12. Nonnegative matrices

- matrix norms and spectral radius
- linear dynamical systems
- Perron–Frobenius theorem

Norms

recall the three defining properties of a *norm* (of vectors or matrices)

- *positive definiteness:* $\|x\| \geq 0$ for all x, and $\|x\| = 0$ only if $x = 0$
- *homogeneity:* $\|\gamma x\| = |\gamma|\|x\|$ for all x and all scalars γ
- *triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$ for all x, y

Examples of vector norms (on $\mathbf{R}^n$ or $\mathbf{C}^n$)

- Euclidean norm $\|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$ (denoted by $\|x\|$ in this course)
- Chebyshev norm (infinity-norm): $\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$
- 1-norm: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

Examples of matrix norms (on $\mathbf{R}^{m\times n}$ or $\mathbf{C}^{m\times n}$)

- Frobenius norm $\|X\|_F = \sum_{i=1}^{m} \sum_{j=1}^{n} |X_{ij}|^2$
- matrix 2-norm (spectral norm) $\|X\|_2 = \sigma_1(X)$ (maximum singular value)
- max-row-sum norm $\|X\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |X_{ij}|$

Submultiplicative matrix norms

for matrix norms on $\mathbf{R}^{n\times n}$ or $\mathbf{C}^{n\times n}$ one often requires a fourth property:

$$\|AB\| \leq \|A\|\|B\| \quad \text{for all } A, B$$

- such a norm is called *submultiplicative*
- for a submultiplicative norm, $\|A^k\| \leq \|A\|^k$ for positive integer k

Examples: the following matrix norms are submultiplicative

- Frobenius norm
- matrix 2-norm
- max-row-sum norm

Exercise: show that the following matrix norm is not submultiplicative

$$\|X\| = \max_{i=1,\ldots,n} \max_{j=1,\ldots,n} |X_{ij}|$$

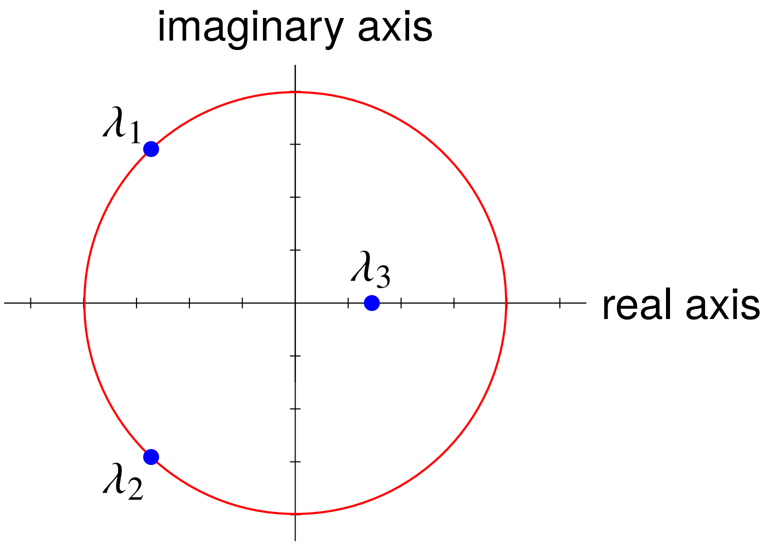
Spectral radius

the *spectral radius* of an $n \times n$ matrix A is defined as

$$\rho(A) = \max_{i=1,\ldots,n} |\lambda_i(A)|$$

Example

$$A = \begin{bmatrix} -1 & -2 & 6 \\ -2 & 1 & 5 \\ 2 & -5 & -4 \end{bmatrix}$$



- eigenvalues of A:

$$\lambda_1 = -2.72 + 2.91\mathrm{j}, \qquad \lambda_2 = -2.72 - 2.91\mathrm{j}, \qquad \lambda_3 = 1.45$$

- spectral radius is $\rho(A) = |\lambda_1| = |\lambda_2| = 3.99$

Spectral radius of normal matrix

for a *normal* matrix,

$$\rho(A) = \|A\|_2$$

- recall definition of normal matrix on page 10.12: $AA^H = A^H A$
- normal matrices include symmetric, skew-symmetric, orthogonal matrices
- Schur decomposition of normal matrix is of the form

$$A = UDU^H \qquad \text{with } U \text{ unitary, } D \text{ diagonal}$$

  diagonal entries of D are the eigenvalues λ_i of A

- therefore, singular values σ_k are the absolute values of the eigenvalues, so

$$\rho(A) = \max_{i=1,\ldots,n} |\lambda_i| = \max_{k=1,\ldots,n} \sigma_k = \|A\|_2$$

for non-normal matrices $\rho(A)$ and $\|A\|_2$ can be very different: for A on page 12.4,

$$\rho(A) = 3.99, \qquad \|A\|_2 = 9.41$$

Spectral radius is not a matrix norm

the spectral radius is not a norm on $\mathbf{R}^{n\times n}$ or $\mathbf{C}^{n\times n}$

- nonzero matrix can have zero spectral radius (all eigenvalues are zero)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- triangle inequality does not hold for spectral radius

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix}, \qquad A + B = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

here, $\rho(A) = \rho(B) = 1$, but

$$\rho(A + B) = 3 > \rho(A) + \rho(B)$$

Norm bound on spectral radius

for any submultiplicative matrix norm,

$$\rho(A) \leq \|A\| \tag{1}$$

Proof

- choose a vector norm that is *consistent* with the matrix norm, *i.e.*,

$$\|Ax\| \leq \|A\|\|x\| \quad \text{for all } A \text{ and all } x$$

- for example, define vector norm as $\|x\| = \|xy^T\|$ where $y \neq 0$ is a fixed n-vector:

$$\|Ax\| = \|Axy^T\| \leq \|A\|\|xy^T\| = \|A\|\|x\|$$

  inequality follows from submultiplicative property of the matrix norm

- now, if x is eigenvector with eigenvalue λ and $|\lambda| = \rho(A)$, then (1) follows from:

$$\rho(A)\|x\| = |\lambda|\|x\| = \|\lambda x\| = \|Ax\| \leq \|A\|\|x\|$$

Outline

- matrix norms and spectral radius
- **linear dynamical systems**
- Perron–Frobenius theorem

Discrete-time linear dynamical system

$$x(t+1) = A(t)x(t), \quad t = 0, 1, 2, \ldots$$

- t denotes time or period
- n-vector $x(t)$ is the *state* at time t
- $n \times n$ matrix $A(t)$ is the *dynamics matrix* at time t
- current state is a *linear* function of the previous state
- system is *time-invariant* if $A(t) = A$ does not depend on time

for a time-invariant system,

$$x(t) = A^t x(0), \quad t = 0, 1, 2, \ldots$$

- properties of matrix power A^t determine dynamical behavior
- in this lecture we will be interested in systems with $A_{ij} \geq 0$ for all i, j

Autoregressive model

$$x(t+1) = A_1 x(t) + A_2 x(t-1) + \cdots + A_p x(t-p+1)$$

- state evolution is described by a homogeneous difference equation
- current state is a linear function of the p previous states
- linear dynamical system $x(t+1) = Ax(t)$ is a special case with $p = 1$
- for $p > 1$, can be written as a linear dynamical system $\tilde{x}(t+1) = \tilde{A}\tilde{x}(t)$:

$$\underbrace{\begin{bmatrix} x(t+1) \\ x(t) \\ x(t-1) \\ \vdots \\ x(t-p+3) \\ x(t-p+2) \end{bmatrix}}_{\tilde{x}(t+1)} = \underbrace{\begin{bmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x(t) \\ x(t-1) \\ x(t-2) \\ \vdots \\ x(t-p+2) \\ x(t-p+1) \end{bmatrix}}_{\tilde{x}(t)}$$

Population dynamics

- $x(t) \in \mathbf{R}^{100}$ gives age distribution of population in year t (say, on January 1)
- $x_i(t)$ for $i < 100$ is the number of people with age $i - 1$
- $x_{100}(t)$ is the number of people aged 99 and above
- we exclude changes due to immigration

Linear dynamical model

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ \vdots \\ x_{100}(t+1) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & \cdots & b_{99} & b_{100} \\ 1-d_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-d_{99} & 1-d_{100} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{100}(t) \end{bmatrix}$$

- $b_i \geq 0$ is number of births per person with age $i - 1$
- $0 < d_i < 1$ is number of deaths per person with age $i - 1$

Epidemic dynamics

4-vector $x(t)$ gives proportion of population in 4 infection states

- susceptible: can acquire the disease the next day
- infected: have the disease
- recovered: had the disease, recovered, now immune
- deceased: had the disease, and unfortunately died

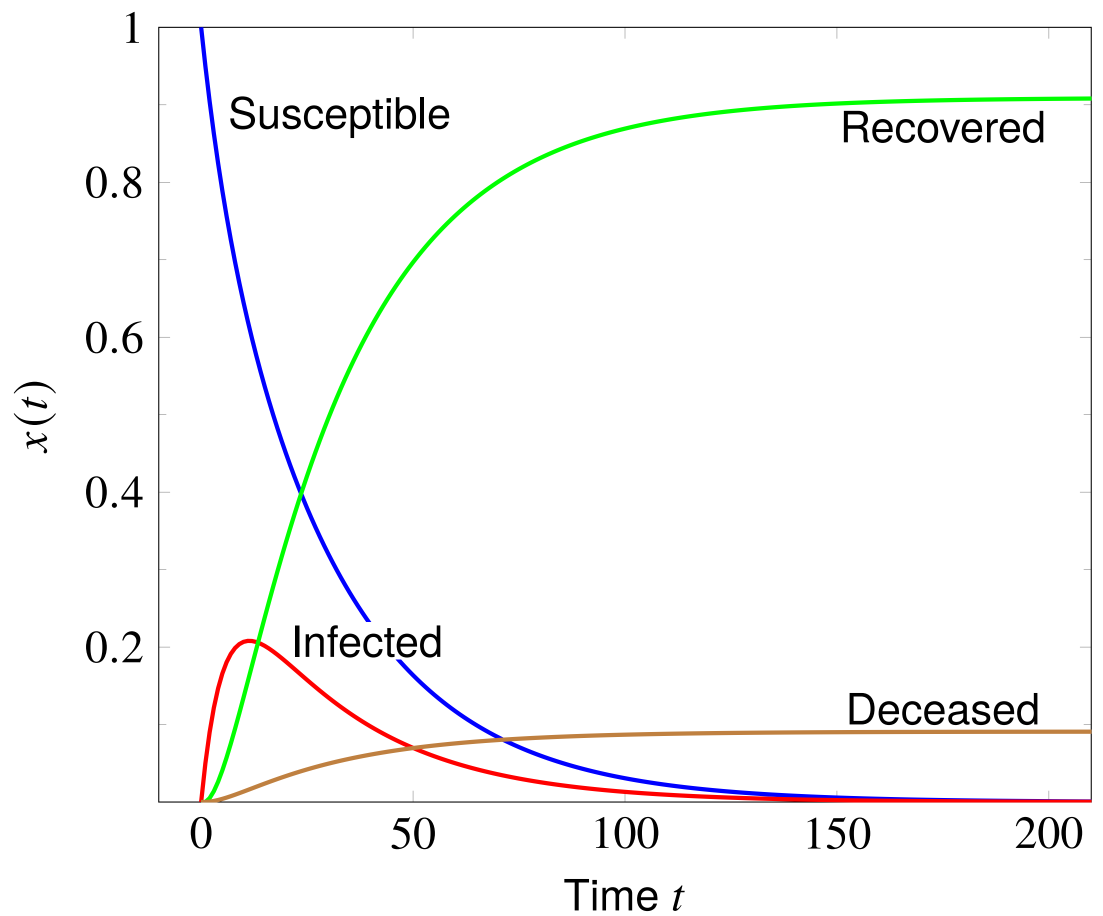
Example

$$\begin{bmatrix} x_{\mathrm{s}}(t+1) \\ x_{\mathrm{i}}(t+1) \\ x_{\mathrm{r}}(t+1) \\ x_{\mathrm{d}}(t+1) \end{bmatrix} = \begin{bmatrix} 0.95 & 0.04 & 0 & 0 \\ 0.05 & 0.85 & 0 & 0 \\ 0 & 0.10 & 1 & 0 \\ 0 & 0.01 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\mathrm{s}}(t) \\ x_{\mathrm{i}}(t) \\ x_{\mathrm{r}}(t) \\ x_{\mathrm{d}}(t) \end{bmatrix}$$

each day,

- 5% of susceptible population acquires the disease, 95% remains susceptible
- 4% of infected recover without immunity, 10% with immunity, 1% die

Simulation of epidemic dynamics



- figure shows $x(t)$ starting at $x(1) = (1, 0, 0, 0)$
- $x(t) \to (0.0, 0.0, 0.91, 0.09)$ (an eigenvector of A with eigenvalue 1)

Finite Markov chain

- $s(t)$ is a random variable, with possible values $1, 2, \ldots, n$ (the n "states")
- $x_i(t)$ is the probability that $s(t) = i$
- A_{ij} is probability that $s(t+1) = i$, given that $s(t) = j$:

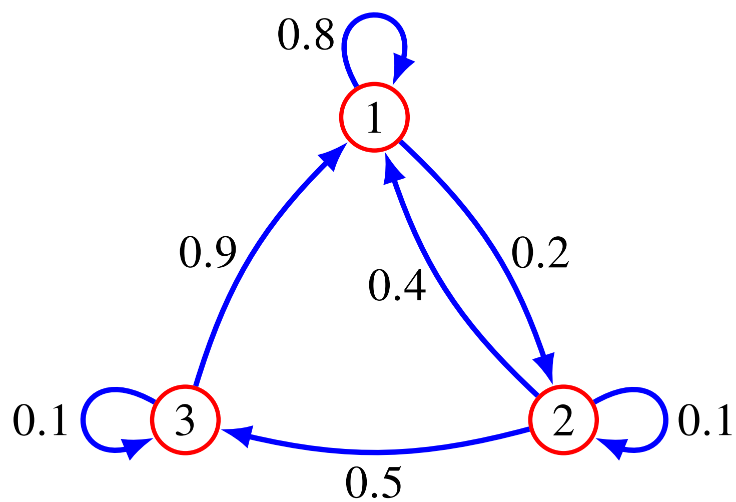
$$x_i(t+1) = A_{i1}x_1(t) + A_{i2}x_2(t) + \cdots + A_{in}x_n(t)$$

  hence, A is nonnegative, with $A_{1j} + A_{2j} + \cdots + A_{nj} = 1$ in each column

- matrix A^T is called the *state transition matrix*

Example

$$A = \begin{bmatrix} 0.8 & 0.4 & 0.9 \\ 0.2 & 0.1 & 0 \\ 0 & 0.5 & 0.1 \end{bmatrix}$$



Eigenvalues of matrix power

let A be a square matrix with Schur decomposition

$$A = UTU^H$$

diagonal elements of T are eigenvalues $\lambda_1, \ldots, \lambda_n$ of A

- if k is a positive integer, then $A^k = UT^kU^H$
- this is a Schur decomposition: T^k is upper triangular with diagonal elements

$$T_{11}^k, \quad T_{22}^k, \quad \ldots, \quad T_{nn}^k$$

- eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$
- spectral radius is $\rho(A^k) = \rho(A)^k$
- one can also show that

$$\lim_{k\to\infty} A^k = 0 \qquad \Longleftrightarrow \qquad \rho(A) < 1$$

Outline

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- **Perron–Frobenius theorem**

Perron–Frobenius theorem

suppose $A \in \mathbf{R}^{n \times n}$ is *nonnegative* (componentwise) and *irreducible*

Perron root

- $\rho(A)$ is an eigenvalue of A, called the *Perron root*
- $\rho(A > 0$
- the eigenvalue $\rho(A)$ has algebraic (hence, geometric) multiplicity one

Perron vector

- A has a positive eigenvector with eigenvalue $\rho(A)$
- normalized to satisfy $\mathbf{1}^T x = 1$ this vector is unique, and called the *Perron vector:*

$$Ax = \rho(A)x, \qquad x > 0, \qquad \mathbf{1}^T x = 1$$

- A has no other nonnegative eigenvectors (except multiples of the Perron vector)

Other eigenvalues: if A is positive, eigenvalues $\lambda \neq \rho(A)$ satisfy $|\lambda| < \rho(A)$

Nonnegative matrices

some weaker results hold if A is nonnegative, but not irreducible

- $\rho(A)$ is an eigenvalue of A
- A has a nonnegative eigenvector with eigenvalue $\rho(A)$

Finite Markov chain

consider a finite Markov chain (page 12.13), and assume A is irreducible

- the state transition matrix A^T is also irreducible
- since $\sum_i A_{ij} = 1$ for all j, the vector $\mathbf{1}$ is an eigenvector of A^T with eigenvalue 1:

$$A^T\mathbf{1} = \mathbf{1}$$

- from the definition of $\rho(A)$ and the norm inequality on page 12.7

$$1 \leq \rho(A) = \rho(A^T) \leq \|A^T\|_\infty = 1$$

  hence, $\rho(A) = 1$

- there is a unique positive stationary probability vector (the Perron vector of A):

$$Ax = x, \qquad x > 0, \qquad \mathbf{1}^T x = 1$$

for the example on page 12.13, the stationary probability distribution is

$$x = (0.743,\ 0.165,\ 0.092)$$