# 11. Geršgorin bounds

- Geršgorin theorem
- diagonal dominance
- extensions

# Geršgorin disks

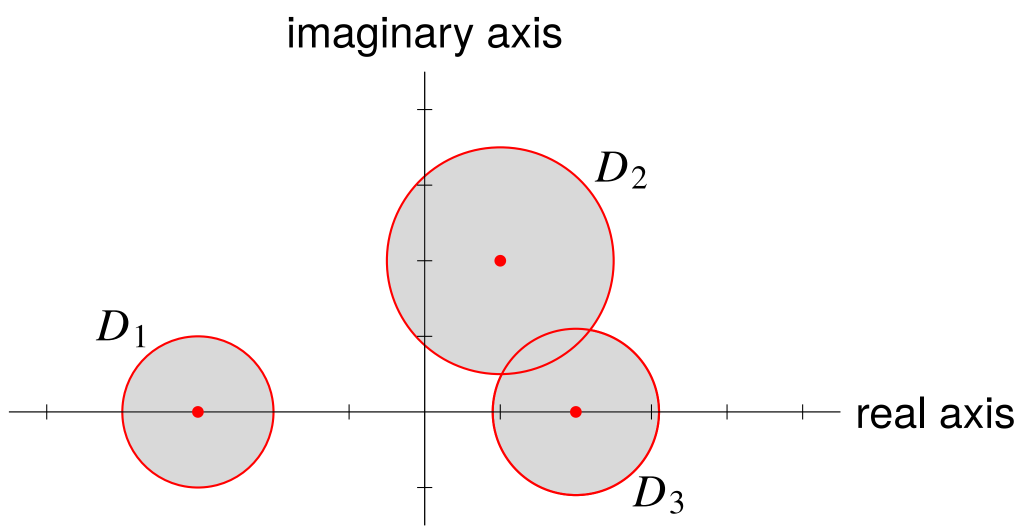
given a matrix $A \in \mathbf{C}^{n\times n}$, we define $n$ disks in the complex plane

$$D_i = \{z \in \mathbf{C} \mid |z - A_{ii}| \leq R_i\} \quad \text{where } R_i = \sum_{j\neq i} |A_{ij}|$$

- center of disk $D_i$ is diagonal entry $A_{ii}$
- radius is sum of absolute values of non-diagonal entries $A_{ij}$ in row $i$

## Example

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0.5 & 1+2\mathrm{j} & -1 \\ 1 & 0.1 & 2 \end{bmatrix}$$



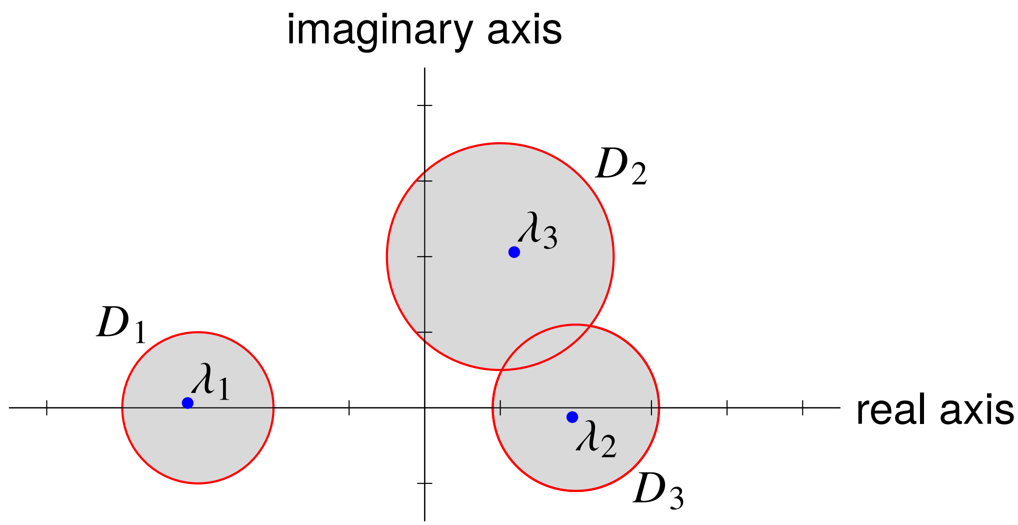
# Geršgorin theorem

the union of the disks $D_1, \ldots, D_n$ contains all the eigenvalues of $A$

(proof follows on page 11.6)

**Example**

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0.5 & 1+2j & -1 \\ 1 & 0.1 & 2 \end{bmatrix}$$




eigenvalues of $A$:

$$\lambda_1 = 2.92 - 0.18j, \qquad \lambda_2 = 2.17 + 0.43j, \qquad \lambda_3 = 0.91 + 1.75j$$

# Outline

- Geršgorin theorem
- **diagonal dominance**
- extensions

# Diagonal dominance

- an $n \times n$ matrix $A$ is *diagonally dominant* if

$$|A_{ii}| \geq \sum_{j \neq i} |A_{ij}|, \quad i = 1, \ldots, n$$

- an $n \times n$ matrix $A$ is *strictly diagonally dominant* if

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}|, \quad i = 1, \ldots, n$$

**Examples**

$$A_1 = \begin{bmatrix} 2 & \mathrm{j} & 0 \\ -\mathrm{j} & -3 & 1 \\ 1 & -2 & 4\mathrm{j} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- $A_1, A_2$ are strictly diagonally dominant
- $A_3$ is diagonally dominant but not strictly

# Strict diagonal dominance theorem

a strictly diagonally dominant matrix is nonsingular

*Proof:* suppose $Ax = 0$ for some $x \neq 0$

- normalize $x$ so that $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} = 1$
- let $i$ be an index with $|x_i| = \|x\|_\infty = 1$
- the $i$th equation in $Ax = 0$ is

$$A_{ii}x_i = -\sum_{j \neq i} A_{ij}x_j$$

- taking absolute values we get

$$|A_{ii}| = |\sum_{j \neq i} A_{ij}x_j| \leq \sum_{j \neq i} |A_{ij}||x_j| \leq \sum_{j \neq i} |A_{ij}|$$

  this contradicts strict diagonal dominance

# Proof of Geršgorin theorem

Geršgorin theorem: if $\lambda$ is an eigenvalue of $A$ then

$$|\lambda - A_{ii}| \leq \sum_{j \neq i} |A_{ij}| \quad \text{for at least one } i = 1, \ldots, n$$

*Proof:* suppose $\lambda$ is a complex number outside the Gershgorin disks,

$$|\lambda - A_{ii}| > \sum_{j \neq i} |A_{ij}| \quad \text{for all } i = 1, \ldots, n$$

- then the matrix $\lambda I - A$ is strictly diagonally dominant
- therefore, $\lambda I - A$ is nonsingular, so $\lambda$ is not an eigenvalue

# Matrices and directed graphs

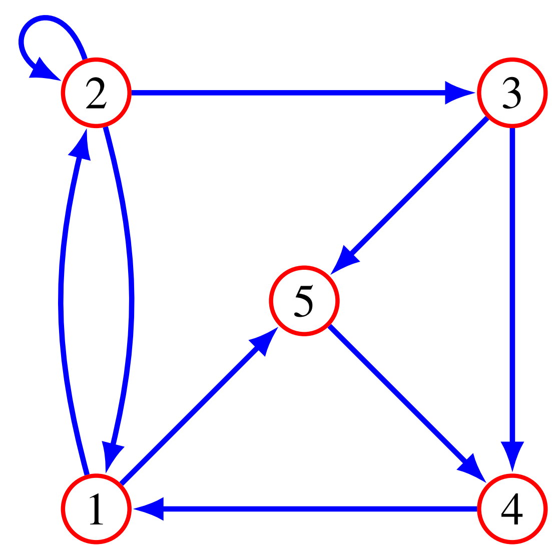
we represent the zero–nonzero pattern of $n \times n$ matrix $A$ by a directed graph $G_A$

- $n$ vertices numbered $1, \ldots, n$
- directed edge $i \to j$ indicates that $A_{ij} \neq 0$
- some authors use the other orientation of the edges ($j \to i$ if $A_{ij} \neq 0$)

## Example

$$A = \begin{bmatrix} 0 & \bullet & 0 & 0 & \bullet \\ \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & \bullet & \bullet \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bullet & 0 \end{bmatrix}$$

dots represent non-zero entries



# Irreducible matrix

an $n \times n$ matrix $A$ is *irreducible* if the graph $G_A$ is *strongly connected*

- there is a *directed path* between every ordered pair of distinct vertices
- equivalently (from ECE133A homework), all entries of the matrix

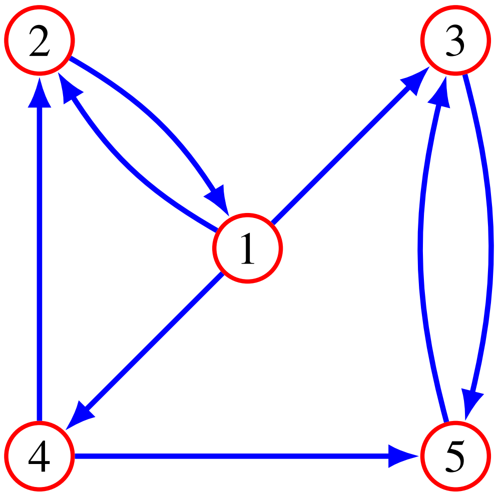
$$(I + |A|)^{n-1}$$

  are positive, where $|A|$ is the matrix with $(i, j)$ entry $|A_{ij}|$

**Examples**

- the matrix on the previous page is irreducible
- the following matrix is not irreducible

$$A = \begin{bmatrix} 0 & \bullet & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & \bullet & 0 & 0 & \bullet \\ 0 & 0 & \bullet & 0 & 0 \end{bmatrix}$$



# Extension of strict diagonal dominance theorem

an $n \times n$ matrix $A$ that satisfies the following three properties is nonsingular

1. $A$ is diagonally dominant:

$$|A_{ii}| \geq \sum_{j \neq i} |A_{ij}| \quad \text{for all } i$$

2. strict inequality holds for at least one row:

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}| \quad \text{for at least one } i$$

3. $A$ is irreducible

**Example**

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

*Proof:* assume $A$ satisfies the three conditions, and $Ax = 0$ for some $x \neq 0$

- without loss of generality, assume $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} = 1$
- define $I \subseteq \{1, 2, \ldots, n\}$ by

$$|x_i| = 1 \quad \text{for } i \in I, \qquad |x_i| < 1 \quad \text{for } i \notin I$$

- consider equation $i$ in $Ax = 0$ for a row index $i \in I$:

$$|A_{ii}| = |A_{ii}||x_i| = |\sum_{j \neq i} A_{ij} x_j| \leq \sum_{j \neq i} |A_{ij}||x_j| \leq \sum_{j \neq i} |A_{ij}| \tag{1}$$

- since $A$ is diagonally dominant, the two inequalities in (1) hold with equality
- we have equality in the second inequality if $|x_j| = 1$ when $A_{ij} \neq 0$:

$$i \in I, \quad A_{ij} \neq 0 \qquad \Longrightarrow \qquad j \in I$$

- this means that directed paths that start at vertices in $I$ end at vertices in $I$
- since $G_A$ is strongly connected, $I = \{1, 2, \ldots, n\}$
- but then (1) holds (with equality) for all $i$, which contradicts property 2

# Implication for Geršgorin theorem

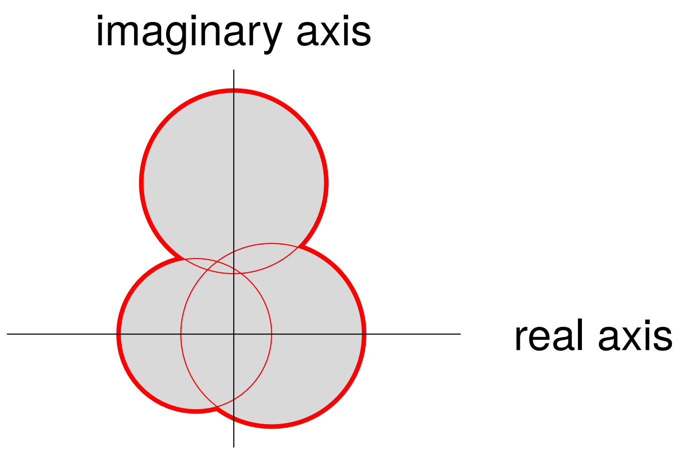
suppose $A$ is irreducible and $\lambda$ is an eigenvalue on the boundary of

$$D_1 \cup D_2 \cup \cdots \cup D_n$$

then $\lambda$ is on the boundary of each disk $D_i$:

$$|\lambda - A_{ii}| = \sum_{j \neq i} |A_{ij}| \quad \text{for } i = 1, \ldots, n$$

**Example:** suppose $A$ is irreducible with Geršgorin disks shown in the figure




eigenvalues of $A$ are in interior of the shaded area (excluding the heavy red line)

*Proof* (of result on page 11.11): suppose $A$ is irreducible

- consider a complex number $\lambda$ on the boundary of $D_1 \cup \cdots \cup D_n$
- for each $i = 1, \ldots, n$, either

$$|\lambda I - A_{ii}| = \sum_{j \neq i} |A_{ij}| \qquad (2)$$

  or

$$|\lambda I - A_{ii}| > \sum_{j \neq i} |A_{ij}| \qquad (3)$$

- if (3) holds for at least one $i$, then $\lambda I - A$ satisfies the 3 properties on page 11.9
- by the result on page 11.9, $\lambda I - A$ is nonsingular, so $\lambda$ is not an eigenvalue of $A$
- in other words, if $\lambda$ is an eigenvalue, it must satisfy (2) for each $i = 1, \ldots, n$

# Outline

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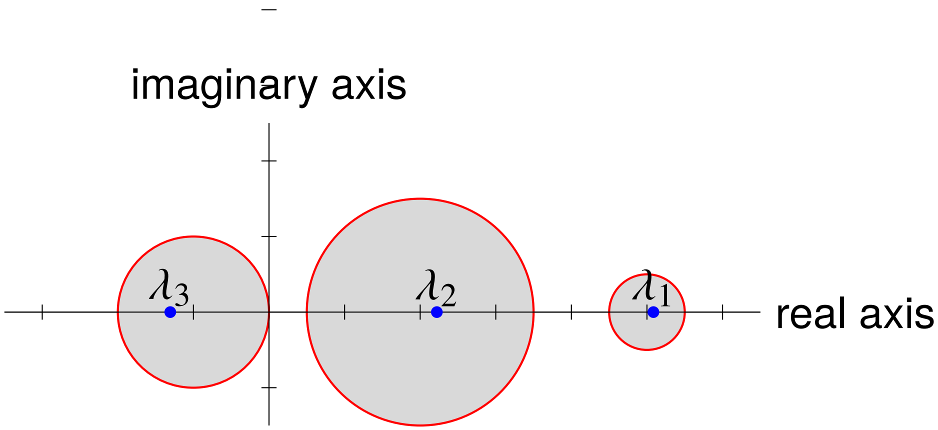
# Disjoint Geršgorin disks

if the Geršgorin disks are disjoint, they each contain one eigenvalue

**Example**

$$A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 2 & 1/2 \\ 0 & 1/2 & 5 \end{bmatrix}$$




eigenvalues of $A$ are

$$\lambda_1 = 5.09, \qquad \lambda_2 = 2.22, \qquad \lambda_3 = -1.31$$

*Proof*

consider a family of matrices

$$A(t) = B + t(A - B) \qquad \text{where } B = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{bmatrix}$$

- for $t = 0$, eigenvalues are $A_{11}, A_{22}, \ldots, A_{nn}$
- for $t = 1$, eigenvalues are eigenvalues of $A$
- for $0 \leq t \leq 1$, eigenvalues of $A(t)$ are in Geršgorin disks for $A(t)$:

$$D_i(t) = \{z \in \mathbf{C} \mid |z - A_{ii}| \leq t \sum_{j \neq i} |A_{ij}|\}, \quad i = 1, \ldots, n$$

- it can be shown that the eigenvalues of $A(t)$ are continuous functions of $t$
- hence, if disks $D_i(t)$ are disjoint they each contain one eigenvalue of $A(t)$

# Geršgorin bounds for transpose

Geršgorin disks for $A^T$ are

$$\tilde{D}_i = \{z \in \mathbf{C} \mid |z - A_{ii}| \leq \tilde{R}_i\} \quad \text{where } \tilde{R}_i = \sum_{j \neq i} |A_{ji}|$$

$A$ and $A^T$ have the same eigenvalues, so the eigenvalues are also contained in

$$\tilde{D}_1 \cup \tilde{D}_2 \cup \cdots \cup \tilde{D}_n$$

# Weighted Gerğorin bounds

- let $W$ be a diagonal matrix with positive diagonal elements $w_1, \ldots, w_n$
- Geršgorin disks for $W^{-1}AW$ are

$$\hat{D}_i = \{z \in \mathbf{C} \mid |z - A_{ii}| \leq \hat{R}_i\} \quad \text{where } \hat{R}_i = \frac{1}{w_i} \sum_{j \neq i} w_j |A_{ij}|$$

$A$ and $W^{-1}AW$ have the same eigenvalues, so eigenvalues are also contained in

$$\hat{D}_1 \cup \hat{D}_2 \cup \cdots \cup \hat{D}_n$$