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Local factorization of trajectory lifting morphisms for single-input affine control systems☆

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Abstract

Trajectory preserving and lifting maps have been implicitly used in many recursive or hierarchical control design techniques. Well known systems theoretic concepts such as differential flatness or more recent ones such as simulation and bisimulation can be also understood through the trajectory lifting maps they define. In this paper we initiate a study of trajectory preserving and lifting maps between affine control systems. Our main result shows that any trajectory lifting map between two single-input control affine systems can be locally factored as the composition of two special trajectory lifting maps: a projection onto a quotient system followed by a differentially flat output with respect to another control system. We use this decomposition result to show that under mild regularity conditions, trajectory preserving maps between single-input affine control systems also lift trajectories. As an additional application of the main result, we also show how the hierarchical stabilization method known as back-stepping can be used based on the existence of a trajectory preserving and lifting map having a feedback stabilizable control system as codomain.

Keywords: Trajectory lifting morphisms; Local factorization; Affine control systems

1. Introduction

This paper initiates the study of a special class of maps between control systems having the property of preserving and lifting (or reflecting) trajectories. The importance of this class of maps can be recognized by realizing that several hierarchical or recursive control design techniques are implicitly based on the existence of such maps. Understanding the structure of these maps has the potential to extend the applicability of existing hierarchical design methods to larger classes of systems.

One of the most popular examples of hierarchical or recursive design is probably back-stepping [19]. This design method refines a stabilizing controller for a system of the form:

$$\dot{y} = f(y) + g(y)v \tag{1.1}$$

with $y \in \mathbb{R}^n$ being the state and $v \in \mathbb{R}$ being the input, to a stabilizing controller for the larger system

$$\begin{aligned} \dot{y} &= f(y) + g(y)v, \\ \dot{v} &= f'(y, v) + g'(y, v)u \end{aligned} \tag{1.2}$$

where $(y, v) \in \mathbb{R}^{n+1}$ is now the state, $u \in \mathbb{R}$ the input and g' is assumed to be non-zero in the region of interest. What is interesting in this design technique, from the perspective of this paper, is that we can define the map $\phi(y, v) = y$ from the state space of (1.2) to the state space (1.1) with the following two remarkable properties:

1. For any state trajectory $x(t)=(y(t), v(t))$ of (1.2), $\phi(x(t))= y(t)$ is a state trajectory of (1.1);
2. For any trajectory $y(t)$ of (1.1) there exists a trajectory $x(t)$ of (1.2) such that $\phi(x(t)) = y(t)$.

Indeed, if $x(t) = (y(t), v(t))$ is a trajectory of (1.2) then $y(t) = \phi(y(t), v(t))$ is the trajectory of (1.1) corresponding to input $v(t)$. Conversely, if $y(t)$ is a trajectory of (1.1) then $(y(t), v(t))$

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is the trajectory of (1.2) corresponding to input

$$\frac{\dot{v}(t) - f'(y(t), v(t))}{g'(y(t), v(t))}$$

and satisfying $\phi(y(t), v(t)) = y(t)$.

A different scenario where trajectory preserving and lifting maps also appear is in the study of abstractions of control systems initiated by Pappas and co-workers [17,18]. Here, one starts with a control system Σ_F defined on some manifold M and a map $\phi : M \to N$ to some lower dimensional manifold and one seeks to construct a control system Σ_G with state space N such that ϕ has property (1). The motivation behind the construction of Σ_G is that the lower dimensionality of Σ_G renders its analysis simpler and hopefully properties studied in Σ_G will lift to Σ_F under the right technical assumptions. An instance of this approach is described in [21] where the problem of designing trajectories for Σ_F joining point a to point b is converted into the problem of designing trajectories for Σ_G joining point $\phi(a)$ to point $\phi(b)$ followed by a constructive procedure lifting designed trajectories from Σ_G to Σ_F. Essential to this work is the fact that ϕ satisfies properties (1) and (2) for certain classes of trajectories.

Differential flatness (see [5] for an introduction to this concept) can also be understood under the light of trajectory preserving and lifting maps. Given a differentially flat system Σ_F equipped with a flat output $\phi : \mathbb{R}^m \to \mathbb{R}^n$ we can always construct the trivial control system Σ_G on $\mathbb{R}^n$ defined by $\dot{y} = v$ where $y \in \mathbb{R}^n$ is the state and $v \in \mathbb{R}^n$ the input. Since any curve in $\mathbb{R}^n$ is a trajectory of Σ_G we immediately have that ϕ satisfies property (1). Furthermore, being ϕ a flat output we also know that for every trajectory $y(t)$ there exists a trajectory $x(t)$ of Σ_F satisfying $\phi(x(t)) = y(t)$ which shows that (2) is also satisfied. However, more is true in this case. Not only trajectories of Σ_G can be lifted to trajectories of Σ_F as this lifting operation is unique, that is, for every trajectory $y(t)$ of Σ_G there is one and only one trajectory of Σ_F mapping to $y(t)$ under ϕ. Note that this uniqueness property also holds for systems in triangular form to which back-stepping is applicable. On the other extreme we have bisimilar control systems. If Σ_F is bisimilar to control system Σ_G through a relation defined by the graph of a map $\phi : M \to N$, then by definition[1] of bisimulation, (1) is satisfied and every trajectory ofΣ_G can be lifted not to one but to a family of trajectories. In more detail we have that for every trajectory $y(t)$ of Σ_G and for every point $x \in M$ satisfying $\phi(x) = y(0)$ there exists a lifting trajectory $x_x(t)$ of Σ_F satisfying $\phi(x_x(t)) = y(t)$ and $x_x(0) = x$. The situations just described correspond to two extreme cases since in general a trajectory preserving and lifting map does not admit unique liftings neither admits lifting for every possible initial condition. However, as we prove in this paper, *every trajectory preserving and lifting map between single-input control affine systems can be locally factored as the composition of two trajectory preserving and lifting maps of the kinds just described.* Based on this decomposition result we show that *under certain rank conditions trajectory preserving maps are also trajectory lifting maps.* Since these rank conditions are easily locally satisfiable *we place the hierarchical stabilization method known as back-stepping in a more general context based on the existence of a trajectory preserving and lifting map having a feedback stabilizable control system as codomain.* This reformulation of back-stepping can be seen as a simple illustration of the more ambitious goal of reformulating hierarchical design algorithms in terms of trajectory preserving (and lifting) maps. By emphasizing the structure of the maps between control systems rather than the structure of the systems per se we hope to gain in versatility and generality when applying existing hierarchical design algorithms.

A related line of inquiry is the study of maps satisfying property (2) but not necessarily property (1) as was done in [6] for the extreme case where trajectories can be lifted for all possible initial conditions. We believe that the results presented in this paper also offer some insight into this "one-sided" aspect of the question of which kinematic reductions [3,2] can be seen as particular examples.

The results presented in this paper rely on the so called geometric approach to nonlinear control [11,14] and are presented in the setting of category theory [12]. Even though category theory only plays a moderate role in the proof of our results, it provides a convenient conceptual setting to study many problems in systems and control theory. Such approach has already been proved useful in the study of quotients [22], bisimulations for dynamical, control and hybrid systems [7,8], mechanical control systems [13] as well as other problems in systems and control theory [4].

2. Notational preliminaries

We follow standard terminology and notation in differential geometry [1]. We will assume all objects to be smooth unless stated otherwise and by smooth we mean infinitely differentiable. We will denote by TM the tangent bundle of a manifold M and by T_xM the tangent space of M at $x \in M$ spanned by $\{\partial/\partial x_1, \ldots, \partial/\partial x_m\}$ where $(x_1, \ldots, x_m)$ are the coordinates of x. Similarly, we denote by Tf the tangent map of a map $f : M \to N$ while T_xf denotes the tangent map of f evaluated at $x \in M$. Recall that T_xf maps tangent vectors in $X \in T_xM$ to tangent vectors $T_xf \cdot X = Y \in T_{f(x)}N$. For each $x \in M$, $T_xf \in L(\mathbb{R}^m, \mathbb{R}^n)$ where $L(\mathbb{R}^m, \mathbb{R}^n)$ denotes the space of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $m = \dim(M)$, $n = \dim(n)$. The kernel of T_xf is denoted by $\ker(T_xf)$ and when its dimension does not change with x we say that f has constant rank. If, furthermore, T_xf is surjective then f is called a submersion. When f has constant rank $f^{-1}(y) = \{x \in M \mid f(x) = y\}$ is a submanifold of M called the fiber of f over y. We say that f has connected fibers when the manifolds $f^{-1}(y)$ are connected for every $y \in N$. By an affine distribution we will mean a function assigning to each $x \in M$ an affine subspace of T_xM. Recall that a subset S of a vector space is said to be an affine space when for any $s, s' \in S$ we have $\lambda s + \lambda' s' \in S$ for any $\lambda + \lambda' = 1$ and $\lambda, \lambda \in \mathbb{R}$. Similarly a function $f(x, y)$ is said to be affine in y when $f(x, \lambda y + \lambda' y') = \lambda f(x, y) + \lambda' f(x, y')$ in which case

[1] See for example [23,20,15] for a discussion of bisimulation in a systems theoretic context.

can be written as $f(x, y) = \alpha(x) + \beta(x)y$. The exterior derivative of a real valued map f will be denoted by $\mathbf{d}f$ while the Lie derivative of f along vector field X will be denoted by $L_X f$. Iterated Lie derivatives are defined by the recursion $L_X^0 f = f$ and $L_X^{i+1} f = L_X(L_X^i f)$. Finally, the notation $[X, Y]$ will be used to denote the Lie bracket between vector fields X and Y.

3. The category of affine control systems

Informally speaking, a category is a collection of *objects* and *morphisms* between the objects relating the structure of the objects. If one is interested in understanding vector spaces, it is natural to consider vector spaces as objects and linear maps as morphisms since they preserve the vector space structure. This choice for objects and morphisms defines **Vect**, the category of vector spaces. Choosing manifolds for objects leads to the natural choice of smooth maps for morphisms and defines **Man**, the category of smooth manifolds. In this section we introduce the category of affine control systems which we regard as the natural framework to study trajectory lifting morphisms. Besides providing an elegant language to describe the constructions to be presented, category theory also offers a conceptual methodology for the study of objects, affine control systems in this case. Since our results are of local nature we define affine control systems directly on open subsets of Euclidean space.

Definition 3.1. A local affine control system $\Sigma = (M, \mathbb{R}^o, F)$ is defined by the following elements:

(1) The state space M, an open subset of $\mathbb{R}^m$;
(2) The input space $\mathbb{R}^o$;
(3) The system map $F : M \times \mathbb{R}^o \to TM$ defined by

$$F(x, u) = X(x) + \sum_{i=1}^{o} Z_i(x)u_i,$$

where $x \in M$, $u = (u_1, \ldots, u_o) \in \mathbb{R}^o$, X is a vector field on M and $Z_1, \ldots Z_o$ are linearly independent vector fields on M.

A local affine control system is said to be single-input when $o = 1$.

Since we are working locally there is no loss in generality in assuming that vector fields $X, Z_1, \ldots, Z_o$ are globally defined in M. The linear independence assumption also results in no loss of generality when the distribution spanned by $Z_1, \ldots, Z_o$ has constant rank. In this case if, for example, vector field Z_o is linearly dependent on the remaining vector fields $Z_1, \ldots, Z_{o-1}$ we have $Z_o(x) = \sum_{i=1}^{o-1} c_i(x)Z_i(x)$, for smooth functions c_i, and the feedback $u_i = -c_i(x) + u'_i$ can be used to cancel Z_o. The resulting control system $F'(x, u') = X(x) + \sum_{i=1}^{o-1} Z_i(x)u'_i$ can now be identified with a control system with input space $\mathbb{R}^{o-1}$ where the linear independence assumption is valid.

Definition 3.2. Let $\Sigma_F = (M, \mathbb{R}^o, F)$ and $\Sigma_G = (N, \mathbb{R}^p, G)$ be affine control systems. A map $f = (f_1, f_2) : M \times \mathbb{R}^o \to N \times \mathbb{R}^p$ with $f_1 : M \to N$ and $f_2 : M \times \mathbb{R}^o \to \mathbb{R}^p$ is a morphism from Σ_F to Σ_G if the following diagram commutes:

$$\begin{array}{ccc} M \times \mathbb{R}^o & \xrightarrow{f} & N \times \mathbb{R}^p \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ TM & \xrightarrow{Tf_1} & TN \end{array} \tag{3.1}$$

that is, if the following equality holds:

$$T_x f_1(x) \cdot F(x, u) = G(f_1(x), f_2(x, u)). \tag{3.2}$$

The notion of morphism generalizes the notion of feedback equivalence so many times used in systems and control theory. Recall that control systems Σ_F and Σ_G, defined by $F(x, u) = X(x) + \sum_{i=1}^{o} Z_i(x)u_i$ and $G(y, v) = Y(y) + \sum_{i=1}^{o} W_i(y)v_i$, respectively, are said to be feedback equivalent when there exists a diffeomorphism in the state space $g(x) = y$ and a feedback $h(x, u) = v = h_x(u)$ with $(x, u) \mapsto (g(x), h(x, u))$ invertible and making the feedback transformed system:

$$\begin{aligned} F'(y, v) &= T_{g^{-1}(y)}g \cdot X \circ g^{-1}(y) \\ &\quad + \sum_{i=1}^{o} T_{g^{-1}(y)}g \cdot Z_i \circ g^{-1}(y)h^{-1}_{g^{-1}(y)}(v) \end{aligned}$$

equal to $G(y, v)$. Note that by using $x = g^{-1}(y)$ and $u = h^{-1}_{g^{-1}(y)}(v)$ the equality between $F'(y, v)$ and $G(y, v)$ can be written as

$$\begin{aligned} T_x g \cdot X(x) &+ \sum_{i=1}^{o} T_x g \cdot Z_i(x)u \\ &= Y \circ g(x) + \sum_{i=1}^{o} W \circ g(x)h(x, u) \end{aligned}$$

which is no more than (3.2) with $f_1(x) = g(x)$ and $f_2(x, u) = h(x, u)$.

Local affine control systems introduced in Definition 3.1 and morphisms between local affine control systems introduced in Definition 3.2 define the category of local affine control systems denoted by $\mathbf{ACon_l}$. It follows from the affine nature of the considered control systems that morphism are also affine in the following sense:

Proposition 3.3. *Let* $\Sigma_F \xrightarrow{f} \Sigma_G$ *be a morphism in* $\mathbf{ACon_l}$. *Then,* $f_2(x, u) = \alpha(x) + \beta(x) \cdot u$ *where* $\alpha : M \to \mathbb{R}^p$ *and for each* $x \in M$, $\beta(x) \in L(\mathbb{R}^o, \mathbb{R}^p)$.

Proof. We need to show that f_2 is affine in u, that is, for any $\lambda, \lambda' \in \mathbb{R}$ satisfying $\lambda + \lambda' = 1$ we have $f_2(x, \lambda u + \lambda' u') = \lambda f_2(x, u) + \lambda' f_2(x, u')$. Since $F(x, u)$ is affine in u

we have

$$\begin{aligned}
&G(f_1(x), f_2(x, \lambda u + \lambda' u'))\\
&\quad = T_x f_1(x) \cdot F(x, \lambda u + \lambda' u')\\
&\quad = T_x f_1(x) \cdot (\lambda F(x, u) + \lambda' F(x, u'))\\
&\quad = \lambda T_x f_1(x) \cdot F(x, u) + \lambda' T_x f_1(x) \cdot F(x, u')\\
&\quad = \lambda G(f_1(x), f_2(x, u)) + \lambda' G(f_1(x), f_2(x, u')) \qquad (3.3)
\end{aligned}$$

which shows that $G(f_1(x), f_2(x, u))$ is an affine function of u. But $G(z, v)$ is affine in v and this leads to

$$\begin{aligned}
&G(f_1(x), f_2(x, \lambda u + \lambda' u'))\\
&\quad = \lambda G(f_1(x), f_2(x, u)) + \lambda' G(f_1(x), f_2(x, u'))\\
&\quad = G(f_1(x), \lambda f_2(x, u) + \lambda' f_2(x, u')).
\end{aligned}$$

It now follows from injectivity of $G(z, v)$ on v (recall that vector fields $W_1, \ldots, W_o$ are linearly independent) that the above equality implies $f_2(x, \lambda u+\lambda' u')=\lambda f_2(x, u)+\lambda' f_2(x, u')$ which concludes the proof. □

Properties of affine control systems are sometimes easily studied with the help of a naturally induced affine distribution.

Definition 3.4. With each local affine control system Σ_F we associate an affine distribution $\mathscr{A}_F$ defined by

$$\mathscr{A}_F(x) = X(x) + \mathrm{span}_{\mathbb{R}}\{Z_1(x), \ldots, Z_o(x)\}.$$

Example 3.5. We now provide an example of a morphism between control systems. Consider control system Σ_F defined by

$$\dot{x}_1 = x_2(1 + x_2 - 2x_3) + x_3(x_1 - 1) + u,$$

$$\dot{x}_2 = x_2^2 + x_3(-1 + x_1 - 2x_2) + u,$$

$$\dot{x}_3 = x_2^2 + x_3(x_1 - 2x_2) + u,$$

$$\dot{x}_4 = -(1 + x_3^2)x_4 + (x_2 - x_3)u$$

and control system Σ_G defined by

$$\dot{y} = y + v.$$

We claim that the pair of maps:

$$f_1(x) = x_1 - x_3, \quad f_2(x, u) = x_2 - x_1$$

defines a morphism from Σ_F to Σ_G. Note that f_2 is of the form $\alpha(x) + \beta(x)u$ with $\beta(x) = 0$. The left-hand side of (3.2) is

$$T_x f_1 \cdot F(x, u) = (\mathbf{d}x_1 - \mathbf{d}x_3)F(x, u) = x_2 - x_3$$

while the right-hand side is

$$y \circ f_1(x) + v \circ f_2(x, u) = x_1 - x_3 + x_2 - x_1 = x_2 - x_3.$$

We thus conclude that $f = (f_1, f_2)$ is indeed a morphism from Σ_F to Σ_G.

4. Trajectories of affine control systems

4.1. The path subcategory

Even though we have already introduced the objects of study, affine control systems, and presented some of its properties, we have not yet defined the fundamental notion of trajectory. Once again we will follow a categorical approach following Joyal's and co-workers work on bisimulation [10]. There are two main reasons for following this approach. One, is that this approach has already proved useful in studying notions of bisimulation for dynamical, control and hybrid systems [7,8]. The other reason, is that by altering the notion of path objects, defined below, we can use similar techniques to study different properties lifted by morphisms.

Definition 4.1. An object Σ_T of $\mathbf{ACon_l}$ is a path object if the following hold:

(1) M is a connected subset of $\mathbb{R}$ containing the origin;
(2) The input space is $\mathbb{R}^0 = \{0\}$;
(3) The system map T is given by $T(t) = (t, 1)$.

A path or trajectory in a local affine control system Σ_F is a morphism $\Sigma_T \xrightarrow{p} \Sigma_F$.

Morphism $p=(p_1, p_2) : \Sigma_T \to \Sigma_F$ captures the usual notion of trajectory since equality (3.2) reduces to

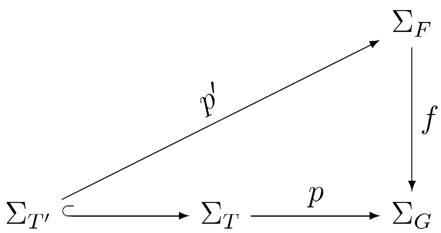
$$\frac{\mathrm{d}}{\mathrm{d}t} p_1(t) = T_t p_1(t) \cdot 1 = F(p_1(t), p_2(t)),$$

where we have identified the function p_2 defined on $M \times \{0\}$ with a function p_2 defined on M. The above definition is no more than an elegant way of expressing trajectories through the use of morphisms. At this point it is important to show that morphisms of control systems have property (1) mentioned in the Introduction. This immediately follows from our definition since given a path $\Sigma_T \xrightarrow{p} \Sigma_F$ in Σ_F and a morphism $\Sigma_F \xrightarrow{f} \Sigma_G$ from Σ_F to Σ_G it follows immediately that $f \circ p$ is a morphism from Σ_T to Σ_G, therefore a path in Σ_G. We will also need to consider path sub-objects since finite explosion times may require reducing the interval where trajectories are defined. We will say that $\Sigma_{T'}$ is a sub-object of Σ_T, denoted by $\Sigma_{T'} \hookrightarrow \Sigma_T$, if $\Sigma_{T'}$ is a path object with $M' \subseteq M$ and $T' = T \circ i$ for the natural inclusion $i : M' \to M$.

4.2. Path lifting morphisms

Although morphisms in $\mathbf{ACon_l}$ preserve trajectories by construction not every morphism reflects or lifts trajectories.

Definition 4.2. Let $\Sigma_F \xrightarrow{f} \Sigma_G$ be a morphism in $\mathbf{ACon_l}$. Morphism f is said to be path lifting if for any path object Σ_T and any morphism $\Sigma_T \xrightarrow{p} \Sigma_G$ there exists a path sub-object $\Sigma_{T'} \hookrightarrow \Sigma_T$ and a morphism $\Sigma_{T'} \xrightarrow{p'} \Sigma_F$ making the following diagram commutative:



(4.1)

A path lifting morphism f is said to be

- *Singular* when p' is unique;
- *Total* when for every $x \in f_1^{-1}(p_1(0))$ there exists a morphism $\Sigma_T \xrightarrow{p'_x} \Sigma_F$ making diagram (4.1) commutative and satisfying $p'_{x1}(0) = x$.

When p' has the same (time) domain as p we can simply take $\Sigma_{T'} = \Sigma_T$ in diagram (4.1). However, this will not be possible, in general, since trajectories are only guaranteed to exist for sufficiently small time intervals. It follows immediately from diagram (4.1) that a necessary condition for f to be a path lifting morphism is surjectivity of f_1. In addition to surjectivity other conditions must hold for a morphism to be path lifting. The study of such conditions requires the use of extensions of affine control systems introduced in the next section.

5. Extensions

The operation of extension allows to increase the state space dimension of a control system while retaining many of its properties. Extensions will play an important role in the factorization of path lifting morphisms.

Definition 5.1. Let $\Sigma = (M, \mathbb{R}^o, F)$ be a local affine control system. The extension of Σ, denoted by Σ^e, is defined by $\Sigma^e = (M^e, \mathbb{R}^o, F^e)$ where

(1) $M^e = M \times \mathbb{R}^o$;
(2) $F^e((x, u), v) = X(x) + \sum_{i=1}^{o} Z_i(x)u_i + \sum_{i=1}^{o} v_i \frac{\partial}{\partial v_i}$.

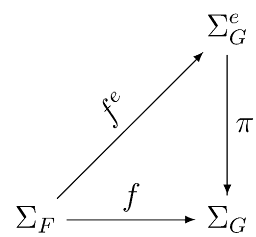
The extension of a control system models the addition of a pre-integrator to the original dynamics. If we start with a system of the form $\dot{x} = X(x) + Z_1(x)u_1 + \cdots + Z_o(x)u_o$ its extension is described by

$$\begin{aligned}
\dot{x} &= X(x) + Z_1(x)u_1 + \cdots + Z_o(x)u_o,\\
\dot{u}_1 &= v_1,\\
&\vdots\\
\dot{u}_o &= v_o,
\end{aligned}$$

where $u_1, \ldots, u_o$ are now regarded as states and $v_1, \ldots, v_o$ are new inputs.

Note that the extension Σ^e of a local affine control system comes equipped with a morphism $\Sigma^e \xrightarrow{\pi} \Sigma$ defined by $\pi_1(x, u) = x$ and $\pi_2((x, u), v) = u$. Furthermore, morphism π is a singular path lifting morphism since any trajectory $p(t) = (p_1(t), p_2(t))$ in Σ defines a unique trajectory $p^e(t) = ((p_1(t), p_2(t)), (\mathrm{d}/\mathrm{d}t)p_2(t))$ in Σ^e satisfying $\pi \circ p^e = p$. The need will arise in the remainder of the paper to compute iterated extensions of the same control system Σ and we shall adopt the following notational conventions: $\Sigma^{e^0} = \Sigma$, $\Sigma^{e^1} = \Sigma^e$ and $\Sigma^{e^k} = (\Sigma^{e^{k-1}})^e$. For the kth extension Σ^{e^k} the natural projection morphism is denoted by $\Sigma^{e^k} \xrightarrow{\pi^k} \Sigma$ and π^0 is simply the identity morphism from Σ to Σ. Note that π^k is still a singular path lifting morphism.

Proposition 5.2. *Let $\Sigma_F \xrightarrow{f} \Sigma_G$ be a morphism in* **ACon**$_\mathbf{l}$ *and assume that $Tf_1 \cdot Z_i = 0$ for $i = 1, \ldots, o$. Then, there exists a unique morphism f^e making the following diagram commutative*:



Proof. Since f is a morphism we have $T_x f_1(x) \cdot F(x, u) = G(f_1(x), f_2(x, u))$ and assumption $Tf_1 \cdot Z_i = 0$ implies that $T_x f_1(x) \cdot F(x, u) = T_x f_1(x) \cdot X(x)$. Therefore, for any $u, u' \in \mathbb{R}^o$ it follows that $G(f_1(x), f_2(x, u)) = G(f_1(x), f_2(x, u'))$. From injectivity of $G(y, v)$ in v we conclude that $f_2(x, u) = f_2(x, u')$ so that we can identify f_2 with a function on M. In particular this means that we can regard $f_1^e = f$ as a map from M to N. To complete the definition of f^e we set $f_2^e = Tf_2 \cdot F$ and show that f^e satisfies equality (3.2):

$$\begin{aligned}
T_x f_1^e \cdot F(x, u) &= (T_x f_1 \cdot F(x, u), T_x f_2 \cdot F(x, u))\\
&= (G(f_1(x), f_2(x)), f_2^e(x, u))\\
&= G^e(f_1^e(x), f_2^e(x, u)).
\end{aligned}$$

To conclude uniqueness of f^e assume that g is another morphism satisfying $\pi \circ g = f$. Since $\pi \circ g = g_1$ we conclude that $g_1 = f = f_1^e$ and as f_e^2 is uniquely determined by $f_1^e = g_1$ and F it follows that $f^e = g$. $\square$

6. Factorization of path lifting morphisms

In this section we present and prove a local factorization result for path lifting morphisms. Note that according to the proof of Proposition 5.2, when $Tf_1 \cdot Z = 0$, f_2 can be identified with a function defined on M. This observation is important to understand the following variation on the notion of relative degree usually found in the geometric control theory literature [9,14]. The slightly different notion presented here will simplify the statement of the main results.

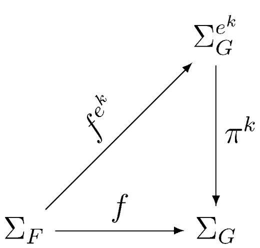
Definition 6.1. Let $\Sigma_F \xrightarrow{f} \Sigma_G$ be a morphism in **ACon**$_\mathbf{l}$ where Σ_F and Σ_G are single-input systems. The relative degree of Σ_F

at $x \in M$ with respect to f is the natural number k satisfying

(1) $k=0$ if $T_x f_1 \cdot Z(x) \neq 0$;
(2) $k=1$ if $T_x f_1 \cdot Z(x)=0$ and $L_Z f_2(x) \neq 0$;
(3) $k=i+1$ if $T_x f_1 \cdot Z(x)=0$, $L_Z L_X^j f_2(x)=0$ for $j=0,\ldots,i-1$ and $L_Z L_X^i f_2(x) \neq 0$.

Note that the relative degree is not necessarily the same at every point in the state space. However, we will only consider systems for which the relative degree does not depend on the particular point $x \in M$.

Theorem 6.2. *Let $\Sigma_F \xrightarrow{f} \Sigma_G$ be a morphism in* **ACon$_\mathbf{I}$** *with f_1 surjective where Σ_F and Σ_G are single-input systems and Σ_F has relative degree k at every $x \in M$ with respect to f. Then, there exist a unique total path lifting morphism $\Sigma_F \xrightarrow{f^{e^k}} \Sigma_G^{e^k}$ making the following diagram commutative*:



Proof. We start by considering the case where $Tf_1 \cdot Z \neq 0$, that is $k=0$. Let $F(x,u)=X(x)+Z(x)u$, $G(y,v)=Y(y)+W(y)v$ and recall that by Proposition 3.3, $f_2(x,u)=\alpha(x)+\beta(x)u$. Evaluating $T_x f_1 \cdot F(x,u)=G(f_1(x), f_2(x,u))$ at $u=0$ provides

$$T_x f_1 \cdot X(x) = Y \circ f_1(x) + W \circ f_1(x)\alpha(x). \tag{6.1}$$

Evaluating now $T_x f_1 \cdot F(x,u) = G(f_1(x), f_2(x,u))$ for an arbitrary $u \in \mathbb{R}$ and using (6.1) we obtain

$$T_x f_1 \cdot Z(x) = W \circ f_1(x)\beta(x).$$

Since the left-hand side is, by assumption, nonzero it follows that $\beta(x)$ must also be nonzero. We can therefore consider the feedback equivalent system $\Sigma_{F'}$ defined by

$$F'(x,u') = F\left(x, \frac{u'-\alpha(x)}{\beta(x)}\right) = X'(x) + Z'(x)u'.$$

Note that f is also a morphism from $\Sigma_{F'}$ to Σ_G and equality $T_x f_1 \cdot F'(x,u') = G(f_1(x), f_2(x,u'))$ now reduces to

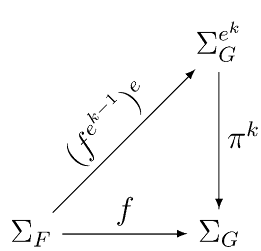
$$\begin{aligned} &T_x f_1 \cdot X'(x) + T_x f_1 \cdot Z'(x)u' \\ &\quad = Y \circ f_1(x) + W \circ f_1(x)\alpha(x) + W \circ f_1(x)\beta(x)\frac{u'-\alpha(x)}{\beta(x)} \\ &\quad = Y \circ f_1(x) + W \circ f_1(x)u'. \end{aligned} \tag{6.2}$$

Let now $q(t)=(q_1(t), q_2(t))$ be any trajectory in Σ_G starting at any $y \in N$, that is, $q_1(0)=y$. Consider also the trajectory $p'(t)$ in $\Sigma_{F'}$ starting at any $x \in M$ such that $f_1(x)=y$, that is, $p_1'(0)=x$ and satisfying also $p_2'=q_2$. Differentiating $f_1 \circ p_1'(t)$ with respect to time and using (6.2) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f \circ p_1'(t) &= T_{p_1'(t)} f_1 \cdot X' \circ p_1'(t) + Z' \circ p_1'(t) p_2'(t) \\ &= Y \circ f_1(p_1'(t)) + W \circ f_1(p_1'(t)) p_2'(t) \\ &= Y \circ f_1(p_1'(t)) + W \circ f_1(p_1'(t)) q_2(t) \end{aligned}$$

thus showing that $f \circ p_1'(t)$ is the trajectory of Σ_G corresponding to input $q_2(t)$. Since trajectories are necessarily unique it follows that we must have $f_1 \circ p_1'(t) = q_1(t)$ from which we conclude that for every trajectory $q(t)$ in Σ_G starting at any $y \in N$ and for any $x \in M$ satisfying $f_1(x)=y$ there exists a trajectory $p'(t)$ in $\Sigma_{F'}$ starting at x and satisfying $f \circ p' = q$. Morphism f is therefore a total path lifting morphism from $\Sigma_{F'}$ to Σ_G and therefore also a total path lifting morphism from Σ_F to Σ_G as Σ_F is isomorphic to $\Sigma_{F'}$. The result thus follows by setting $f^{e^0} = f$.

We now consider the case where $Tf_1 \cdot Z = 0$. By applying Proposition 5.2 we can factor $\Sigma_F \xrightarrow{f} \Sigma_G$ as $\Sigma_F \xrightarrow{f^e} \Sigma_G^e \xrightarrow{\pi} \Sigma_G$. If the relative degree k is one, then f^e is a total path lifting morphism since the relative degree of Σ_F with respect to f^e is zero as $Tf_1^e \cdot Z = (Tf_1 \cdot Z, Tf_2 \cdot Z)$ and $Tf_2 \cdot Z = L_Z f_2 \neq 0$ by definition of relative degree. If $k=2$, then $L_Z f_2 = 0$ which combined with $Tf_1 \cdot Z = 0$ implies $Tf_1^e \cdot Z = 0$. We can therefore apply Proposition 5.2 again to factor $\Sigma_F \xrightarrow{f^e} \Sigma_G^e \xrightarrow{\pi} \Sigma_G$ as $\Sigma_F \xrightarrow{(f^e)^e} \Sigma_G^{e^2} \xrightarrow{\pi^2} \Sigma_G$. We now have accordingto Proposition 5.2, $(f_1^e)^e = f^e = (f, Tf_2 \cdot F)$. If we set $f^{e^2} = (f^e)^e$ we see that the conclusions of the theorem are satisfied since Σ_F has relative degree zero with respect to f^{e^2} and this implies, via the first part of the proof for $k=0$, that f^{e^2} is a total path lifting morphism. If $k=3$ it follows that $L_Z L_X f_2 = 0 = L_Z(Tf_2 \cdot X) = L_Z(f_2^e) = Tf_2^e \cdot Z = 0$ leading to $Tf_1^{e^2} \cdot Z = 0$. We can thus apply Proposition 5.2 repeatedly for a total of k times after which $Tf_1^{e^k} \cdot Z \neq 0$ since $f_1^{e^k} = (f_1^{e^{k-1}}, L_X^{k-1} f_2)$ and by definition of relative degree we have $L_Z L_X^{k-1} f_2 \neq 0$. This leads to the following commutative diagram:



The proof is finished by defining $f^{e^k} = (f^{e^{k-1}})^e$ and noting that f^{e^k} is a total path lifting morphism (by the argument for $k=0$) since the relative degree of Σ_F with respect to f^{e^k} is zero. $\square$

The previous Theorem has the following obvious corollary showing that under the mild regularity conditions captured by the notion of relative degree, every morphism f with f_1 surjective is a path lifting morphism.

Corollary 6.3. *Let* $\Sigma_F \xrightarrow{f} \Sigma_G$ *be a morphism in* **ACon**$_\mathbf{l}$ *with* f_1 *surjective where* Σ_F *and* Σ_G *are single-input systems. If* Σ_F *has relative degree k at every* $x \in M$ *with respect to f, then f is a path lifting morphism.*

Even though, when the relative degree is well defined, a morphism $\Sigma_F \xrightarrow{f} \Sigma_G$ with f_1 surjective is guaranteed to be path lifting, it is neither total nor singular. Theorem 6.2 asserts that f can be uniquely factored into a composition of singular and total path lifting morphisms. This decomposition allows one to regard Σ_F as a control system that is differentially flat with respect to Σ_G up to symmetries. As explained in [20], a total path lifting morphism $\Sigma_F \xrightarrow{f^{e^k}} \Sigma_G^{e^k}$ that is also a surjective submersion necessarily corresponds to the projection from Σ_F onto the quotient control system $\Sigma_G^{e^k}$ obtained from Σ_F by factoring out the controlled invariant distribution defined by all the vector fields K satisfying $Tf_1^{e^k} \cdot K = 0$. Once this controlled invariant distribution, describing symmetries of Σ_F, is factored out we obtain a singular path lifting morphism $\Sigma_G^{e^k} \xrightarrow{\pi^k} \Sigma_G$ that can be regarded as a differentially flat output with respect to Σ_G in the sense that any trajectory of Σ_G lifts uniquely to a trajectory of $\Sigma_G^{e^k}$. The special cases of singular and total path lifting morphisms correspond to the cases where f^{e^k} or π^k are the identity morphisms, respectively, as we now summarize in the following corollary.

Corollary 6.4. *Let* $\Sigma_F \xrightarrow{f} \Sigma_G$ *be a morphism in* **ACon**$_\mathbf{l}$ *with* f_1 *surjective where* Σ_F *and* Σ_G *are single-input systems and* Σ_F *has relative degree k at every* $x \in M$ *with respect to f.*

(1) *If* $k = 0$ *then f is a total path lifting morphism*;
(2) *If* $k = \dim(M) - \dim(N) > 0$ *then f is a singular path lifting morphism.*

Example 6.5. We now revisit Example 3.5 with the purpose of illustrating Theorem 6.2. We first determine if Σ_F has well defined relative degree with respect to f by computing:

$$\begin{aligned} Tf_1 \cdot Z &= (\mathbf{d}x_1 - \mathbf{d}x_3)(Z) = 0, \\ L_Z f_2 &= (\mathbf{d}x_2 - \mathbf{d}x_1)(Z) = 0, \\ L_Z L_X f_2 &= L_Z(-x_2) = (-\mathbf{d}x_2)(Z) = -1 \neq 0. \end{aligned} \tag{6.3}$$

According to Definition 6.1, Σ_F has relative degree $k = 2$ at every $x \in \mathbb{R}^4$ with respect to f. We can thus employ Theorem 6.2 to factor $\Sigma_F \xrightarrow{f} \Sigma_G$ as $\Sigma_F \xrightarrow{f^{e^2}} \Sigma_G^{e^2} \xrightarrow{\pi^2} \Sigma_G$. Control system $\Sigma_G^{e^2}$ is the second extension of Σ_G and is thus defined by

$$\begin{aligned} \dot{y} &= y + v, \\ \dot{v} &= w, \\ \dot{w} &= z \end{aligned}$$

with state $(y, v, w) \in \mathbb{R}^3$ and input $z \in \mathbb{R}$. Morphism $\pi^2 = (\pi_1^2, \pi_2^2)$ is defined by $\pi_1^2(y, v, w) = y$ and $\pi_2^2((y, v, w), z) = v$. The $f_1^{e^2}$ part of morphism f^{e^2} is given by

$$f_1^{e^2} = (f_1, f_2, L_X f_2) = (x_1 - x_3, x_2 - x_1, -x_2)$$

and since $\ker(Tf_1^{e^2}) = \mathrm{span}\{\partial/\partial x_4\}$ we can regard Σ_F as being the linear control system $\Sigma_G^{e^2}$ up to the symmetries defined by the controlled invariant distribution $\mathrm{span}\{\partial/\partial x_4\}$. In other words, we can obtain $\Sigma_G^{e^k}$ from Σ_F by factoring out the controlled invariant distribution $\mathrm{span}\{\partial/\partial x_4\}$ as described in [20].

7. Hierarchical stabilization

Since every morphism f with f_1 surjective is path lifting, under the mild assumption of well defined relative degree, it is natural to try to reformulate hierarchical or recursive design methods based on the existence of path lifting morphisms. In this section we show how this can be done for the particular case of back-stepping. Since we will be dealing with stabilization we shall assume in this section that control systems Σ_F and Σ_G satisfy $X(0) = 0$ and $Y(0) = 0$, respectively. We shall also say that control systems Σ_F and Σ_G are feedback stabilizable if there exist smooth feedback laws $v = l(y)$ and $u = k(x)$ satisfying $l(0) = 0$ and $k(0) = 0$, and rendering $G(y, l(y))$ and $F(x, k(x))$ asymptotically stable, respectively.

For a singular path lifting morphism $\Sigma_F \xrightarrow{f} \Sigma_G$ Theorem 6.2 shows that Σ_F can be regarded as the kth extension of Σ_G. We can thus apply back-steping to construct a feedback controller for Σ_F, provided that a feedback controller stabilizing Σ_G exists, since the kth extension of Σ_G is a system in triangular form. However, since path lifting morphisms are not necessarily singular we cannot directly use back-stepping arguments to infer feedback stabilizability from the existence of a path lifting morphism $\Sigma_F \xrightarrow{f} \Sigma_G$ to a feedback stabilizable system Σ_G. We now show how to overcome this difficulty based on the notion of zero-detectability.

Definition 7.1. Let $\Sigma_F \xrightarrow{f} \Sigma_G$ be a morphism in **ACon**$_\mathbf{l}$. Control system Σ_F is said to be 0-detectable with respect to f if any path $\Sigma_T \xrightarrow{p} \Sigma_F$ with $f \circ p(t) = 0$ for $t \geqslant 0$ implies $\lim_{t\to\infty} p_1(t) = 0$.

This notion of detectability is the key ingredient to deal with arbitrary path lifting morphisms.

Proposition 7.2. *Let* $\Sigma_F \xrightarrow{f} \Sigma_G$ *be a path lifting morphism in* **ACon**$_\mathbf{l}$ *where* Σ_F *and* Σ_G *are single-input systems*, Σ_F *has relative degree k at every* $x \in M$ *with respect to f and* Σ_G *is feedback stabilizable. Then,* Σ_F *is feedback stabilizable if it is* 0-*state detectable with respect to morphism* f^{e^k} *defined in Theorem* 6.2.

Proof. Consider the diagram $\Sigma_F \xrightarrow{f^{e^k}} \Sigma_G^{e^k} \xrightarrow{\pi^k} \Sigma_G$ whose existence is asserted by Theorem 6.2. Control system Σ_G is asymptotically stabilizable by assumption and since $\Sigma_G^{e^k}$ is the kth extension of Σ_G we can use back-stepping (see [19]) to construct a smooth feedback controller $v = l(y)$ rendering $G^{e^k}(y, l(y))$ asymptotically stable. Note that $l(0) = 0$. Let us denote by $\Sigma_F(k)$ and $\Sigma_G^{e^k}(l)$ the dynamical systems resulting from composing

Σ_F and $\Sigma_G^{e^k}$ with feedback controllers $u = k(x)$ and $v = l(y)$, respectively. We now claim existence of a controller $u = k(x)$ for Σ_F such that for any path $\Sigma_T \xrightarrow{p} \Sigma_F(k)$, $f^{e^k} \circ p$ is a path in $\Sigma_G^{e^k}(l)$. The desired controller $u = k(x)$ is given by

$$k(x) = \frac{l \circ f_1^{e^k}(x) - \alpha(x)}{\beta(x)}, \tag{7.1}$$

where $\alpha(x) + \beta(x)u = f_2^{e^k}(x, u)$. Note that $k(x)$ is well defined since Σ_F has relative degree 0 with respect to f^{e^k} which implies that $Tf_1^{e^k} \cdot Y \neq 0$ and as shown in the proof of Theorem 6.2 this leads to $\beta(x) \neq 0$. We now show that $k(x)$ has the desired properties. Let p be a path in $\Sigma_F(k)$. Differentiating $f_1^{e^k} \circ p_1$ with respect to time we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f_1^{e^k} \circ p_1 &= T_{p_1} f_1^{e^k} \cdot F(p_1, k \circ p_1) \\
&= T_{p_1} f_1^{e^k} \cdot X \circ p_1 + (T_{p_1} f_1^{e^k} \cdot Z \circ p_1) k \circ p_1 \\
&= Y \circ f_1^{e^k} \circ p_1 + (W \circ f_1^{e^k} \circ p_1)\alpha \circ p_1 \\
&\quad + (W \circ f_1^{e^k} \circ p_1)(\beta \circ p_1) k \circ p_1 \\
&= Y \circ f_1^{e^k} \circ p_1 + (W \circ f_1^{e^k} \circ p_1)\alpha \circ p_1 \\
&\quad + (W \circ f_1^{e^k} \circ p_1)(\beta \circ p_1) \frac{l \circ f_1^{e^k} \circ p_1 - \alpha \circ p_1}{\beta \circ p_1} \\
&= Y \circ f_1^{e^k} \circ p_1 + (W \circ f_1^{e^k} \circ p_1) l \circ f_1^{e^k} \circ p_1 \\
&= G^{e^k}(f_1^{e^k} \circ p_1, l \circ f_1^{e^k} \circ p_1)
\end{aligned}$$

which shows that $f^{e^k} \circ p$ is a path q in $\Sigma_G^{e^k}(l)$. It now follows from asymptotic stability of $G^{e^k}(y, l(y))$ that for every path $\Sigma_T \xrightarrow{p} \Sigma_F$ we have

$$\lim_{t \to \infty} f_1^{e^k} \circ p_1(t) = \lim_{t \to \infty} q_1(t) = 0,$$

$$\begin{aligned}
\lim_{t \to \infty} f_2^{e^k}(p_1(t), k \circ p_1(t)) &= \lim_{t \to \infty} l \circ f_1^{e^k} \circ p_1(t) \\
&= \lim_{t \to \infty} l \circ q_1(t) = l(0) = 0.
\end{aligned}$$

The result now follows from a straightforward application of LaSalle's invariance principle to the set defined by $(f^{e^k})^{-1}(0)$ combined with the 0-state detectability assumption. □

This reformulation of back-stepping shows that determining stabilizability of a single-input affine control system can be done by determining the existence of a morphism to a stabilizable single-input affine control system. We thus see that path lifting morphisms allow us to relate or to transfer stabilizability properties between control systems. Proposition 7.2 can also be seen as a generalization of the results in [16] to the nonlinear single-input case.

Example 7.3. Consider again Example 3.5. Control system Σ_G is clearly feedback stabilizable and $v = -2y$ is one possible stabilizing feedback. Following the back-steping stabilization scheme with unit gains we obtain the following stabilizing controller for $\Sigma_G^{e^2}$:

$$z = -10y - 9v - 4w. \tag{7.2}$$

To construct a stabilizing feedback for Σ_F from (7.2) we can use Proposition 7.2 provided that Σ_F is 0-state detectable. To show 0-state detectability we start by computing $f_2^{e^k}$ using equality (3.2) to obtain $f_2^{e^k}(x, u) = \alpha(x) + \beta(x)u$ with $\alpha(x) = -x_2^2 - x_3(-1 + x_1 - 2x_2)$ and $\beta(x) = -1$. We now note that $f_1^{e^k}(x) = 0$ implies $x_1 = x_2 = x_3 = 0$. Therefore, $f_2^{e^k}(x, u) = 0$ implies $u = 0$. Restricting the dynamics of x_4 to the set $(f^{e^k})^{-1}(0)$ defined by $u = x_1 = x_2 = x_3 = 0$ we obtain $\dot{x}_4 = -x_4$ from which we immediately conclude that Σ_F is 0-state detectable with respect to f^{e^k}. The resulting stabilizing feedback controller is then given by (7.1) and is equal to

$$\begin{aligned}
k(x) &= \frac{-10(x_1 - x_3) - 9(x_2 - x_1) + 4x_2 - \alpha(x)}{\beta(x)} \\
&= x_1 + 5x_2 - x_2^2 - x_3(x_1 - 2x_2 + 11).
\end{aligned}$$

8. Conclusions

The results described in this paper constitute the first step to understand and place in a broader setting the many existing hierarchical and recursive control design algorithms. Even though only the single-input case has been discussed we believe that a similar decomposition result should hold also for the multi-input case. In addition to a study of the multi-input case, ongoing research is focusing on the study of weaker forms of path lifting in order to extend hierarchical and recursive control design techniques to broader classes of systems.

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