15. Problem condition

• condition of a mathematical problem

• matrix norm

• condition number
Sources of error in numerical computation

Example: evaluate a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) at a given \( x \)

sources of error in the result:

- \( x \) is not exactly known
  - measurement errors
  - errors in previous computations

  \( \rightarrow \) how sensitive is \( f(x) \) to errors in \( x \)?

- the algorithm for computing \( f(x) \) is not exact
  - discretization (\( e.g. \), algorithm uses a table to look up function values)
  - truncation (\( e.g. \), function is evaluated by truncating a Taylor series)
  - rounding error during the computation

  \( \rightarrow \) how large is the error introduced by the algorithm?
Condition (conditioning) of a problem

describes sensitivity of the solution to changes in the problem data

- **well-conditioned problem:**
  
  small changes in the data produce small changes in the solution

- **ill-conditioned (badly conditioned) problem:**
  
  small changes in the data can produce large changes in the solution

a rigorous definition depends on what “large error” means

- absolute or relative error, which norm is used, ...

- the informal definition is sufficient for our purposes
Example: function evaluation

here the problem is: given $x$, evaluate $y = f(x)$

- if $x$ is changed to $x + \Delta x$, solution changes to

  $y + \Delta y = f(x + \Delta x)$

- condition with respect to absolute error in $x$ and $y$

  $|\Delta y| \approx |f'(x)||\Delta x|$

  problem is ill-conditioned with respect to absolute error if $|f'(x)|$ is very large

- condition with respect to relative errors in $x$ and $y$

  $\frac{|\Delta y|}{|y|} \approx \frac{|f'(x)||x| |\Delta x|}{|f(x)||x|}$

  ill-conditioned with respect to relative error if $|f'(x)||x|/|f(x)|$ is very large

Problem condition 15.4
Roots of a polynomial

\[ p(x) = (x - 1)(x - 2) \cdots (x - 10) + \delta \cdot x^{10} \]

roots of \( p \) computed by MATLAB for two values of \( \delta \)

\[ \delta = 10^{-5} \]

\[ \delta = 10^{-3} \]

roots are very sensitive to errors in the coefficients
Condition of a set of linear equations

- assume $A$ is nonsingular and $Ax = b$

- if we change $b$ to $b + \Delta b$, the new solution is $x + \Delta x$ with

$$A(x + \Delta x) = b + \Delta b$$

- the change in $x$ is

$$\Delta x = A^{-1}\Delta b$$

Condition

- the equations are *well-conditioned* if small $\Delta b$ results in small $\Delta x$
- the equations are *ill-conditioned* if small $\Delta b$ can result in large $\Delta x$
Example of ill-conditioned equations

\[ A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 + 10^{-10} & 1 - 10^{-10} \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 - 10^{10} & 10^{10} \\ 1 + 10^{10} & -10^{10} \end{bmatrix} \]

- solution for \( b = (1, 1) \) is \( x = (1, 1) \)

- change in \( x \) if we change \( b \) to \( b + \Delta b \):

\[ \Delta x = A^{-1} \Delta b = \begin{bmatrix} \Delta b_1 - 10^{10}(\Delta b_1 - \Delta b_2) \\ \Delta b_1 + 10^{10}(\Delta b_1 - \Delta b_2) \end{bmatrix} \]

small \( \Delta b \) can lead to extremely large \( \Delta x \)
Outline

• condition of a mathematical problem

• matrix norm

• condition number
Matrix norms

the Frobenius norm of an $m \times n$ matrix $A$ is defined as

$$
\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}
$$

- denoted $\|A\|$ in the textbook
- in MATLAB: `norm(A, 'fro')`; in Julia: `norm(A)`

the 2-norm or spectral norm is defined as

$$
\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}
$$

- the norms $\|Ax\|$ and $\|x\|$ are Euclidean norms of vectors
- no simple explicit expression, except for special $A$
- readily computed numerically (in MATLAB: `norm(A)`; in Julia: `opnorm(A)`)

Problem condition 15.8
Interpretation of 2-norm

the $m \times n$ matrix $A$ defines a linear function $f(x) = Ax$

$$x \rightarrow A \rightarrow y = f(x) = Ax$$

- $\|Ax\|/\|x\|$ gives the amplification factor or gain for input $x$

- the gain only depends on the direction of $x$

- the 2-norm of $A$ is the maximum gain over all directions:

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$
Computing the 2-norm of a matrix

**Simple matrices:** sometimes it is easy to maximize $\|Ax\|/\|x\|$

- zero matrix: $\|0\|_2 = 0$
- identity matrix: $\|I\|_2 = 1$
- diagonal matrix:

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}, \quad \|A\|_2 = \max_{i=1,\ldots,n} |A_{ii}|$$

- matrix with orthonormal columns: $\|A\|_2 = 1$

**General matrices:** $\|A\|_2$ must be computed by numerical algorithms
Properties of the matrix norm

Properties satisfied by all matrix norms

- nonnegative: $\|A\|_2 \geq 0$ for all $A$
- positive definiteness: $\|A\|_2 = 0$ only if $A = 0$
- homogeneity: $\|\beta A\|_2 = |\beta| \|A\|_2$
- triangle inequality: $\|A + B\|_2 \leq \|A\|_2 + \|B\|_2$

Additional properties satisfied by the 2-norm

- $\|Ax\| \leq \|A\|_2 \|x\|$ if the product $Ax$ exists
- $\|AB\|_2 \leq \|A\|_2 \|B\|_2$ if the product $AB$ exists
- if $A$ is nonsingular: $\|A\|_2 \|A^{-1}\|_2 \geq 1$
- if $A$ is nonsingular: $1/\|A^{-1}\|_2 = \min_{x \neq 0} (\|Ax\|_2/\|x\|)$
- $\|A^T\|_2 = \|A\|_2$
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Bound on absolute error

suppose $A$ is nonsingular and define

$$x = A^{-1}b, \quad \Delta x = A^{-1}\Delta b$$

**Upper bound** on $||\Delta x||$:

$$||\Delta x|| \leq ||A^{-1}||_2 ||\Delta b||$$

- follows from property 4 on page 15.11
- small $||A^{-1}||_2$ means that $||\Delta x||$ is small when $||\Delta b||$ is small
- large $||A^{-1}||_2$ means that $||\Delta x||$ can be large, even when $||\Delta b||$ is small
- for every $A$, there exists nonzero $\Delta b$ such that $||\Delta x|| = ||A^{-1}||_2 ||\Delta b||$
suppose in addition that \( b \neq 0 \); hence \( x \neq 0 \)

**Upper bound** on \( \| \Delta x \| / \| x \| \):

\[
\frac{\| \Delta x \|}{\| x \|} \leq \| A \|_2 \| A^{-1} \|_2 \frac{\| \Delta b \|}{\| b \|} \tag{1}
\]

- follows from \( \| \Delta x \| \leq \| A^{-1} \|_2 \| \Delta b \| \) and \( \| b \| \leq \| A \|_2 \| x \| \)

- \( \| A \|_2 \| A^{-1} \|_2 \) small means \( \| \Delta x \| / \| x \| \) is small when \( \| \Delta b \| / \| b \| \) is small

- \( \| A \|_2 \| A^{-1} \|_2 \) large means \( \| \Delta x \| / \| x \| \) can be much larger than \( \| \Delta b \| / \| b \| \)

- for every \( A \), there exist nonzero \( b, \Delta b \) such that equality holds in (1)
Condition number

Definition: the condition number of a nonsingular matrix $A$ is

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$$

Properties

• $\kappa(A) \geq 1$ for all $A$ (last property on page 15.11)

• $A$ is a well-conditioned matrix if $\kappa(A)$ is small (close to 1):
  the relative error in $x$ is not much larger than the relative error in $b$

• $A$ is badly conditioned or ill-conditioned if $\kappa(A)$ is large:
  the relative error in $x$ can be much larger than the relative error in $b$
Example

- $A$ is blurring matrix, nonsingular with condition number $\approx 10^9$
- we apply $A$ to image $x$

$$y_1 = Ax$$

blurred image

$$y_2 = Ax + \text{small noise}$$

blurred and noisy image
Example

we solve $Ax = y$ for the two blurred images

- illustrates ill conditioning of $A$
- explains need for regularization in deblurring algorithms
Exercises

Exercise 1

\[ A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 + a & 1 - a \end{bmatrix}, \quad A^{-1} = \frac{1}{a} \begin{bmatrix} a - 1 & 1 \\ a + 1 & -1 \end{bmatrix} \]

\( a \) is small and nonzero \((a = 10^{-10} \text{ on page } 15.7)\); show that \( \kappa(A) \geq 1/|a| \)

Exercise 2

suppose \( A = UBV \) with \( U, V \) orthogonal, and \( B \) nonsingular; show that \( \kappa(A) = \kappa(B) \)

Exercise 3

suppose \( A = uv^T \) where \( u \) and \( v \) are vectors; show that \( \|A\|_2 = \|u\| \|v\| \)
Exercises

Exercise 4 (ex. 15.3)

• Let \( u \) be a vector; show that

\[
\|u\| = \max_{v \neq 0} \frac{v^T u}{\|v\|}
\]

• Let \( A \) be a matrix; show that

\[
\|A\|_2 = \max_{y \neq 0, x \neq 0} \frac{y^T Ax}{\|x\| \|y\|}
\]

Therefore \( \|A\|_2 = \|A^T\|_2 \)