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ON EQUIVALENCE OF THE MOVING MASS AND
MOVING OSCILLATOR PROBLEMS

Alexander V. Pesterev
Institute for Systems Analysis
Russian Academy of Sciences
Moscow, 117312
Russia
Email: pesterev@isa.ru

Lawrence A. Bergman
Aeronautical and Astronautical
Engineering Department
University of Illinois
Urbana, Illinois 61801
Email: lbergman@uiuc.edu

Chin An Tan
Department of Mechanical
Engineering, Wayne
State University
Detroit, Michigan 48202
Email: tan@tan.eng.wayne.edu

T.-C. Tsao
Mechanical and Aerospace
Engineering Department
University of California in Los Angeles
Los Angeles, California 90095
Email: ttsao@seas.ucla.edu

Bingen Yang
Department of Mechanical
Engineering, University of
Southern California
Los Angeles, California 90089
Email: bingen@almaak.usc.edu

ABSTRACT

Asymptotic behavior of the solution of the moving oscillator problem is examined for large values of the spring stiffness for the general case of nonzero beam initial conditions. In the limit of infinite spring stiffness, the moving oscillator problem for a simply supported beam is shown to be not equivalent in a strict sense to the moving mass problem; i.e., beam displacements obtained by solving the two problems are the same, but the higher-order derivatives of the two solutions are different. In the general case, the force acting on the beam from the oscillator is shown to contain a high-frequency component, which does not vanish, or even grows, when the spring coefficient tends to infinity. The magnitude of this force and its dependence on the oscillator parameters can be estimated by considering the asymptotics of the solution for the initial stage of the oscillator motion. For the case of a simply supported beam, the magnitude of the high-frequency force linearly depends on the oscillator eigenfrequency and velocity. The deficiency of the moving mass model is noted in that it fails to predict stresses in the bridge structure. Results of numerical experiments are presented.

1 INTRODUCTION

The problem of calculation of the dynamic response of a distributed parameter system carrying one or more traveling subsystems is very important in many engineering applications related, for example, to the analysis and design of highway and railway bridges, cable-railways, and the like. Two simple models of moving subsystems are generally accepted in the studies on this subject where the emphasis is put on the dynamics of the distributed parameter system rather than on that of the moving subsystem: *moving mass* and *moving oscillator* models. The difference between the two models is that the stiffness of the coupling between the moving subsystem and the continuum in the former model is assumed infinite. In what follows, the problems of the vibration of the distributed parameter system due to the moving mass or oscillator will be referred to as the moving mass or moving oscillator problems, respectively. There is a large body of literature devoted to these problems, and a number of methods for solving them have been developed during last several decades. We refer the interested reader to the ample lists of references in Yang at al. (2000) and Pesterev at al. (2000a), as well to those in other

works cited throughout this paper. Some discussion of papers on the moving mass problem related to the main subject discussed in this study is given in Section 7.

The purpose of this study is to examine the asymptotic behavior of the solution of the moving oscillator problem for large values of the spring stiffness and to establish the relationship between the moving oscillator and moving mass problems. The reasons for this study are as follows.

It is commonly accepted that the moving oscillator problem in the limit of infinite coupling stiffness is equivalent to the moving mass one (see, e.g., Yang and Lin, 1995; Yang and Yau, 1997). One can also meet the statements in some papers that the authors used large values of the spring stiffness in their numerical experiments modeling thus the moving mass problem. At first glance, the assumption about the equivalence of the two problems seems to be valid taking into account the fact that the amplitude of the oscillator vibration vanishes when the spring stiffness goes to infinity. It is also substantiated by numerous results of numerical experiments presented in the literature, which show the convergence of the solution of the moving oscillator problem as the spring stiffness grows. To the authors' best knowledge, however, the validity of this assertion has never been proved in the literature but was taken for granted.

On the other hand, when modeling the multiple moving oscillator problem, one can observe that the force acting on the beam from the second oscillator is very different from that from the first one. Figure 1 illustrates this; it shows the time history of the forces acting on the unit dimensionless simply supported beam with zero initial conditions traversed by two identical high-frequency oscillators of unit weight moving with the velocity $v = \pi/2$ (more details about this example, as well as about the subsequent illustrations, are given in Section 8). The second oscillator enters the left end of the beam at the moment when the first oscillator leaves the beam. Both oscillators are assumed to have zero initial conditions at the moment when they enter the beam. Thus, the problem is decomposed into two problems of one moving oscillator (in the intervals $[0, 2/\pi]$ and $[2/\pi, 4/\pi]$, respectively). The only difference between these two problems is that the beam initial conditions in the second problem are nonzero. The difference in the two forces is easily seen: whereas the force from the first oscillator may be associated with the inertia of the moving mass, the force from the second oscillator contains a high-frequency component of large magnitude, which is not typical to a moving mass solution. This phenomenon implies that the solution to the problem considered cannot adequately be approximated by any moving mass solution.

The above was a motivation to more closely examine the effect of nonzero beam initial conditions on the moving oscillator solution and to compare the latter solution with the corresponding moving mass one. It will be shown that the moving oscillator solution does not tend to the corresponding moving mass one in a strict sense and that the force acting on the beam from the oscil-
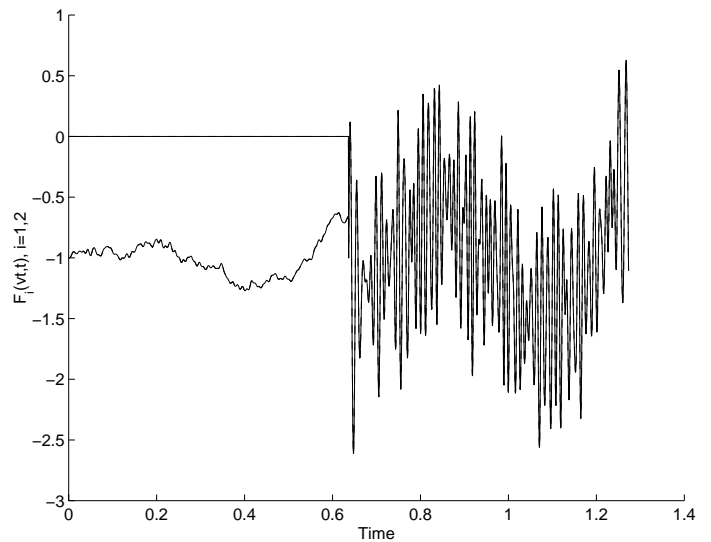

Figure 1. Forces from two oscillators traversing the SS beam one after another

lator contains a high-frequency component, which does not vanish, or even grows, when the spring stiffness increases. In spite of this, the two solutions are still equivalent in terms of the beam displacement (which is further referred to as *weak equivalence*).

The examination of the equivalence problem brought us to the conclusion about *deficiency of the moving mass model,* which is discussed in Section 7. While the moving mass model can be used to accurately approximate the displacement of a long bridge due to a real vehicle with a stiff suspension, it fails to predict stresses. The issue of the deficiency of the moving mass problems comes to existence when the beam initial conditions are allowed to be nonzero and originates from the fact that the moving mass problem statement is physically incorrect in this case. Note that, in most of publications on the moving mass problem, the initial conditions for the beam either are not discussed at all or are assumed to be zero. Even if the governing equation is written for arbitrary initial conditions, the discussion is usually reduced to zero ones by means of the *magic words* "without loss of generality." Indeed, in many cases, the consideration of zero initial conditions simplifies calculations and results in no loss of generality. However, this expedient does not work in the case of the moving mass problem, which is an idealization obtained by assuming infinitely large stiffness of coupling between the subsystems.

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2 PROBLEM STATEMENT

The vibration of a uniform beam traversed by an oscillator of mass m_0 attached to the beam through a spring of stiffness k moving with a constant velocity v is governed by the equations

$$\rho \frac{\partial^2}{\partial t^2} w + EI \frac{\partial^4}{\partial x^4} w = -(m_0 g + k(w(vt,t) - z(t)))\delta(x - vt), \quad (1)$$

$$m_0 \ddot{z} = k(w(vt,t) - z(t)), \ 0 \le x \le L, \ 0 \le t \le L/v. \quad (2)$$

subject to given boundary and initial conditions, where $z(t)$ is the absolute displacement of the lumped mass. The beam ends are assumed to be fixed (simply supported or clamped).

The equations governing the moving mass solution $w_{mm}(x,t)$ is well known to be

$$\rho \frac{\partial^2}{\partial t^2} w_{mm} + EI \frac{\partial^4}{\partial x^4} w_{mm} =$$
$$- \left[m_0 g + m_0 \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right)^2 w_{mm}(x,t) \Big|_{x=vt} \right] \delta(x - vt) \quad (3)$$

subject to given boundary and initial conditions.

The basic purpose of this study is to examine the asymptotics of the solution of equation (1), (2) for large values of the spring stiffness k, to find out what new phenomena are associated with nonzero beam initial conditions, and to understand whether the moving mass and moving oscillator problems are equivalent in the limit of infinite spring stiffness.

3 ASYMPTOTIC FORMULA FOR THE INTERACTION FORCE

Let beam initial conditions be arbitrary, and let k be large. If we compare equations (1), (2) with (3), we see that the solution to the former equations depends on two additional parameters, the initial vertical displacement $z(0)$ and velocity $\dot{z}(0)$ of the oscillator. Under the assumption that the oscillator initial conditions are due to external forces of finite magnitude acting on the oscillator before it enters the beam, it can be easily shown that we should confine our consideration to the case where

$$z(0) = O\left(\frac{1}{\omega_0^2}\right), \ \dot{z}(0) = O\left(\frac{1}{\omega_0}\right). \quad (4)$$

Let us rewrite equation (2) in the form

$$\ddot{z} + \omega_0^2 z = \omega_0^2 w(vt,t), \quad (5)$$

where $\omega_0 = \sqrt{k/m_0}$ is the oscillator eigenfrequency. The solution to equation (5) is the sum of the solution to the homogeneous

equation (5) satisfying given initial conditions and the particular solution satisfying zero initial conditions and is given by

$$z(t) = z(0) \cos \omega_0 t + \frac{1}{\omega_0} \dot{z}(0) \sin \omega_0 t$$
$$+ \omega_0 \int_0^t w(v\tau, \tau) \sin \omega_0 (t - \tau) \, d\tau. \quad (6)$$

Taking the integral on the right-hand side of (6) by parts four times with regard to the condition $w(0,0) = 0$ and dropping the terms of order less than $\frac{1}{\omega_0^2}$, we get

$$z(t) = z(0) \cos \omega_0 t + \frac{1}{\omega_0} \dot{z}(0) \sin \omega_0 t - \frac{1}{\omega_0} \dot{w}(0,0) \sin \omega_0 t$$
$$+ w(vt,t) - \frac{1}{\omega_0^2} \ddot{w}(vt,t) + \frac{1}{\omega_0^2} \ddot{w}(0,0) \cos \omega_0 t + o\left(\frac{1}{\omega_0^2}\right), \quad (7)$$

where $\dot{w}(vt,t)$ and $\ddot{w}(vt,t)$ are the convective derivatives,

$$\dot{w}(vt,t) \equiv \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) w(x,t) \Big|_{x=vt},$$
$$\ddot{w}(vt,t) \equiv \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right)^2 w(x,t) \Big|_{x=vt}.$$

Note that, for $t = 0$, we have

$$\dot{w}(0,0) = v w_x(0,0), \ \ddot{w}(0,0) = v^2 w_{xx}(0,0) + 2v w_{xt}(0,0), \quad (8)$$

where w_{xx} and w_{xt} are the partial derivatives. It immediately follows from equations (7) and (4) that the relative oscillator displacement $z(t) - w(vt,t)$ vanishes when the spring stiffness goes to infinity.

Further, multiplying both sides of equation (7) by the spring stiffness k, we find the elastic force of interaction between the beam and oscillator

$$f(t) \equiv -k(w(vt,t) - z(t)) = -m_0 \omega_0 (\dot{w}(0,0) - \dot{z}(0)) \sin \omega_0 t +$$
$$m_0 (\ddot{w}(0,0) + \omega_0^2 z(0)) \cos \omega_0 t - m_0 \ddot{w}(vt,t) + o(1). \quad (9)$$

As can be seen, in the general case of the oscillator and beam initial conditions, the elastic interaction force contains two harmonic components with the frequency ω_0 due to the eigenvibration of the oscillator, which do not vanish when k goes to infinity. Moreover, the amplitude of the first of them grows infinitely as the spring stiffness tends to infinity. This implies that
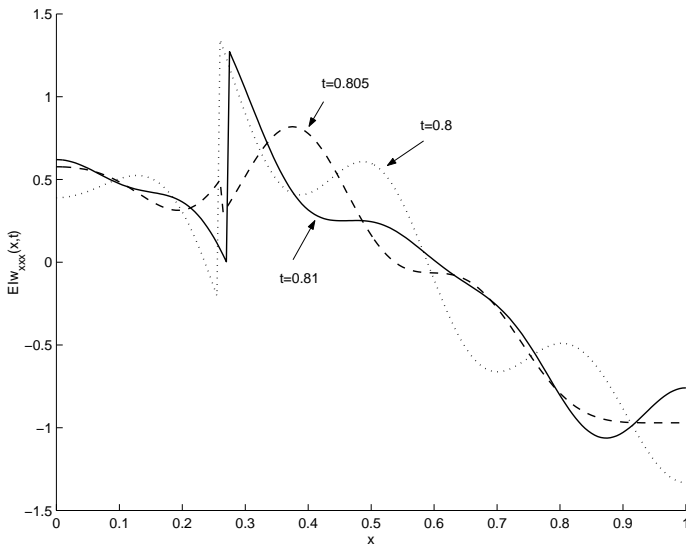
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Figure 2. Shear force distributions at three close instants in the two-oscillator problem (the first oscillator is outside the beam).

the *moving mass and moving oscillator problems are generally not equivalent* in the limit of infinite spring stiffness since the force on the right-hand side of (1) does not tend to that of equation (3). The oscillating character of the interaction force in the moving oscillator problem implies, in particular, that the picture of the shear force distribution rapidly changes in time, especially in the vicinity of the moving oscillator attachment point, since the concentrated force acting on the beam is equal to the jump in the shear force at that point. This phenomenon is illustrated in Fig. 2, which shows the shear force distributions at three close instants, $t = 0.8, 0.805$, and 0.81, for the example discussed above (see explanations to Fig. 1). It demonstrates that the jump in the shear force distribution considerably changes in a fraction of time, which may imply that the beam is subject to "damaging" stresses.

4 CONDITIONS OF EQUIVALENCE OF THE MOVING OSCILLATOR AND MOVING MASS PROBLEMS

Since the operators on the left-hand sides of equations (1) and (3) are exactly the same, the solution of the moving oscillator problem, together with all its derivatives, tends to that of the moving mass problem if the right-hand side of (1) tends to that of equation (3) as $\omega_0 \to \infty$. It follows from equation (9) that the latter condition holds only if oscillator initial conditions satisfy

the relations

$$z(0) = -\frac{\ddot{w}(0,0)}{\omega_0^2}, \quad \dot{z}(0) = \dot{w}(0,0) \equiv v w_x(0,0). \qquad (10)$$

The second condition (10) implies that the velocity of the oscillator at $t = 0$ is directed along the tangent line to the beam at $x = 0$, and the first condition (10) implies that the spring is prestressed to make the force acting on the beam at $t = 0$ equal to the initial inertia force inherent in the moving mass problem.

In the case of zero beam initial conditions, equations (10) are satisfied by taking zero oscillator initial conditions. Let now beam initial conditions be nonzero. If the left beam end is simply supported, the initial slope is generally a certain finite number not depending on the oscillator eigenfrequency, and the second condition (10) cannot be satisfied since, by virtue of (4), $\dot{z}(0)$ must vanish when $\omega_0 \to \infty$. This implies that the moving oscillator problem in the limit of infinite spring stiffness *is not equivalent in the strict sense* to the moving mass problem.

Thus, we may conclude that, except for the case of zero initial conditions, there is no strict equivalence between the two problems in the limit of infinite spring stiffness if the left beam end is simply supported. If the beam left end is clamped, it is always possible to choose oscillator initial conditions such that the moving oscillator problem is strictly equivalent to the moving mass problem in the limit of infinite spring stiffness. If the oscillator initial conditions do not match well the beam initial conditions (do not satisfy (10)), the two problems are not equivalent.

Remark. The incorporation of damping into the oscillator model considerably complicates all calculations and makes the analysis more involved. It is for this reason that we consider here the undamped oscillator. It can be shown however that all basic findings of this study remain valid for the damped case. The elastic force in the damped case is described by the equation similar to equation (9), with the functions $\sin \omega_0 t$ and $\cos \omega_0 t$ being replaced by the $e^{-\alpha t} \sin \sqrt{\omega_0^2 - \alpha^2}$ and $e^{-\alpha t} \cos \sqrt{\omega_0^2 - \alpha^2}$, respectively, where α is the damping coefficient. The coefficients of the latter functions are more complicated functions of the initial conditions and the spring and damper coefficients. It can be proved however that these coefficients vanish if the oscillator initial conditions satisfy the same equations (10). Although the interaction force vanishes shortly after the oscillator enters the beam, it still exists during finite time such that the two problems, as in the undamped case, are not equivalent in the mathematical sense. From the engineering standpoint, it is important that the magnitude of that force immediately after the oscillator enters the beam may be considerable, and the picture of the shear force distribution for an initial stage of the oscillator motion will be similar to that depicted in Fig. 2.

The above analysis shows that high-frequency oscillations

of the elastic (dynamic) force appear when a stiff oscillator enters an already vibrating beam. In the general case, the oscillator initial conditions cannot be adjusted to satisfy (10), and we need to examine the effect of the high-frequency component of the dynamic force on the beam vibration. We will show that the beam displacement is not sensitive to the oscillator initial conditions as long as they satisfy (4) and that, in spite of the "stress" nonequivalence, the two problems are still equivalent in terms of the beam displacements (i.e., the response of the beam due to the moving oscillator tends to that due to the moving mass as $k \to \infty$). We will call this *weak equivalence.*

5 WEAK EQUIVALENCE OF THE MOVING MASS AND MOVING OSCILLATOR PROBLEMS

Let oscillator initial conditions be arbitrary. Equation (9) can be written in the form

$$f(t) = -c_1 m_0 \omega_0 \sin \omega_0 t + c_2 m_0 \cos \omega_0 t - m_0 \ddot{w}(vt,t) + o(1), \tag{11}$$

where c_1 and c_2 are determined by the beam and oscillator initial conditions and do not depend on ω_0. Let us represent the solution to equation (1) in the form

$$w(x,t) = w_{mm}(x,t) + \widetilde{w}(x,t), \tag{12}$$

where $w_{mm}(x,t)$ is the solution to the corresponding moving mass problem (3) satisfying the given initial conditions

$$w_{mm}(x,0) = w(x,0), \frac{\partial}{\partial t} w_{mm}(x,t) \mid_{t=0} = \frac{\partial}{\partial t} w(x,t) \Big|_{t=0},$$

and

$$\widetilde{w}(x,0) = \frac{\partial}{\partial t} \widetilde{w}(x,t) \mid_{t=0} = 0.$$

Substituting (12) into equation (1) with regard to (11), dropping the small-order term, and taking into account that $w_{mm}(x,t)$ satisfies (3), we find that $\widetilde{w}(x,t)$ is governed by the equation

$$\rho \frac{\partial^2}{\partial t^2} \widetilde{w} + EI \frac{\partial^4}{\partial x^4} \widetilde{w} + m_0 \ddot{\widetilde{w}}(vt,t)\delta(x-vt) = -(c_1 m_0 \omega_0 \sin \omega_0 t - c_2 m_0 \cos \omega_0 t)\delta(x-vt), \tag{13}$$

subject to the given boundary conditions and zero initial conditions. The last equation describes the vibration of the beam with the rigidly attached weightless mass m_0 that moves along the beam with the velocity v (or, in other words, the beam with

the mass distribution given by $\rho + m_0 \delta(x - vt))$ excited by the moving harmonic force.

It is evident that the second harmonic force on the right-hand side of equation (13) can be dropped for sufficiently large values of ω_0 (it remains constant when $\omega_0 \to \infty$, whereas the first force increases infinitely). Thus, for simplicity of the notation, it is sufficient to prove that the solution to the equation

$$\rho \frac{\partial^2}{\partial t^2} \widetilde{w} + EI \frac{\partial^4}{\partial x^4} \widetilde{w} + m_0 \ddot{\widetilde{w}}(vt,t)\delta(x-vt) = -c_1 m_0 \omega_0 \sin(\omega_0 t)\delta(x-vt) \tag{14}$$

satisfying the given boundary and zero initial conditions tends to zero when $\omega_0 \to \infty$. The solution to (14) is given by

$$\widetilde{w}(x,t) = -\int_0^t \int_0^L g(x,\xi;t-\tau) c_1 m_0 \omega_0 \sin(\omega_0 \tau)\delta(\xi - v\tau) \, d\xi d\tau$$

$$= -c_1 m_0 \omega_0 \int_0^t g(x,v\tau;t-\tau) \sin \omega_0 \tau d\tau,$$

where $g(x,\xi,t)$ is the dynamic Green's function of the system governed by the left-hand side of equation (14) with regard to the boundary conditions. Although its closed-form representation is not available (and hardly can be found), it is sufficient for our purposes that such a function exists (which follows from physical considerations). Taking the last integral by parts twice, we get

$$\widetilde{w}(x,t) = c_1 m_0 \int_0^t g(x,v\tau;t-\tau) d \cos \omega_0 \tau =$$

$$c_1 m_0 g(x,v\tau;t-\tau) \cos \omega_0 \tau|_{\tau=0}^t - \frac{c_1 m_0}{\omega_0} \frac{d}{d\tau} g(x,v\tau,t-\tau) \sin \omega_0 \tau \mid_{\tau=0}^t$$

$$+ \frac{c_1 m_0}{\omega_0} \int_0^t \frac{d^2}{d\tau^2} g(x,v\tau;t-\tau) \sin \omega_0 \tau d\tau.$$

The first term on the right-hand side of the equation vanishes since $g(x,0;t) = 0$ (fixed left end) and $g(x,\xi;0) = 0$ (the deflection of the system at $t = 0$ due to the unit impulse applied at $t = 0$ is zero), and we finally arrive at the following equation:

$$\widetilde{w}(x,t) = -\frac{c_1 m_0}{\omega_0} \frac{d}{d\tau} g(x,v\tau,t-\tau) \mid_{\tau=t} \sin \omega_0 t$$

$$+ \frac{c_1 m_0}{\omega_0} \int_0^t \frac{d^2}{d\tau^2} g(x,v\tau;t-\tau) \sin \omega_0 \tau d\tau. \tag{15}$$

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It follows from the last equation that, in view of the finiteness of the first and second partial derivatives of the Green's function, the solution to equation (14) tends to zero as $\omega_0 \to 0$,

$$\widetilde{w}(x,t) = O\left(\frac{1}{\omega_0}\right),$$

which proves the weak equivalence of the moving mass and moving oscillator problems in the limit of infinite spring stiffness.

From the physical standpoint, small effect of the additional harmonic forces on the beam vibration is explained by the fact that these forces have high frequency and excite only high-order eigenvibrations of the beam, the contribution of which into the beam response is negligible. Moreover, no resonance phenomena can take place because of the finiteness of the passage time and since the time-varying system governed by the left-hand side of equation (14) has no fixed resonance frequencies.

Although the high-frequency component of the dynamic interaction force does not result in any increase of the beam displacement, the magnitude of this force (or, to be more specific, the magnitude of the total dynamic force) is of great significance, in particular, to pavement wear (DIVINE report, 1998). The magnitude of the additional high-frequency force is determined by the oscillator parameters and the beam and oscillator initial conditions and may be considerable. Thus, it is important to establish the dependence of this force on the oscillator parameters and initial conditions and to find a priori estimates of the peak values of the concentrated force acting on the beam. In the next section, we will show that the magnitude of the dynamic force $f(t)$ can be estimated by considering its asymptotics for small values of time.

6 ESTIMATES FOR THE INTERACTION FORCE

As was shown above, when a stiff oscillator enters the already vibrating beam, there appear a harmonic high-frequency component in the elastic interaction force with the frequency ω_0 and the accompanying force associated with the high-frequency low-amplitude vibration $\widetilde{w}(x,t)$ of the beam excited by the former force. Although, as was shown in Section 5, $\widetilde{w}(x,t)$ tends to zero as $\omega_0 \to \infty$, the force $m_0 \ddot{\widetilde{w}}(x,t)$ associated with this vibration does not (note that this is clearly seen in Fig. 2). These forces are added to the force associated with the inertia of the moving mass such that the resulting dynamic force acting on the beam from the oscillator can be represented as

$$f(t) = -m_0 \ddot{w}_{mm}(vt,t) + f_{ad}(t). \quad (16)$$

By virtue of (9) and (12), the equation for the additional force is given by

$$f_{ad}(t) = -m_0 \omega_0 c_1 \sin \omega_0 t + m_0 c_2 \cos \omega_0 t + m_0 \ddot{\widetilde{w}}(vt,t), \quad (17)$$

where c_1 and c_2 are determined by the oscillator and beam initial conditions and do not depend on ω_0.

If $c_1 \neq 0$ and ω_0 is sufficiently large, the amplitude of the harmonic force can be very well approximated by the first term only. The inertia term $m_0 \ddot{w}_{mm}(vt,t)$ associated with the moving mass solution can also be neglected for large ω_0. The last term in equation (17) is a rather complicated function and can accurately be calculated only numerically. As can be seen from equation (15), its magnitude linearly depends on the amplitude of the exciting harmonic force (and, hence, on ω_0), and, thus, it cannot be generally neglected. Note, however, that, due to the damping inherent in any real problem, the high-frequency vibration of the beam and oscillator decrease rapidly. Thus, from the practical standpoint, it is sufficient to be able to estimate the peak values of the dynamic force on the initial stage of the oscillator motion. For small t such that $vt \ll L$ (the oscillator is close to the left end of the beam), the third term on the right-hand side of (17) can be dropped. This can formally be proved if we take into account that the harmonic force with the frequency ω_0 excites mainly the beam eigenvibrations at the eigenfrequencies ω_n close to ω_0 and that the wave numbers λ_n are proportional to the square roots of the eigenfrequencies, $\sqrt{\omega_n}$. Then it follows that the maximum magnitude of the dynamic force $f(t)$ can be very well approximated by that for small t, which is given by

$$\max_t |f(t)| \approx m_0 \omega_0 v \left(w_x(0,0) - \frac{\dot{z}(0)}{v} \right), \; vt \ll L. \quad (18)$$

Note that the expression in the parentheses is the difference between the initial slope of the beam at $x = 0$ and the direction of the oscillator velocity. As can be seen from (18), the magnitude of the dynamic force is proportional to the oscillator eigenfrequency and velocity.

If c_1 is small (the difference between the initial beam slope and the direction of the oscillator velocity is small) or the spring stiffness is not large enough, the estimate can be improved by taking the second term on the right-hand side of (17) into account. Moreover, we can also take the first term in (16) into account by noting that, for $vt \ll L$, it can be considered constant, $\ddot{w}_{mm}(vt,t) \approx \ddot{w}(0,0)$. Thus, e.g., for a simply supported beam and zero oscillator initial conditions, we get the following approximate formula for the concentrated force acting on the beam

$$F(t) \approx -m_0 g - m_0 \omega_0 v w_x(0,0) \sin \omega_0 t$$
$$-2m_0 v w_{xt}(0,0)(1 - \cos \omega_0 t), \; vt \ll L. \quad (19)$$

Figure 3. Oscillator with $\omega_0 = 200$ traversing the SS beam at $v = \pi/2$.
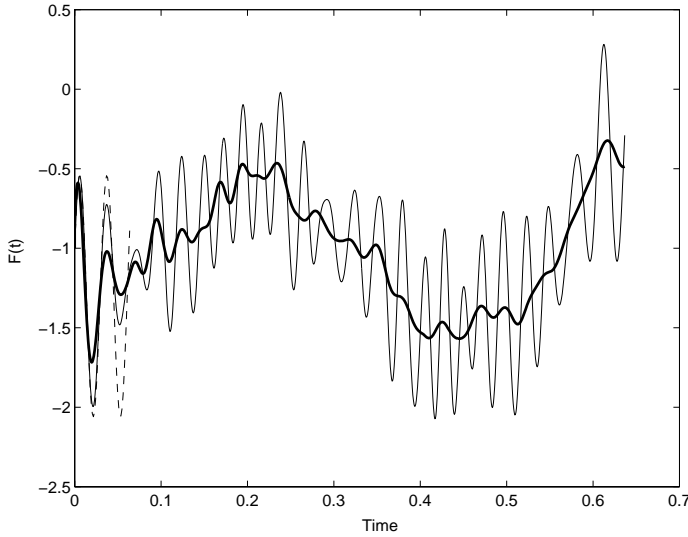

Figure 4. Oscillator with $\omega_0 = 400$ traversing the SS beam at $v = \pi/2$.

Figures 3–5 illustrate the above-said. The concentrated forces $F(t)$ acting on the beam from the undamped oscillators are depicted by the thin solid lines. Figures 3 ($\omega_0 = 200, v = \pi/2$) and 4 ($\omega_0 = 400, v = \pi/2$) demonstrate that the magnitude of the elastic force grows with the increase of the oscillator eigenfrequency. Figures 4 and 5($\omega = 400, v = \pi$) show that it is a linear function of the oscillator velocity. Approximations of the concentrated force for small values of time by means of (19) are shown in Figs. 3–5 by the dashed lines. The bold lines show the forces acting on the beam in the case of the damped oscillators with the damping coefficients $c_0 = 6$ (Fig. 3) and $c_0 = 12$ (Figs. 4 and 5) (for both oscillators, damping is about 15% of the critical damping). These figures clearly demonstrate that (i) the high-frequency oscillations of the elastic force reduce rapidly and (ii) the approximate equation (19) can be used to adequately estimate upper bounds of the peak values of the elastic force.

If c_1 is zero, the dynamic force does not depend on ω_0; in this case, the inertia force cannot generally be neglected, and estimates (18) and (19) are not applicable. In particular, they cannot be applied to the case of an oscillator with zero initial conditions entering a vibrating clamped-clamped beam. Let us show how to estimate the concentrated interaction force without solving numerically the differential equations involved for a particular case of the problem where the clamped-clamped beam freely vibrates at the moment when the oscillator enters the beam (no other oscillators on the beam). Clearly, at the initial stage of the oscillator motion, the effect of the moving oscillator on the beam vibration can be neglected, and we may assume that $w_{mm}(vt, t) \approx w_h(vt, t)$, where $w_h(x, t)$ is the solution to the homogeneous equation (1). The latter solution is easily calculated given the initial displacements $w(x, 0)$ and velocities $w_t(x, 0)$ of the beam points and is given by
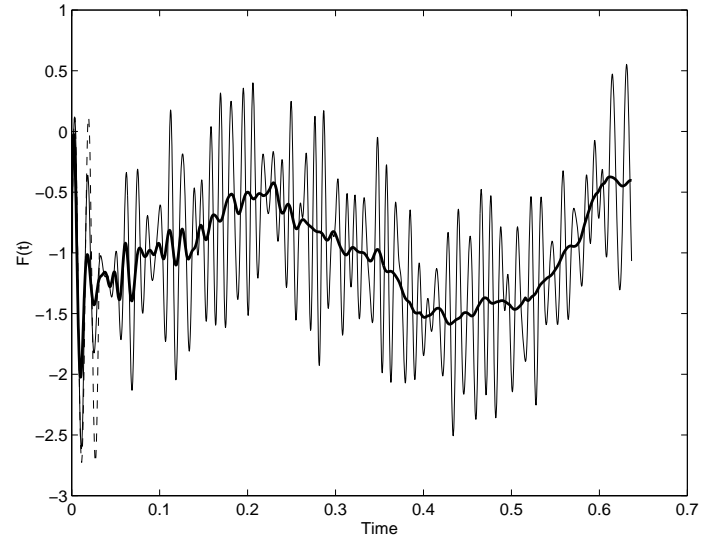
$$w_h(x, t) = \sum_{n=1}^{N} \left[q_{n1} \cos \omega_n t + \frac{q_{n2}}{\omega_n} \sin \omega_n t \right] \varphi_n(x), \qquad (20)$$

where ω_n and $\varphi_n(x)$ are eigenfrequencies and eigenfunctions of the beam, N is the number of the series terms taken into account, and q_{n1} and q_{n2} are the Fourier coefficients of the expansions of the initial functions in terms of the eigenfunctions of the beam,

$$w(x, 0) = \sum_{n=1}^{N} q_{n1} \varphi_n(x), \ w_t(x, 0) = \sum_{n=1}^{N} q_{n2} \varphi_n(x).$$

Substituting $x = vt$ into (20) and differentiating the resulting equation twice, we get the desired approximation for the concentrated interaction force. Figure 6 illustrates this, it shows the calculated numerically force $F(t)$ (solid line) acting on the clamped-clamped beam from the oscillator of eigenfrequency $\omega_0 = 200$ traversing the beam with the velocity $v = \pi/6$ and its approximation by the function $-m_0 g - m_0 \ddot{w}_h(vt, t)$ (dashed line). The beam was assumed to vibrate at its fundamental frequency before the oscillator entered it.

7 DEFICIENCY OF THE MOVING MASS MODEL

The moving mass model is an idealization of the moving oscillator model obtained by assuming infinitely large stiffness of coupling between the subsystems. However, if the beam initial conditions are nonzero, it cannot be obtained from the moving
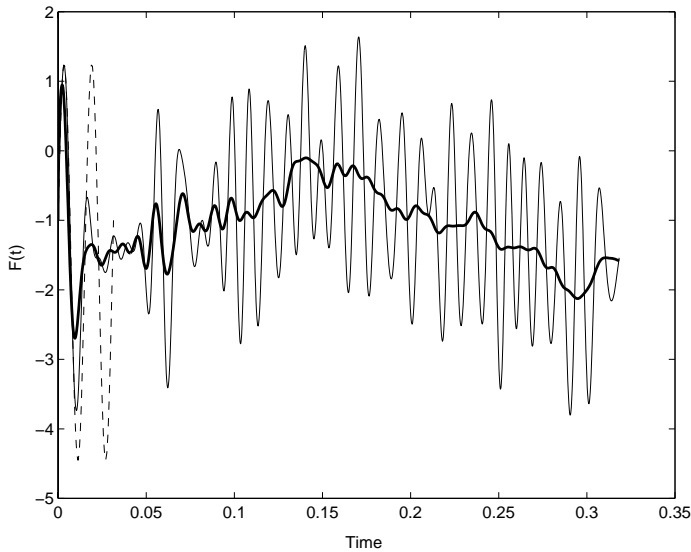
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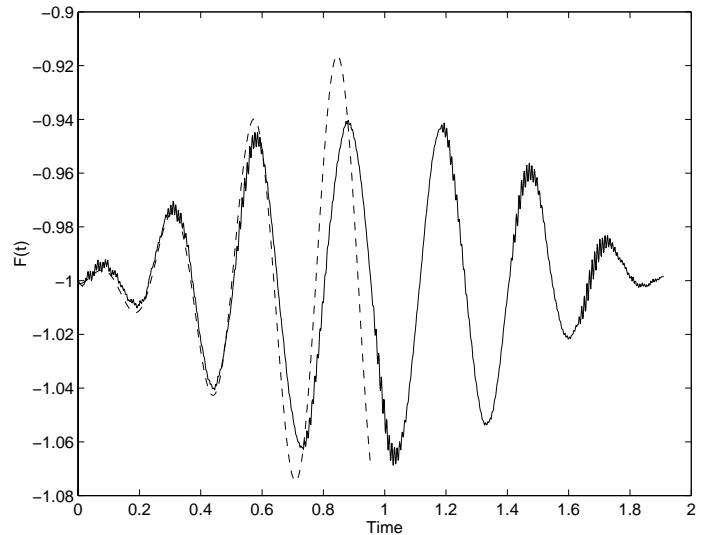Figure 5. Oscillator with $\omega_0 = 400$ traversing the SS beam at $v = \pi$.

Figure 6. Oscillator with $\omega_0 = 200$ traversing the CC beam at $v = \pi/6$.

oscillator model without assuming infinitely large forces acting on the mass, which follows from the second condition in (10). This implies that the moving mass model is *physically incorrect* if initial conditions for a simply supported beam are allowed to be nonzero.

As shown in Section 5, the use of the moving mass model is still justified when we need to calculate the beam displacement. The "cost" of this model incorrectness is that it fails to accurately calculate the concentrated force acting on the beam from the vehicle and, thus, to predict stresses in the beam. Indeed, when a vehicle with a stiff suspension enters an already vibrating bridge, its initial conditions are generally not "in agreement" with those of the bridge (i.e., do not satisfy (10)). As shown in Section 3, this results in the appearance of a high-frequency component in the dynamic interaction force. In certain circumstances, the magnitude of this force may be considerable and exceed that of the inertia force associated with the moving vehicle. Thus, neglecting this force, we are not able to accurately calculate stresses in the bridge. When using the moving mass model, the high-frequency component of the force is missing, which implies the *deficiency of the moving mass model* in that it fails to predict stresses in the bridge.

The deficiency of the moving mass model becomes even more evident when it is applied to solving the problem of several vehicles passing a bridge represented by a simply supported beam. Assuming that a "rigid" vehicle approaches the beam moving along the rigid horizontal surface, we see that the vertical velocity of the vehicle at $x = -0$ is zero. When the vehicle enters the beam, we *must* admit that, in the framework of the moving mass model, its vertical velocity at $x = +0$ is $vw_x(0,0)$, which

implies that the velocity has been instantly changed, which, in turn, suggests infinite force acting on the mass (beam).

In view of the above, it is not surprising that the multiple moving mass problem almost has not been considered in the literature and the problem of stress calculation is not discussed at all. Moreover, in most of publications on the moving mass problem, only zero initial conditions for the beam are considered (e.g., Stanišić (1985) and Sadiku and Leipholz (1987)). In Gbadeyan and Oni (1995), the governing equation is written for the case of several moving masses, and it is stated that the method is applicable to arbitrary initial conditions; however, by means of the *universal expedient* "without loss of generality," the analysis is reduced to one moving mass and zero initial conditions, and no numerical results related to several masses are presented. Moreover, the right-hand side of the governing equation in that paper suggests that all masses enter the beam at the same moment. In Stanišić and Hardin (1969), all masses are also assumed to enter the beam at the same moment and the initial conditions are zero. Lee (1996) examines the case of high velocities of the moving mass and considers the effect of the separation between the mass and beam. At the instant of recontact in that problem, the situation is the same as that at the moment when a moving mass enters the already vibrating beam in the problem considered in this paper. For simplicity, the impact effect in that paper is neglected and the concentrated force is assumed to have a jump at the instant of recontact. Since the paper examines only the beam displacement rather than stresses in the beam, such an approach seems to be justified in view of the analysis given in Section 5 of this paper. In several papers (Nelson and Conover, 1971; Benedetti, 1974; Rao, 2000), results of numerical experiments with several moving masses are presented; however, no

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discussions are given about what happens when a mass enters the already vibrating beam, the initial conditions are not presented, and the problem of stress calculation is not discussed.

8 NUMERICAL EXAMPLES

In our numerical experiments, we employed the unit dimensionless beam ($\rho = EI = L = 1$). The fundamental frequencies of the simply supported and clamped-clamped dimensionless beams are $\omega_1 = \pi^2$ and $\omega_1 \approx 22.37$, respectively. The nondimensionalization procedure is pretty standard (Yang at al., 2000) and is not presented here to save room. The dimensionless oscillator velocity is given in terms of π in order to compare it with the "critical" velocity $v = \pi$ inherent in the moving force problem (Meirovitch, 1967). The oscillator weight was taken equal to one; i.e., $m_0 = 1/g \approx 0.1$, where g is the acceleration of gravity. All figures show the resulting concentrated force $F(t) \equiv -m_0 g + f(t)$ acting on the beam from the oscillator.

The results depicted in Figs. 1 and 2 (two-moving-oscillator problem) correspond to zero beam initial conditions and the oscillator parameters $\omega_0 = 400$ and $v = \pi/2$. In all other experiments with simply supported beam (Figs. 3–5), the beam initial conditions were nonzero and the same to make it possible to examine the effect of variation of the oscillator parameters on the magnitude of the elastic force. To avoid the danger of choosing "unrealistic" initial conditions and the question of what initial functions should be considered as "appropriate," we did the following. First, we numerically solved the one moving oscillator problem for the beam with zero initial conditions and the oscillator parameters $\omega_0 = 400$ and $v = \pi/2$. The functions $w(x,t)$ and $w_t(x,t)$ have been calculated at the moment when the oscillator was at the right end of the beam and were taken as the initial functions $w(x,0)$ and $w_t(x,0)$ for all following experiments. In the experiment with the clamped ends (Fig. 6), the beam was assumed to vibrate at its fundamental frequency at the moment when the oscillator entered it.

To numerically solve the moving oscillator problem, the method described in Pesterev and Bergman (1997) (one oscillator) and Pesterev et al. (2001a) (extension to the multiple moving oscillator problem) was used, which is based on the expansion of the solution in the series in terms of the beam eigenfunctions. The number of the series terms used in all calculations was equal to eight, which is quite sufficient to make the results reliable (note that the maximum oscillator eigenfrequency considered, $\omega_0 = 400$, is less than the seventh beam eigenfrequency $\omega_7 \approx 484$ of the simply supported beam).

The conventional series expansion is known to converge poorly when applied to calculation of the higher-order derivatives of the response. To accurately calculate the shear force distributions depicted in Fig. 2, an improved series expansion suggested in Pesterev et al. (2001b) was employed, which makes use of the beam static Green's function and gives an exact value of the shear force jump.

9 CONCLUSIONS

Asymptotic behavior of the moving oscillator problem has been examined for large values of the spring stiffness for the general case of nonzero beam initial conditions.

1. It has been shown that, in the case of a simply supported beam with nonzero initial conditions, the moving oscillator problem is mathematically not equivalent to the moving mass problem in the limit of infinite spring stiffness. In the case of a clamped beam, the two problems are equivalent only under appropriate choice of the oscillator initial conditions. Nevertheless, when the spring stiffness goes to infinity, the beam displacement obtained by solving the moving oscillator problem tends to that due to the moving mass. Thus, for sufficiently large spring stiffness, the beam displacement is a function of the beam initial conditions and the oscillator mass and velocity but is not sensitive to the spring coefficient and to the oscillator initial conditions. The two solutions differ by their higher-order derivatives and by the dynamic force acting on the beam from the mass.

2. The magnitude of the high-frequency component in the concentrated force has been shown to linearly depend on the oscillator eigenfrequency and velocity if the vector of the oscillator velocity is not directed along the tangent line to the beam at its left end. Asymptotic formulas (18) and (19) for the concentrated force acting on a simply supported beam on the initial stage of the oscillator motion have been derived, which provide a priori estimates for the maximum magnitude of the dynamic force acting on the beam for the damped case. Note that the high-frequency component of the dynamic force appears not only at the moment when the oscillator enters the beam. Clearly, such a component appears any time when the oscillator passes a point where the function describing the "road profile" is not smooth (the first derivative has a jump).

3. The existence of the high-frequency component of the interaction force in the moving oscillator problem results in a rapidly changing "picture of stresses" in the vicinity of the oscillator attachment point (Fig. 2). This effect seems to considerably affect the pavement wear. If so, it follows from the results obtained that a vehicle with a softer suspension is more road-friendly than that with a stiffer suspension. This result agrees well with the conclusion made in the DIVINE report (1998) that "pavement profile deteriorates more rapidly under a steel suspension than under an air suspension carrying the same load."

4. The adequacy of the moving mass model for modeling real vehicles and its physical incorrectness when applied to the case of a simply supported beam with nonzero initial conditions have been discussed. The deficiency of the model has been noted in that it fails to predict stresses in the bridge structure.

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