4. Matrix inverses

- left and right inverse
- linear independence
- nonsingular matrices
- matrices with linearly independent columns
- matrices with linearly independent rows
Left and right inverse

$AB \neq BA$ in general, so we have to distinguish two types of inverses

**Left inverse:** $X$ is a *left inverse* of $A$ if

$$XA = I$$

$A$ is *left-invertible* if it has at least one left inverse

**Right inverse:** $X$ is a *right inverse* of $A$ if

$$AX = I$$

$A$ is *right-invertible* if it has at least one right inverse
Examples

\[ A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

- \( A \) is left-invertible; the following matrices are left inverses:

\[ \frac{1}{9} \begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1/2 & 3 \\ 0 & 1/2 & -2 \end{bmatrix} \]

- \( B \) is right-invertible; the following matrices are right inverses:

\[ \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \]
Some immediate properties

Dimensions

a left or right inverse of an $m \times n$ matrix must have size $n \times m$

Left and right inverse of (conjugate) transpose

- $X$ is a left inverse of $A$ if and only if $X^T$ is a right inverse of $A^T$
  \[ A^T X^T = (XA)^T = I \]

- $X$ is a left inverse of $A$ if and only if $X^H$ is a right inverse of $A^H$
  \[ A^H X^H = (XA)^H = I \]
Inverse

If $A$ has a left and a right inverse, then they are equal and unique:

$$XA = I, \quad AY = I \quad \implies \quad X = X(AY) = (XA)Y = Y$$

- In this case, we call $X = Y$ the inverse of $A$ (notation: $A^{-1}$)
- $A$ is invertible if its inverse exists

Example

$$A = \begin{bmatrix} -1 & 1 & -3 \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 4 & 1 \\ 0 & -2 & 1 \\ -2 & -2 & 0 \end{bmatrix}$$
Linear equations

set of \( m \) linear equations in \( n \) variables

\[
\begin{align*}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\
\vdots \\
A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= b_m
\end{align*}
\]

- in matrix form: \( Ax = b \)
- may have no solution, a unique solution, infinitely many solutions
Linear equations and matrix inverse

**Left-invertible matrix:** if $X$ is a left inverse of $A$, then

\[ Ax = b \implies x = XAx = Xb \]

there is *at most one* solution (if there is a solution, it must be equal to $Xb$)

**Right-invertible matrix:** if $X$ is a right inverse of $A$, then

\[ x = Xb \implies Ax = AXb = b \]

there is *at least one* solution (namely, $x = Xb$)

**Invertible matrix:** if $A$ is invertible, then

\[ Ax = b \iff x = A^{-1}b \]

there is a *unique* solution
Outline

- left and right inverse
- **linear independence**
- nonsingular matrices
- matrices with linearly independent columns
- matrices with linearly independent rows
Linear combination

A linear combination of vectors $a_1, \ldots, a_n$ is a sum of scalar–vector products

$$x_1a_1 + x_2a_2 + \cdots + x_na_n$$

- the scalars $x_i$ are the *coefficients* of the linear combination
- can be written as a matrix–vector product

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- the *trivial* linear combination has coefficients $x_1 = \cdots = x_n = 0$

(same definition holds for real and complex vectors/scalars)
Linearly independent vectors

vectors $a_1, \ldots, a_n$ are *linearly independent* if

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0 \implies x_1 = x_2 = \cdots = x_n = 0$$

- in matrix–vector notation, with $A = [a_1 \ a_2 \ \cdots \ a_n]$,

  $$Ax = 0 \implies x = 0$$

- $a_1, \ldots, a_n$ are *linearly dependent* if there exist $x_1, \ldots, x_n$, not all zero, such that

  $$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$$

  at least one vector is a linear combination of the other vectors: if $x_i \neq 0$, then

  $$a_i = \frac{x_1}{x_i} a_1 - \cdots - \frac{x_{i-1}}{x_i} a_{i-1} - \frac{x_{i+1}}{x_i} a_{i+1} - \cdots - \frac{x_n}{x_i} a_n$$

- linear (in)dependence is a property of the set of vectors $\{a_1, \ldots, a_n\}$

  (by convention, the empty set is linearly independent)
Example

the vectors

\[ a_1 = \begin{bmatrix} 0.2 \\ -7 \\ 8.6 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -0.1 \\ 2 \\ -1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ -1 \\ 2.2 \end{bmatrix} \]

are linearly dependent

- 0 can be expressed as a nontrivial linear combination of \( a_1, a_2, a_3 \):

\[ 0 = a_1 + 2a_2 - 3a_3 \]

- \( a_1 \) can be expressed as a linear combination of \( a_2, a_3 \):

\[ a_1 = -2a_2 + 3a_3 \]

(and similarly \( a_2 \) and \( a_3 \))
Example

the vectors

\[
\begin{align*}
a_1 &= \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, &
 a_2 &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, &
 a_3 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

are linearly independent:

\[
x_1a_1 + x_2a_2 + x_3a_3 = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_3 \\ x_2 + x_3 \end{bmatrix} = 0
\]

holds only if \(x_1 = x_2 = x_3 = 0\)
Dimension inequality

if \( n \) vectors \( a_1, a_2, \ldots, a_n \) of length \( m \) are linearly independent, then

\[
n \leq m
\]

(proof is in textbook)

- if an \( m \times n \) matrix has linearly independent columns then \( m \geq n \)
- if \( A \) is wide, the columns are linearly dependent: the homogeneous equation
  \[
  Ax = 0
  \]
  has nontrivial solutions \((x \neq 0)\)
- if an \( m \times n \) matrix has linearly independent rows then \( m \leq n \)
- if \( A \) is tall, its rows are linearly dependent
Outline

- left and right inverse
- linear independence
- **nonsingular matrices**
- matrices with linearly independent columns
- matrices with linearly independent rows
Nonsingular matrix

for a **square** matrix $A$ the following four properties are equivalent

1. $A$ is left-invertible
2. the columns of $A$ are linearly independent
3. $A$ is right-invertible
4. the rows of $A$ are linearly independent

a square matrix with these properties is called **nonsingular**

Nonsingular = invertible

- if properties 1 and 3 hold, then $A$ is invertible (page 4.5)
- if $A$ is invertible, properties 1 and 3 hold (by definition of invertibility)
Proof

\[
\begin{align*}
\text{left-invertible} & \quad \xrightarrow{(a)} \quad \text{linearly independent columns} \\
& \quad \uparrow^{(b')} \quad \downarrow^{(b)} \\
\text{linearly independent rows} & \quad \xleftarrow{(a')} \quad \text{right-invertible}
\end{align*}
\]

- we show that (a) holds in general
- we show that (b) holds for square matrices
- (a’) and (b’) follow from (a) and (b) applied to \(A^T\)
**Part a:** suppose $A$ is left-invertible

- if $B$ is a left inverse of $A$ (satisfies $BA = I$), then

\[Ax = 0 \implies BAx = 0\]
\[\implies x = 0\]

- this means that the columns of $A$ are linearly independent: if

\[A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\]

then

\[x_1a_1 + x_2a_2 + \cdots + x_na_n = 0\]

holds only for the trivial linear combination $x_1 = x_2 = \cdots = x_n = 0$
Part b: suppose $A$ is square with linearly independent columns $a_1, \ldots, a_n$ 

- for every $n$-vector $b$ the vectors $a_1, \ldots, a_n, b$ are linearly dependent 
  (from dimension inequality on page 4.12) 
- hence for every $b$ there exists a nontrivial linear combination 

$$x_1a_1 + x_2a_2 + \cdots + x_na_n + x_{n+1}b = 0$$

- we must have $x_{n+1} \neq 0$ because $a_1, \ldots, a_n$ are linearly independent 
- hence every $b$ can be written as a linear combination of $a_1, \ldots, a_n$ 
- in particular, there exist $n$-vectors $c_1, \ldots, c_n$ such that 

$$Ac_1 = e_1, \quad Ac_2 = e_2, \quad \ldots, \quad Ac_n = e_n,$$

- the matrix $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$ is a right inverse of $A$: 

$$A \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = I$$
Examples

\[ A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \]

• \( A \) is nonsingular because its columns are linearly independent:

\[
\begin{align*}
x_1 - x_2 + x_3 &= 0, \\
-x_1 + x_2 + x_3 &= 0, \\
x_1 + x_2 - x_3 &= 0
\end{align*}
\]

is only possible if \( x_1 = x_2 = x_3 = 0 \)

• \( B \) is singular because its columns are linearly dependent:

\[ Bx = 0 \quad \text{for} \ x = (1, 1, 1, 1) \]
Example: Vandermonde matrix

\[
A = \begin{bmatrix}
1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\
1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & t_n & t_n^2 & \cdots & t_n^{n-1}
\end{bmatrix}
\]

with \( t_i \neq t_j \) for \( i \neq j \)

we show that \( A \) is nonsingular by showing that \( Ax = 0 \) only if \( x = 0 \)

- \( Ax = 0 \) means \( p(t_1) = p(t_2) = \cdots = p(t_n) = 0 \) where

\[
p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}
\]

\( p(t) \) is a polynomial of degree \( n - 1 \) or less

- if \( x \neq 0 \), then \( p(t) \) can not have more than \( n - 1 \) distinct real roots

- therefore \( p(t_1) = \cdots = p(t_n) = 0 \) is only possible if \( x = 0 \)
Inverse of transpose and product

Transpose and conjugate transpose

if $A$ is nonsingular, then $A^T$ and $A^H$ are nonsingular and

$$(A^T)^{-1} = (A^{-1})^T, \quad (A^H)^{-1} = (A^{-1})^H$$

we write these as $A^{-T}$ and $A^{-H}$

Product

if $A$ and $B$ are nonsingular and of equal size, then $AB$ is nonsingular with

$$(AB)^{-1} = B^{-1}A^{-1}$$
Outline

- left and right inverse
- linear independence
- nonsingular matrices
- matrices with linearly independent columns
- matrices with linearly independent rows
Gram matrix

the Gram matrix associated with a matrix

\[
A = \begin{bmatrix}
a_1 & a_2 & \cdots & a_n
\end{bmatrix}
\]

is the matrix of all pairwise inner products of the column vectors

- for real matrices:

\[
A^T A = \begin{bmatrix}
a_1^T a_1 & a_1^T a_2 & \cdots & a_1^T a_n \\
a_2^T a_1 & a_2^T a_2 & \cdots & a_2^T a_n \\
\vdots & \vdots & \ddots & \vdots \\
a_n^T a_1 & a_n^T a_2 & \cdots & a_n^T a_n
\end{bmatrix}
\]

- for complex matrices:

\[
A^H A = \begin{bmatrix}
a_1^H a_1 & a_1^H a_2 & \cdots & a_1^H a_n \\
a_2^H a_1 & a_2^H a_2 & \cdots & a_2^H a_n \\
\vdots & \vdots & \ddots & \vdots \\
a_n^H a_1 & a_n^H a_2 & \cdots & a_n^H a_n
\end{bmatrix}
\]
Nonsingular Gram matrix

the Gram matrix is nonsingular if only if $A$ has linearly independent columns

• suppose $A \in \mathbb{R}^{m \times n}$ has linearly independent columns:

\[
A^T Ax = 0 \implies x^T A^T Ax = (Ax)^T (Ax) = \|Ax\|^2 = 0
\]

\[
\implies Ax = 0
\]

\[
\implies x = 0
\]

therefore $A^T A$ is nonsingular

• suppose the columns of $A \in \mathbb{R}^{m \times n}$ are linearly dependent

\[
\exists x \neq 0, \ Ax = 0 \implies \exists x \neq 0, \ A^T Ax = 0
\]

therefore $A^T A$ is singular

(for $A \in \mathbb{C}^{m \times n}$, replace $A^T$ with $A^H$ and $x^T$ with $x^H$)
Pseudo-inverse of matrix with independent columns

- suppose \( A \in \mathbb{R}^{m \times n} \) has linearly independent columns
- this implies that \( A \) is tall or square \((m \geq n)\); see page 4.12

the *pseudo-inverse* of \( A \) is defined as

\[
A^\dagger = (A^T A)^{-1} A^T
\]

- this matrix exists, because the Gram matrix \( A^T A \) is nonsingular
- \( A^\dagger \) is a left inverse of \( A \):

\[
A^\dagger A = (A^T A)^{-1} (A^T A) = I
\]

(for complex \( A \) with linearly independent columns, \( A^\dagger = (A^H A)^{-1} A^H \))
Summary

the following three properties are equivalent for a real matrix $A$

1. $A$ is left-invertible
2. the columns of $A$ are linearly independent
3. $A^T A$ is nonsingular

- 1 $\Rightarrow$ 2 was already proved on page 4.15
- 2 $\Rightarrow$ 1: we have seen that the pseudo-inverse is a left inverse
- 2 $\Leftrightarrow$ 3: proved on page 4.21
- a matrix with these properties must be tall or square
- for complex matrices, replace $A^T A$ in property 3 by $A^H A$
Outline

- left and right inverse
- linear independence
- nonsingular matrices
- matrices with linearly independent columns
- matrices with linearly independent rows
Pseudo-inverse of matrix with independent rows

• suppose $A \in \mathbb{R}^{m \times n}$ has linearly independent rows

• this implies that $A$ is wide or square ($m \leq n$); see page 4.12

the pseudo-inverse of $A$ is defined as

$$A^\dagger = A^T (AA^T)^{-1}$$

• $A^T$ has linearly independent columns

• hence its Gram matrix $AA^T$ is nonsingular, so $A^\dagger$ exists

• $A^\dagger$ is a right inverse of $A$:

$$AA^\dagger = (AA^T)(AA^T)^{-1} = I$$

(for complex $A$ with linearly independent rows, $A^\dagger = A^H (AA^H)^{-1}$)
Summary

the following three properties are equivalent

1. $A$ is right-invertible
2. the rows of $A$ are linearly independent
3. $AA^T$ is nonsingular

- $1 \Rightarrow 2$ and $2 \Leftrightarrow 3$: by transposing result on page 4.23
- $2 \Rightarrow 1$: we have seen that the pseudo-inverse is a right inverse
- a matrix with these properties must be wide or square
- for complex matrices, replace $AA^T$ in property 3 by $AA^H$