hence make a step forward towards computability issues of such relations. Secondly, our definition might be too strong for applications, notice that in our setting, the two bisimilar hybrid systems are locked in timing, that is, wherever one gets in time t the other should also be able to simulate in the same time duration t. This condition could be weakened to allow for other equivalence relations similar to weak bisimulation relation in the context of concurrency theory [19]. Another weaker relation could be obtained by allowing a discrete transition a in one hybrid system to be simulated by pre and post time evolution of the other machine during the execution of the event a. We plan to study both of these weaker versions of equivalences and the possibilities of characterizing them in abstract bisimulation framework.

Appendix A. Differential geometry

Our treatment of differential geometry follows that of [10]. For a more thorough introduction to geometry, the reader may wish to consult numerous books on the subject such as [1,26].

A.1. Differentiable manifolds

Recall that a function $h : A \to B$ is a homeomorphism iff h is a bijection and both h and h^{-1} are continuous. In this case, topological spaces A and B are called homeomorphic. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called smooth or C^∞ if all derivatives of any order exist and are continuous. Function f is real analytic or C^ω, if it is C^∞ and for each $x \in \mathbb{R}^n$ there exists a neighborhood U of x, such that the Taylor series expansion of f at x converges to $f(x)$ for all $x \in U$. A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is a collection $(f_1, \ldots, f_m)$ of functions $f_i : \mathbb{R}^n \to \mathbb{R}$. The mapping f is smooth (analytic) if all functions f_i are smooth (analytic).

Definition A.1 (*Manifolds*). A manifold M of dimension n is a Hausdorff and second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

A manifold, which is of great interest to us, is $\mathbb{R}^n$ itself. A subset N of a manifold M which is itself a manifold is called a submanifold of M. Any open subset N of a manifold M is clearly a submanifold, since if M is locally homeomorphic to $\mathbb{R}^n$ then so is N. In particular, an open interval $I \subseteq \mathbb{R}$ is also a manifold.

A coordinate chart on a manifold M is a pair (U, ϕ) where U is an open set of M and ϕ is a homeomorphism of U on an open set of $\mathbb{R}^n$. The function ϕ is also called a coordinate function and can also be written as $(\phi_1, \ldots, \phi_n)$ where $\phi_i : M \longrightarrow \mathbb{R}$. If $p \in U$ then

$$\phi(p) = (\phi_1(p), \ldots, \phi_n(p)) \text{ is called the set of local coordinates in the chart } (U, \phi).$$

When doing operations on a manifold, we must ensure that our results are consistent regardless of the particular chart we use. We must therefore impose some conditions. Two charts (U, ϕ) and (V, ψ) with $U \cap V \neq \emptyset$, are called C^∞ (C^ω) compatible if the map

$$\psi \circ \phi^{-1} : \phi(U \cap V) \subseteq \mathbb{R}^n \longrightarrow \psi(U \cap V) \subseteq \mathbb{R}^n$$

is a C^∞ (C^ω) function. A C^∞ (C^ω) atlas on a manifold M is a collection of charts (U_α, ϕ_α) with $\alpha \in A$ which are C^∞ (C^ω) compatible and such that the open sets U_α cover the manifold M, so $M = \bigcup_{a \in A} U_\alpha$. An atlas is called maximal if it is not contained in any other atlas.

Definition A.2 (*Differentiable manifolds*). A differentiable (analytic) manifold is a manifold with a maximal, C^∞ (C^ω) atlas.

Now that we have imposed this differential structure on our manifold M we can perform calculus on M. In particular let $f : M \longrightarrow \mathbb{R}$ be a map. If (U, ϕ) is a chart on M then the function

$$\hat{f} = f \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$$

is called the local representative of f in the chart (U, ϕ). We therefore define the map f to be smooth (analytic) if its local representative $\hat{f}$ is smooth (analytic). Notice if f is smooth (analytic) in one chart, then it is smooth (analytic) in every chart since we required our charts to be C^∞ (C^ω) compatible and our atlas to be maximal. Hence our results are intrinsic to the manifold and do not depend on the particular chart we use. Similarly, if we have a map $f : M \longrightarrow N$, where M, N are differentiable manifolds, the local representation of f given a chart (U, ϕ) of M and (V, ψ) of N is

$$\hat{f} = \psi \circ f \circ \phi^{-1},$$

which makes sense only if $f(U) \cap V \neq \emptyset$. Again f is smooth (analytic) if $\hat{f}$ is a smooth (analytic) map.

A.2. Tangent spaces

Let p be a point on a manifold M and let $C^\infty(p)$ denote the vector space of all smooth functions in a neighborhood of p. A tangent vector X_p at $p \in M$ is an operator from $C^\infty(p)$ to $\mathbb{R}$ which satisfies for $f, g \in C^\infty(p)$ and $a, b \in \mathbb{R}$, the following properties:
(1) Linearity $X_p(a \cdot f + b \cdot g) = a \cdot X_p(f) + b \cdot X_p(g)$.
(2) Derivation $X_p(f \cdot g) = f(p) \cdot X_p(g) + X_p(f) \cdot g(p)$.
The set of all tangent vectors at $p \in M$ is called the tangent space of M at p and is denoted by T_pM. The tangent space T_pM becomes a vector space over $\mathbb{R}$ if for tangent vectors X_p, Y_p and real numbers c_1, c_2 we define

$$(c_1 \cdot X_p + c_2 \cdot Y_p)(f) = c_1 \cdot X_p(f) + c_2 \cdot Y_p(f)$$

for any smooth function f in the neighborhood of p. The collection of all tangent spaces of the manifold,

$$TM = \bigcup_{p \in M} T_pM$$

is called the tangent bundle. The tangent bundle has a naturally associated projection map $\pi : TM \longrightarrow M$ taking a tangent vector $X_p \in T_pM \subset TM$ to the point $p \in M$. The tangent space T_pM can then be thought of as $\pi^{-1}(p)$.

The tangent space can be thought of as a special case of a more general mathematical object called a fiber bundle. Loosely speaking, a fiber bundle can be thought of as gluing sets at each point of the manifold in a smooth way.

The tangent bundle is a vector bundle and the fiber at each point $p \in M$ is the tangent space T_pM. In particular, the tangent bundle TM has dimension $2n$, where M is n-dimensional.

Now let M be a manifold and let (U, ϕ) be a chart containing the point p. In this chart we can associate the following tangent vectors

$$\frac{\partial}{\partial \phi_1}, \ldots, \frac{\partial}{\partial \phi_n}$$

defined by

$$\frac{\partial}{\partial \phi_i}(f) = \frac{\partial(f \circ \phi^{-1})}{\partial x_i}$$

for any smooth function $f \in C^\infty(p)$. The tangent space T_pM is an n-dimensional vector space and if (U, ϕ) is a local chart around p then the tangent vectors

$$\frac{\partial}{\partial \phi_1}, \ldots, \frac{\partial}{\partial \phi_n}$$

form a basis for T_pM. Therefore if X_p is a tangent vector at p then

$$X_p = \sum_{i=1}^{n} a_i \frac{\partial}{\partial \phi_i},$$

where $a_1, \ldots, a_n$ are real numbers. From the above formula we can see that $X_p(f)$ is an operator which simply takes the directional derivative of f in the direction of $[a_1, \ldots, a_n]$.

Now let M and N be smooth manifolds and $f : M \longrightarrow N$ be a smooth map. Let $p \in M$ and let $q = f(p) \in N$. We wish to push forward tangent vectors from T_pM to T_qN using the map f. The natural way to do this is by defining a map $T_pf : T_pM \longrightarrow T_qN$ by

$$(T_pf(X_p))(g) = X_p(g \circ f)$$

for smooth functions g in the neighborhood of q. One can easily check that $T_pf(X_p)$ is a linear operator and a derivation and thus a tangent vector. The map $T_pf : T_pM \longrightarrow T_{f(p)}N$ is called the push forward map of f. The push forward map $T_pf : T_pM \longrightarrow T_{f(p)}N$ is a linear map, and furthermore if $f : M \longrightarrow N$ and $g : N \longrightarrow K$ then

$$T_p(g \circ f) = T_{f(p)}g \circ T_pf,$$

which is essentially the chain rule.

A.3. Vector fields

A vector field on a manifold M is a smooth map X which places at each point p of M a tangent vector from T_pM. Therefore since a vector field, X, places at each point p a tangent vector $X(p)$ we have that in the chart (U, ϕ) the local expression for the vector field X is

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial \phi_i}.$$

The vector field is smooth (analytic) if and only if $a_i(p)$ is C^∞ (C^ω).

Let $I \subseteq \mathbb{R}$ be an open interval containing the origin. An integral curve of a vector field is a curve $c : I \longrightarrow M$ whose tangent at each point is identically equal to the vector field at that point. Therefore an integral curve satisfies for all $t \in I$,

$$c' = T_t c(t, 1) = X(c).$$

A vector field is called *complete* if the integral curve passing through every $p \in M$ can be extended for all time, that is we can choose $I = \mathbb{R}$. Integral curves of smooth (analytic) vector fields are smooth (analytic).

Definition A.3 (*f-related vector fields*). Let X and Y be vector fields on manifolds M and N respectively and $f : M \longrightarrow N$ be a smooth map. Then X and Y are f-related iff

$$T(f) \circ X = Y \circ f. \qquad \text{(A.1)}$$

If f is not surjective, then X may be f-related to many vector fields on N. If, however, f is surjective, then X can only be f-related to a unique vector field on N.

References

[1] R. Abraham, J. Marsden, T. Ratiu, Manifolds, Tensor Analysis and Applications, Applied Mathematical Sciences, Springer, 1988.

[2] P.J. Antsaklis, A brief introduction to the theory and applications of hybrid systems, Proc. of the IEEE 88 (7) (2000) 879–887 Special issue on Hybrid Systems: Theory and Applications.

[3] P.J. Antsaklis, A. Nerode, Hybrid control systems: an introductory discussion to the special issue, IEEE Trans. Automatic Control 43 (4) (1998) 501–508 Special Issue on Hybrid Systems.

[4] R. Blute, J. Desharnais, A. Edalat, P. Panangaden, Bisimulation for labelled markov processes, in: Logic in Computer Science, IEEE Press, Silver Spring, MD, 1997, pp. 149–158.

[5] W. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, Elsevier Science & Technology Books, 1986.

[6] J. Burch, R. Passerone, A.L. Sangiovanni-Vincentelli, Overcoming heterophobia: modeling concurrency in heterogeneous systems, in: Proc. of the Second Internat. Conf. on Application of Concurrency to System Design, Newcastle upon Tyne, UK, June 2001.

[7] A. Cheng, M. Nielsen, Open maps, behavioural equivalences, and congruences, Theoret. Comput. Sci. 190 (1) (1998) 87–112.

[8] E. Haghverdi, P. Tabuada, G.J. Pappas, Bisimulation relations for dynamical and control systems, in: R. Blute, P. Selinger (Eds.), Category Theory and Computer Science, Vol. 69, Electronic Notes in Theoretical Computer Science, Elsevier Science Publishers, Amsterdam, 2003.

[9] M. Hennessy, Algebraic Theory of Processes, MIT Press, series in the foundations of computing, 1988.

[10] A. Isidori, Nonlinear Control Systems, second ed., Springer, Berlin, 1989.

[11] B. Jacobs, J.J.M.M. Rutten (Eds.), Coalgebraic Methods in Computer Science, Vol. 19, ENTCS, Elsevier Science, Amsterdam, 1999.

[12] A. Joyal, M. Nielsen, G. Winskel, Bisimulation from open maps, Inform. Comput. 127 (2) (1996) 164–185.

[13] R.E. Kalman, P.L. Falb, M.A. Arbib, Topics in Mathematical Systems Theory, McGraw-Hill, New York, 1969.
[14] I. Kolar, P.W. Michor, J. Slovak, Natural Operations in Differential Geometry, Springer, Berlin, 1993.
[15] J. Komenda, Coalgebra and supervisory control with partial observations, in: Mathematical Theory of Networks and Systems, Notre Dame, IN, August 2002.
[16] Kim G. Larsen, A. Skou, Bisimulation through probabilistic testing, Inform. Comput. 94 (1991) 1–28.
[17] E.A. Lee, A. Sangiovanni-Vincentelli, A framework for comparing models of computation, IEEE Trans. Comput. Aided Design 17 (12) (1998) 1217–1229.
[18] H. Lorenco, A. Sernadas, An institution of hybrid systems, in: D. Bert, C. Choppy (Eds.), Recent Trends in Algebraic Development Techniques, Lecture Notes in Computer Science, Vol. 1827, Springer, Berlin, 2000, pp. 219–236.
[19] R. Milner, Communication and Concurrency, Prentice Hall, Englewood Cliffs, NJ, 1989.
[20] R. Milner, D. Sangiorgi, Barbed bisimulation, in: Automata, Languages and Programming, 19th Internat. Colloq. (Proc. ICALP'92), 1992, pp. 685–695.
[21] G.J. Pappas, Bisimilar linear systems, Automatica 39 (12) (2003) 2035–2047.
[22] G.J. Pappas, G. Lafferriere, S. Sastry, Hierarchically consistent control systems, IEEE Trans. Automatic Control 45 (6) (2000) 1144–1160.
[23] J.J.M.M. Rutten, Universal coalgebra: a theory of systems, Theoret. Comput. Sci. 249 (1) (2000) 3–80.
[24] S.S. Sastry, A.S. Morse, C.C. Pantelides, J.M.H. Schumaker, Introduction to the special issue on hybrid systems, Automatica 35 (3) (1999) 347–348 Special issue on Hybrid Systems.
[25] J.P. Serre, Lie Algebras and Lie Groups, Lecture Notes in Mathematics, second ed., Vol. 1500, Springer, Berlin, Heidelberg, 1992.
[26] M. Spivak, A Comprehensive Introduction to Differential Geometry, Publish or Perish, Berkeley, CA, 1979.
[27] P. Tabuada, G.J. Pappas, Bisimilar control affine systems, Syst. Control Lett. (2004) 49–58.
[28] A.J. van der Schaft, J.M. Schumacher, Compositionality issues in discrete, continuous, and hybrid systems, Internat. J. Robust and Nonlinear Control 11 (5) (2001) 417–434.
[29] J.H. van Schuppen, Equivalences of discrete-event systems and of hybrid systems, in: V.D. Blondel et al. (Eds.), Open Problems in Mathematical Systems and Control Theory, Springer, Berlin, 1999, pp. 251–257.
[30] G. Winskel, M. Nielsen, Presheaves as transition systems, in: DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 29, DIMACS, 1997, pp. 129–140.