# 15. Problem condition

- condition of a mathematical problem
- matrix norm
- condition number

# Sources of error in numerical computation

**Example**: evaluate a function $f : \mathbf{R} \to \mathbf{R}$ at a given $x$

sources of error in the result:

- $x$ is not exactly known
  - measurement errors
  - errors in previous computations

  $\longrightarrow$ how sensitive is $f(x)$ to errors in $x$?

- the algorithm for computing $f(x)$ is not exact
  - discretization (*e.g.*, algorithm uses a table to look up function values)
  - truncation (*e.g.*, function is evaluated by truncating a Taylor series)
  - rounding error during the computation

  $\longrightarrow$ how large is the error introduced by the algorithm?

# Condition (conditioning) of a problem

describes sensitivity of the solution to changes in the problem data

- **well-conditioned problem:**

  small changes in the data produce small changes in the solution

- **ill-conditioned (badly conditioned) problem:**

  small changes in the data can produce large changes in the solution

a rigorous definition depends on what “large error” means

- absolute or relative error, which norm is used, . . .
- the informal definition is sufficient for our purposes

# Example: function evaluation

here the problem is: given $x$, evaluate $y = f(x)$

- if $x$ is changed to $x + \Delta x$, solution changes to

$$y + \Delta y = f(x + \Delta x)$$

- condition with respect to absolute error in $x$ and $y$

$$|\Delta y| \approx |f'(x)||\Delta x|$$

  problem is ill-conditioned with respect to absolute error if $|f'(x)|$ is very large

- condition with respect to relative errors in $x$ and $y$

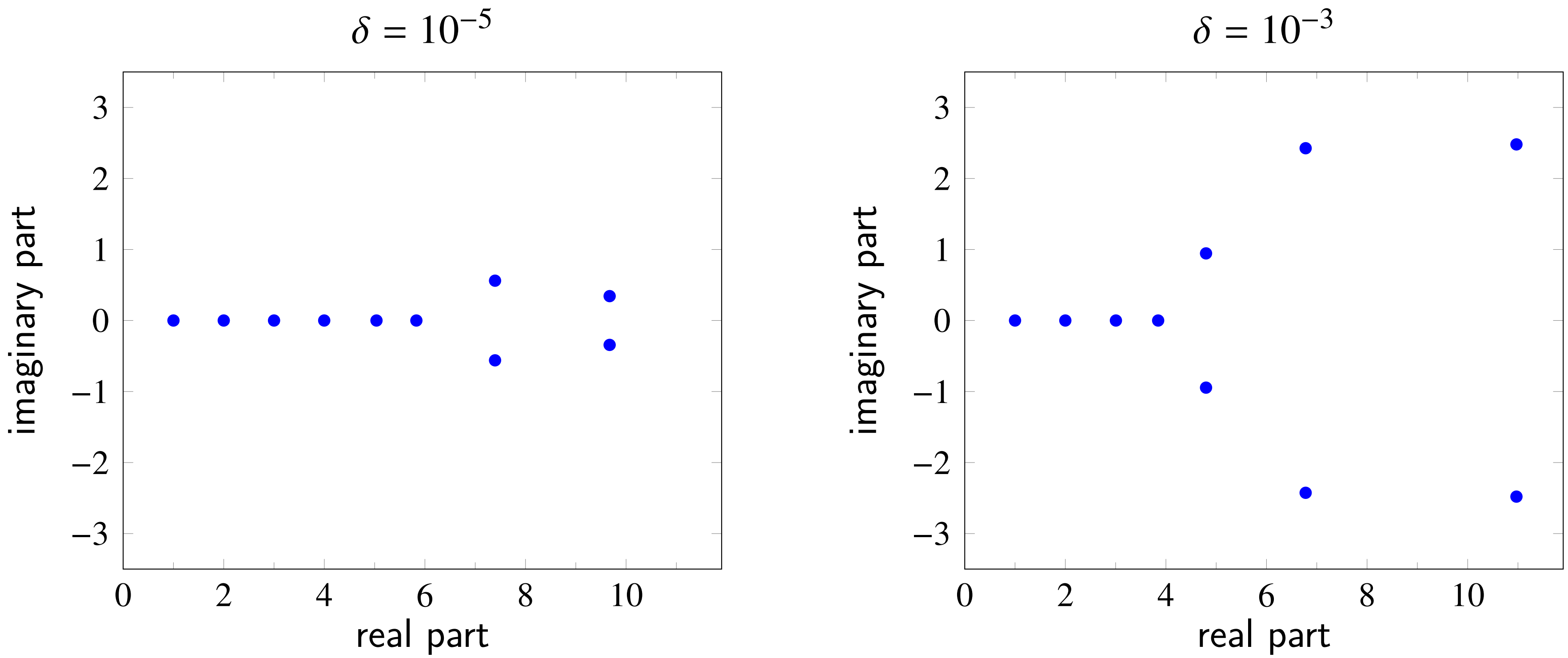
$$\frac{|\Delta y|}{|y|} \approx \frac{|f'(x)||x|}{|f(x)|} \frac{|\Delta x|}{|x|}$$

  ill-conditioned with respect to relative error if $|f'(x)||x|/|f(x)|$ is very large

# Roots of a polynomial

$$p(x) = (x-1)(x-2)\cdots(x-10) + \delta \cdot x^{10}$$

roots of $p$ computed by MATLAB for two values of $\delta$




roots can be very sensitive to errors in the coefficients

# Condition of a set of linear equations

- assume $A$ is nonsingular and $Ax = b$
- if we change $b$ to $b + \Delta b$, the new solution is $x + \Delta x$ with

$$A(x + \Delta x) = b + \Delta b$$

- the change in $x$ is

$$\Delta x = A^{-1} \Delta b$$

## Condition

- the equations are *well-conditioned* if small $\Delta b$ results in small $\Delta x$
- the equations are *ill-conditioned* if small $\Delta b$ can result in large $\Delta x$

# Example of ill-conditioned equations

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$$

- solution for $b = (1,1)$ is $x = (1,1)$
- change in $x$ if we change $b$ to $b + \Delta b$:

$$\Delta x = A^{-1}\Delta b = \begin{bmatrix} \Delta b_1 - 10^{10}(\Delta b_1 - \Delta b_2) \\ \Delta b_1 + 10^{10}(\Delta b_1 - \Delta b_2) \end{bmatrix}$$

small $\Delta b$ can lead to extremely large $\Delta x$

# Outline

- condition of a mathematical problem
- **matrix norm**
- condition number

# Matrix norms

the **Frobenius norm** of an $m \times n$ matrix $A$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}$$

- denoted $\|A\|$ in the textbook
- in MATLAB: `norm(A,'fro')`; in Julia: `norm(A)`

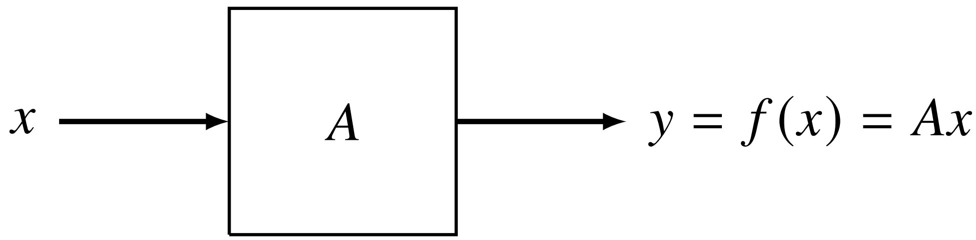
the **2-norm** or **spectral norm** is defined as

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

- the norms $\|Ax\|$ and $\|x\|$ are Euclidean norms of vectors
- no simple explicit expression, except for special $A$
- readily computed numerically (in MATLAB: `norm(A)`; in Julia: `opnorm(A)`)

# Interpretation of 2-norm

the $m \times n$ matrix $A$ defines a linear function $f(x) = Ax$




- $\|Ax\|/\|x\|$ gives the *amplification factor* or *gain* for input $x$
- the gain only depends on the direction of $x$
- the 2-norm of $A$ is the maximum gain over all directions:

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

# Computing the 2-norm of a matrix

**Simple matrices:** sometimes it is easy to maximize $\|Ax\|/\|x\|$

- zero matrix: $\|0\|_2 = 0$
- identity matrix: $\|I\|_2 = 1$
- diagonal matrix:

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}, \qquad \|A\|_2 = \max_{i=1,\ldots,n} |A_{ii}|$$

- matrix with orthonormal columns: $\|A\|_2 = 1$

**General matrices:** $\|A\|_2$ must be computed by numerical algorithms

# Properties of the matrix norm

## Properties satisfied by all matrix norms

- *nonnegative:* $\|A\|_2 \geq 0$ for all $A$
- *positive definiteness:* $\|A\|_2 = 0$ only if $A = 0$
- *homogeneity:* $\|\beta A\|_2 = |\beta|\|A\|_2$
- *triangle inequality:* $\|A + B\|_2 \leq \|A\|_2 + \|B\|_2$

## Additional properties satisfied by the 2-norm

- $\|Ax\| \leq \|A\|_2\|x\|$ if the product $Ax$ exists
- $\|AB\|_2 \leq \|A\|_2\|B\|_2$ if the product $AB$ exists
- if $A$ is nonsingular: $\|A\|_2\|A^{-1}\|_2 \geq 1$
- if $A$ is nonsingular: $1/\|A^{-1}\|_2 = \min_{x\neq 0}\left(\|Ax\|_2/\|x\|\right)$
- $\|A^T\|_2 = \|A\|_2$

# Outline

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# Bound on absolute error

suppose $A$ is nonsingular and define

$$x = A^{-1}b, \qquad \Delta x = A^{-1}\Delta b$$

**Upper bound** on $\|\Delta x\|$:

$$\|\Delta x\| \leq \|A^{-1}\|_2 \, \|\Delta b\|$$

- follows from property 4 on page 15.11
- small $\|A^{-1}\|_2$ means that $\|\Delta x\|$ is small when $\|\Delta b\|$ is small
- large $\|A^{-1}\|_2$ means that $\|\Delta x\|$ can be large, even when $\|\Delta b\|$ is small
- for every $A$, there exists nonzero $\Delta b$ such that $\|\Delta x\| = \|A^{-1}\|_2 \, \|\Delta b\|$

# Bound on relative error

suppose in addition that $b \neq 0$; hence $x \neq 0$

**Upper bound** on $\|\Delta x\|/\|x\|$:

$$\frac{\|\Delta x\|}{\|x\|} \leq \|A\|_2 \|A^{-1}\|_2 \frac{\|\Delta b\|}{\|b\|} \qquad (1)$$

- follows from $\|\Delta x\| \leq \|A^{-1}\|_2 \|\Delta b\|$ and $\|b\| \leq \|A\|_2 \|x\|$
- $\|A\|_2 \|A^{-1}\|_2$ small means $\|\Delta x\|/\|x\|$ is small when $\|\Delta b\|/\|b\|$ is small
- $\|A\|_2 \|A^{-1}\|_2$ large means $\|\Delta x\|/\|x\|$ can be much larger than $\|\Delta b\|/\|b\|$
- for every $A$, there exist nonzero $b$, $\Delta b$ such that equality holds in (1)

# Condition number

**Definition:** the condition number of a nonsingular matrix $A$ is

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$$

## Properties

- $\kappa(A) \geq 1$ for all $A$ (additional property 3 on page 15.11)
- $A$ is a *well-conditioned* matrix if $\kappa(A)$ is small (close to 1):

  the relative error in $x$ is not much larger than the relative error in $b$
- $A$ is *badly conditioned* or *ill-conditioned* if $\kappa(A)$ is large:

  the relative error in $x$ can be much larger than the relative error in $b$

# Example

- $A$ is blurring matrix, nonsingular with condition number $\approx 10^9$
- we apply $A$ to image $x_{\mathrm{ex}}$

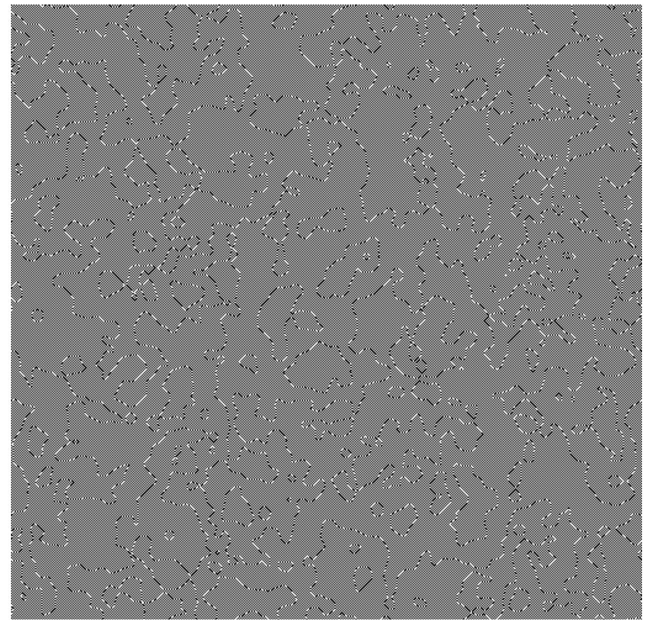
blurred image
$y_1 = Ax_{\mathrm{ex}}$

blurred and noisy image
$y_2 = Ax_{\mathrm{ex}} +$ small noise

# Example

we solve $Ax = y$ for the two blurred images

$A^{-1}y_1$



$A^{-1}y_2$

- illustrates ill conditioning of $A$
- explains need for regularization in deblurring algorithms

# Exercises

## Exercise 1

$$A = \frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ 1+a & 1-a \end{array}\right], \qquad A^{-1} = \frac{1}{a}\left[\begin{array}{cc} a-1 & 1 \\ a+1 & -1 \end{array}\right]$$

$a$ is small and nonzero ($a = 10^{-10}$ on page 15.7); show that $\kappa(A) \geq 1/|a|$

## Exercise 2

suppose $A = UBV$ with $U$, $V$ orthogonal, and $B$ nonsingular; show that

$$\kappa(A) = \kappa(B)$$

## Exercise 3

suppose $A = uv^T$ where $u$ and $v$ are vectors; show that $\|A\|_2 = \|u\|\|v\|$

# Exercises

**Exercise 4 (ex. A15.3)**

- let $u$ be a vector; show that

$$\|u\| = \max_{v\neq 0} \frac{v^T u}{\|v\|}$$

- let $A$ be a matrix; show that

$$\|A\|_2 = \max_{y\neq 0,\, x\neq 0} \frac{y^T A x}{\|x\|\|y\|}$$

therefore $\|A\|_2 = \|A^T\|_2$