4. Matrix inverses

- left and right inverse
- linear independence
- nonsingular matrices
- matrices with linearly independent columns
- matrices with linearly independent rows

Left and right inverse

$AB \neq BA$ in general, so we have to distinguish two types of inverses

Left inverse: X is a *left inverse* of A if

$$XA = I$$

A is *left-invertible* if it has at least one left inverse

Right inverse: X is a *right inverse* of A if

$$AX = I$$

A is *right-invertible* if it has at least one right inverse

Examples

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- A is left-invertible; the following matrices are left inverses:

$$\frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1/2 & 3 \\ 0 & 1/2 & -2 \end{bmatrix}$$

- B is right-invertible; the following matrices are right inverses:

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Some immediate properties

Dimensions

a left or right inverse of an $m \times n$ matrix must have size $n \times m$

Left and right inverse of (conjugate) transpose

- X is a left inverse of A if and only if X^T is a right inverse of A^T

$$A^T X^T = (XA)^T = I$$

- X is a left inverse of A if and only if X^H is a right inverse of A^H

$$A^H X^H = (XA)^H = I$$

Inverse

if A has a left **and** a right inverse, then they are equal and unique:

$$XA = I, \quad AY = I \qquad \Longrightarrow \qquad X = X(AY) = (XA)Y = Y$$

- in this case, we call $X = Y$ the **inverse** of A (notation: A^{-1})
- A is *invertible* if its inverse exists

Example

$$A = \begin{bmatrix} -1 & 1 & -3 \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \qquad A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 4 & 1 \\ 0 & -2 & 1 \\ -2 & -2 & 0 \end{bmatrix}$$

Linear equations

set of m linear equations in n variables

$$
\begin{array}{rcl}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n & = & b_1 \\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n & = & b_2 \\
& \vdots & \\
A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n & = & b_m
\end{array}
$$

- in matrix form: $Ax = b$
- may have no solution, a unique solution, infinitely many solutions

Linear equations and matrix inverse

Left-invertible matrix: if X is a left inverse of A, then

$$Ax = b \qquad \Longrightarrow \qquad x = XAx = Xb$$

there is *at most one* solution (if there is a solution, it must be equal to Xb)

Right-invertible matrix: if X is a right inverse of A, then

$$x = Xb \qquad \Longrightarrow \qquad Ax = AXb = b$$

there is *at least one* solution (namely, $x = Xb$)

Invertible matrix: if A is invertible, then

$$Ax = b \qquad \Longleftrightarrow \qquad x = A^{-1}b$$

there is a *unique* solution

Outline

- left and right inverse
- **linear independence**
- nonsingular matrices
- matrices with linearly independent columns
- matrices with linearly independent rows

Linear combination

a linear combination of vectors $a_1, \ldots, a_n$ is a sum of scalar–vector products

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

- the scalars x_i are the *coefficients* of the linear combination
- can be written as a matrix–vector product

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- the *trivial* linear combination has coefficients $x_1 = \cdots = x_n = 0$

(same definition holds for real and complex vectors/scalars)

Linearly independent vectors

vectors $a_1, \ldots, a_n$ are *linearly independent* if

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0 \qquad \Longrightarrow \qquad x_1 = x_2 = \cdots = x_n = 0$$

- in matrix–vector notation, with $A = [a_1 \ a_2 \ \cdots \ a_n]$,

$$Ax = 0 \qquad \Longrightarrow \qquad x = 0$$

- $a_1, \ldots, a_n$ are *linearly dependent* if there exist $x_1, \ldots, x_n$, not all zero, such that

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$$

  at least one vector is a linear combination of the other vectors: if $x_i \neq 0$, then

$$a_i = -\frac{x_1}{x_i} a_1 - \cdots - \frac{x_{i-1}}{x_i} a_{i-1} - \frac{x_{i+1}}{x_i} a_{i+1} - \cdots - \frac{x_n}{x_i} a_n$$

- linear (in)dependence is a property of the set of vectors $\{a_1, \ldots, a_n\}$

  (by convention, the empty set is linearly independent)

Example

the vectors

$$a_1 = \begin{bmatrix} 0.2 \\ -7 \\ 8.6 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} -0.1 \\ 2 \\ -1 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 0 \\ -1 \\ 2.2 \end{bmatrix}$$

are linearly dependent

- 0 can be expressed as a nontrivial linear combination of a_1, a_2, a_3:

$$0 = a_1 + 2a_2 - 3a_3$$

- a_1 can be expressed as a linear combination of a_2, a_3:

$$a_1 = -2a_2 + 3a_3$$

  (and similarly a_2 and a_3)

Example

the vectors

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

are linearly independent:

$$x_1 a_1 + x_2 a_2 + x_3 a_3 = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_3 \\ x_2 + x_3 \end{bmatrix} = 0$$

holds only if $x_1 = x_2 = x_3 = 0$

Dimension inequality

if n vectors $a_1, a_2, \ldots, a_n$ of length m are linearly independent, then

$$n \leq m$$

(proof is in textbook)

- if an $m \times n$ matrix has linearly independent columns then $m \geq n$
- if A is wide, the columns are linearly dependent: the homogeneous equation

  $$Ax = 0$$

  has nontrivial solutions ($x \neq 0$)
- if an $m \times n$ matrix has linearly independent rows then $m \leq n$
- if A is tall, its rows are linearly dependent

Outline

- left and right inverse
- linear independence
- **nonsingular matrices**
- matrices with linearly independent columns
- matrices with linearly independent rows

Nonsingular matrix

for a **square** matrix A the following four properties are equivalent

1. A is left-invertible
2. the columns of A are linearly independent
3. A is right-invertible
4. the rows of A are linearly independent

a square matrix with these properties is called **nonsingular**

Nonsingular $=$ **invertible**

- if properties 1 and 3 hold, then A is invertible (page 4.5)
- if A is invertible, properties 1 and 3 hold (by definition of invertibility)

Proof

$$\begin{tikzcd}
\text{left-invertible} \arrow[r, Rightarrow, "\text{(a)}"] & \text{linearly independent columns} \arrow[d, Rightarrow, "\text{(b)}"] \\
\text{linearly independent rows} \arrow[u, Rightarrow, "\text{(b')}"'] & \text{right-invertible} \arrow[l, Rightarrow, "\text{(a')}"']
\end{tikzcd}$$

- we show that (a) holds in general
- we show that (b) holds for square matrices
- (a') and (b') follow from (a) and (b) applied to A^T

Part a: suppose A is left-invertible

- if B is a left inverse of A (satisfies $BA = I$), then

$$\begin{aligned} Ax = 0 \quad &\Longrightarrow \quad BAx = 0 \\ &\Longrightarrow \quad x = 0 \end{aligned}$$

- this means that the columns of A are linearly independent: if

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

then

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$$

holds only for the trivial linear combination $x_1 = x_2 = \cdots = x_n = 0$

Part b: suppose A is square with linearly independent columns $a_1, \ldots, a_n$

- for every n-vector b the vectors $a_1, \ldots, a_n, b$ are linearly dependent

  (from dimension inequality on page 4.12)

- hence for every b there exists a nontrivial linear combination

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n + x_{n+1} b = 0$$

- we must have $x_{n+1} \neq 0$ because $a_1, \ldots, a_n$ are linearly independent
- hence every b can be written as a linear combination of $a_1, \ldots, a_n$
- in particular, there exist n-vectors $c_1, \ldots, c_n$ such that

$$Ac_1 = e_1, \qquad Ac_2 = e_2, \qquad \ldots, \qquad Ac_n = e_n,$$

- the matrix $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$ is a right inverse of A:

$$A \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = I$$

Examples

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

- A is nonsingular because its columns are linearly independent:

$$x_1 - x_2 + x_3 = 0, \qquad -x_1 + x_2 + x_3 = 0, \qquad x_1 + x_2 - x_3 = 0$$

  is only possible if $x_1 = x_2 = x_3 = 0$

- B is singular because its columns are linearly dependent:

$$Bx = 0 \quad \text{for } x = (1, 1, 1, 1)$$

Example: Vandermonde matrix

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \qquad \text{with } t_i \neq t_j \text{ for } i \neq j$$

we show that A is nonsingular by showing that $Ax = 0$ only if $x = 0$

- $Ax = 0$ means $p(t_1) = p(t_2) = \cdots = p(t_n) = 0$ where

$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}$$

  $p(t)$ is a polynomial of degree $n - 1$ or less

- if $x \neq 0$, then $p(t)$ can not have more than $n - 1$ distinct real roots
- therefore $p(t_1) = \cdots = p(t_n) = 0$ is only possible if $x = 0$

Inverse of transpose and product

Transpose and conjugate transpose

if A is nonsingular, then A^T and A^H are nonsingular and

$$(A^T)^{-1} = (A^{-1})^T, \qquad (A^H)^{-1} = (A^{-1})^H$$

we write these as A^{-T} and A^{-H}

Product

if A and B are nonsingular and of equal size, then AB is nonsingular with

$$(AB)^{-1} = B^{-1}A^{-1}$$

Outline

- left and right inverse
- linear independence
- nonsingular matrices
- **matrices with linearly independent columns**
- matrices with linearly independent rows

Gram matrix

the *Gram matrix* associated with a matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

is the matrix of all pairwise inner products of the column vectors

- for real matrices:

$$A^T A = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \cdots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \cdots & a_2^T a_n \\ \vdots & \vdots & & \vdots \\ a_n^T a_1 & a_n^T a_2 & \cdots & a_n^T a_n \end{bmatrix}$$

- for complex matrices:

$$A^H A = \begin{bmatrix} a_1^H a_1 & a_1^H a_2 & \cdots & a_1^H a_n \\ a_2^H a_1 & a_2^H a_2 & \cdots & a_2^H a_n \\ \vdots & \vdots & & \vdots \\ a_n^H a_1 & a_n^H a_2 & \cdots & a_n^H a_n \end{bmatrix}$$

Nonsingular Gram matrix

the Gram matrix is nonsingular if only if A has linearly independent columns

- suppose $A \in \mathbf{R}^{m\times n}$ has linearly independent columns:

$$\begin{aligned} A^TAx = 0 \quad &\Longrightarrow \quad x^TA^TAx = (Ax)^T(Ax) = \|Ax\|^2 = 0 \\ &\Longrightarrow \quad Ax = 0 \\ &\Longrightarrow \quad x = 0 \end{aligned}$$

  therefore A^TA is nonsingular

- suppose the columns of $A \in \mathbf{R}^{m\times n}$ are linearly dependent

$$\exists x \neq 0,\ Ax = 0 \qquad \Longrightarrow \qquad \exists x \neq 0,\ A^TAx = 0$$

  therefore A^TA is singular

(for $A \in \mathbf{C}^{m\times n}$, replace A^T with A^H and x^T with x^H)

Pseudo-inverse of matrix with independent columns

- suppose $A \in \mathbf{R}^{m \times n}$ has linearly independent columns
- this implies that A is tall or square $(m \geq n)$; see page 4.12

the *pseudo-inverse* of A is defined as

$$A^\dagger = (A^T A)^{-1} A^T$$

- this matrix exists, because the Gram matrix $A^T A$ is nonsingular
- $A^\dagger$ is a left inverse of A:

$$A^\dagger A = (A^T A)^{-1}(A^T A) = I$$

(for complex A with linearly independent columns, $A^\dagger = (A^H A)^{-1} A^H$)

Summary

the following three properties are equivalent for a real matrix A

1. A is left-invertible
2. the columns of A are linearly independent
3. $A^T A$ is nonsingular

- $1 \Rightarrow 2$ was already proved on page 4.15
- $2 \Rightarrow 1$: we have seen that the pseudo-inverse is a left inverse
- $2 \Leftrightarrow 3$: proved on page 4.21
- a matrix with these properties must be tall or square
- for complex matrices, replace $A^T A$ in property 3 by $A^H A$

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- left and right inverse
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Pseudo-inverse of matrix with independent rows

- suppose $A \in \mathbf{R}^{m \times n}$ has linearly independent rows
- this implies that A is wide or square $(m \leq n)$; see page 4.12

the *pseudo-inverse* of A is defined as

$$A^{\dagger} = A^T (AA^T)^{-1}$$

- A^T has linearly independent columns
- hence its Gram matrix AA^T is nonsingular, so $A^{\dagger}$ exists
- $A^{\dagger}$ is a right inverse of A:

$$AA^{\dagger} = (AA^T)(AA^T)^{-1} = I$$

(for complex A with linearly independent rows, $A^{\dagger} = A^H (AA^H)^{-1}$)

Summary

the following three properties are equivalent

1. A is right-invertible
2. the rows of A are linearly independent
3. AA^T is nonsingular

- $1 \Rightarrow 2$ and $2 \Leftrightarrow 3$: by transposing result on page 4.23
- $2 \Rightarrow 1$: we have seen that the pseudo-inverse is a right inverse
- a matrix with these properties must be wide or square
- for complex matrices, replace AA^T in property 3 by AA^H