Analytic center cutting-plane method

- analytic center cutting-plane method
- computing the analytic center
- pruning constraints
- lower bound and stopping criterion
Analytic center and ACCPM

**analytic center** of a set of inequalities $Ax \leq b$

$$x_{ac} = \arg\min_z - \sum_{i=1}^{m} \log(b_i - a_i^T z)$$

**analytic center cutting-plane method (ACCPM)**

Localization method that

- represents $P_k$ by set of inequalities $A^{(k)}$, $b^{(k)}$
- selects analytic center of $A^{(k)}x \leq b^{(k)}$ as query point $x^{(k+1)}$
ACCPM algorithm outline

given an initial polyhedron $\mathcal{P}_0 = \{ x \mid A^{(0)} x \preceq b^{(0)} \}$ known to contain $C$

repeat for $k = 1, 2, \ldots$

1. compute $x^{(k)}$, the analytic center of $A^{(k-1)} x \preceq b^{(k-1)}$

2. query cutting-plane oracle at $x^{(k)}$

3. if $x^{(k)} \in C$, quit; otherwise, add returned cutting plane $a^T z \leq b$:

\[ A^{(k)} = \begin{bmatrix} A^{(k-1)} \\ a^T \end{bmatrix}, \quad b^{(k)} = \begin{bmatrix} b^{(k-1)} \\ b \end{bmatrix} \]

if $\mathcal{P}_k = \{ x \mid A^{(k)} x \preceq b^{(k)} \} = \emptyset$, quit
Constructing cutting planes

cutting planes for optimal set $C$ of convex problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

- if $x^{(k)}$ is not feasible, say $f_j(x^{(k)}) > 0$, we have (deep) feasibility cut

\[
f_j(x^{(k)}) + g_j^T (z - x^{(k)}) \leq 0 \quad \text{where} \quad g_j \in \partial f_j(x^{(k)})
\]

- if $x^{(k)}$ is feasible, we have (deep) objective cut

\[
g_0^T (z - x^{(k)}) + f_0(x^{(k)}) - f_{\text{best}}^{(k)} \leq 0 \quad \text{where} \quad g_0 \in \partial f_0(x^{(k)})
\]

and $f_{\text{best}}^{(k)} = \min\{f_0(x^{(i)}) \mid i \leq k, \ x^{(i)} \ \text{feasible}\}$
Computing the analytic center

\[
\text{minimize } \phi(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)
\]

\[\text{dom } \phi = \{x \mid a_i^T x < b_i, \ i = 1, \ldots, m\}\]

**challenge**: we are not given a point in \(\text{dom } \phi\)

**some options**

- use phase I to find \(x \in \text{dom } \phi\), followed by standard Newton method
- standard Newton method applied to dual problem
- infeasible start Newton method (EE236B lecture 11, BV §10.3)
Dual Newton method

dual analytic centering problem

maximize \( g(z) = \sum_{i=1}^{m} \log z_i - b^T z + m \)
subject to \( A^T z = 0 \)

optimality conditions

\( x, z \) are primal and dual optimal if

\[ b_i - a_i^T x = 1/z_i, \quad A^T z = 0, \quad z \succ 0, \quad A x \prec b \]
Initialization of dual Newton method

dual method is interesting when a strictly feasible \( z \) is easy to find, \( e.g., \)

\[
A = \begin{bmatrix}
  I \\
  -I \\
  B
\end{bmatrix}
\]

- dual feasibility requires

\[
A^T z = z_1 - z_2 + B^T z_3 = 0, \quad z = (z_1, z_2, z_3) \succeq 0
\]

(for example, can pick any \( z_3 \succeq 0 \) and find corresponding \( z_1, z_2 \))

- this corresponds to variable bounds in (primal) centering problem, \( e.g., \)

\[
\mathcal{P}_0 = \{x \mid l \preceq x \preceq u\}
\]
Dual Newton equation

analytic centering problem

\[
\begin{align*}
\text{minimize} & \quad - \sum_{i=1}^{m} \log z_i + b^T z \\
\text{subject to} & \quad A^T z = 0
\end{align*}
\]

Newton equation

\[
\begin{bmatrix}
- \operatorname{diag}(z)^{-2} & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta z \\
w
\end{bmatrix}
= \begin{bmatrix}
b - \operatorname{diag}(z)^{-1} \mathbf{1} \\
0
\end{bmatrix}
\]

can be solved by elimination of \( \Delta z \): solve

\[
(A^T \operatorname{diag}(z)^2 A) w = A^T (\operatorname{diag}(z)^2 b - z)
\]

and take \( \Delta z = z - \operatorname{diag}(z)^2 (b - Aw) \)
Stopping criterion for dual Newton method

Newton decrement at $z$ is

$$
\lambda(z) = \left( \Delta z^T \nabla g(z) \right)^{1/2} = \| \text{diag}(z)^{-1} \Delta z \|_2
$$

- $\lambda(z) = 0$ implies $w$ is the analytic center:

$$
b - Aw = \text{diag}(z)^{-1} 1
$$

- $\lambda(z) < 1$ implies $x = w$ is primal feasible

$$
b - Aw = \text{diag}(z)^{-1} (1 - \text{diag}(z)^{-1} \Delta z) \succ 0
$$

terminating with small $\lambda(z)$ gives strictly feasible, approximate center
Infeasible start Newton method

reformulated analytic centering problem (variables $x$ and $y$)

minimize $- \sum_{i=1}^{m} \log y_i$, subject to $y = b - Ax$

optimality conditions

$y \succ 0, \quad z \succ 0, \quad r(x, y, z) = \begin{bmatrix} y + Ax - b \\ A^T z \\ z - \text{diag}(y)^{-1} \mathbf{1} \end{bmatrix} = 0$

initialization: can start from any $x$, $z$, and any $y \succ 0$

example: take previous analytic center as $x$, and choose $y$ as

$y_i = b_i - a_i^T x$ if $b_i - a_i^T x > 0$, $y_i = 1$ otherwise
Newton equation for infeasible Newton method

\[
\begin{bmatrix}
  A & I & 0 \\
  0 & 0 & A^T \\
  0 & \text{diag}(y)^{-2} & I
\end{bmatrix}
\begin{bmatrix}
  \Delta x \\
  \Delta y \\
  \Delta z
\end{bmatrix}
= -
\begin{bmatrix}
  y + Ax - b \\
  A^T z \\
  z - \text{diag}(y)^{-1}1
\end{bmatrix}
\]

can be solved by block elimination of $\Delta y$, $\Delta z$: solve

\((A^T \text{diag}(y)^{-2}A)\Delta x = A^T \text{diag}(y)^{-2}(b - Ax - 2y)\)

and take

\[
\Delta y = b - y - Ax - A\Delta x, \quad \Delta z = \text{diag}(y)^{-2}(y - \Delta y) - z
\]
Pruning constraints

enclosing ellipsoid at analytic center

if $x_{ac}$ is the analytic center of $a_i^T x \leq b_i$, $i = 1, \ldots, m$, then the ellipsoid

$$\mathcal{E} = \{ z \mid (z - x_{ac})^T \nabla^2 \phi(x_{ac})(z - x_{ac}) \leq m^2 \}$$

contains $\mathcal{P} = \{ z \mid a_i^T x \leq b_i, \; i = 1, \ldots, m \}$

• proof in BV page 420

• from expression for Hessian,

$$\mathcal{E} = \left\{ z \left| \sum_{i=1}^{m} \left( \frac{a_i^T (z - x_{ac})}{b_i - a_i^T x_{ac}} \right)^2 \leq m^2 \right. \right\}$$
(ir)relevance measure for constraint $a_i^T x \leq b_i$

$$
\eta_i = \frac{b_i - a_i^T x_{ac}}{(a_i^T \nabla^2 \phi(x_{ac})^{-1} a_i)^{1/2}}
$$

if $\eta_i \geq m$, then constraint $a_i^T x \leq b_i$ is redundant

proof: the optimal value of

$$
\begin{align*}
\text{maximize} & \quad a_i^T z \\
\text{subject to} & \quad (z - x_{ac})^T H (z - x_{ac}) \leq m^2
\end{align*}
$$

(with $H = \nabla^2 \phi(x_{ac})$) is

$$
m \sqrt{a_i^T H^{-1} a_i + a_i^T x_{ac}}
$$

the constraint is redundant if this is less than $b_i$
common ACCPM constraint dropping schemes

- drop all constraints with $\eta_i \geq m$ (guaranteed to not change $\mathcal{P}$)
- drop constraints in order of irrelevance, keeping constant number, usually $3n - 5n$
Lower bound in ACCPM

suppose we apply ACCPM to a convex problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Gx \leq h
\end{align*}
\]  

(1)

the inequalities \(A^{(k-1)}x \leq b^{(k-1)}\) at iteration \(k\) can be divided in two sets

- \(A_f x \leq b_f\) includes the constraints \(Gx \leq h\) plus the feasibility cuts
- \(A_o x \leq b_o + c_o\) includes the objective cuts

\[
f_0(x^{(i)}) + g_0^{(i)T}(x - x^{(i)}) \leq f_{\text{best}}^{(i)},
\]

with \(g_0^{(i)T}x^{(i)} - f_0(x^{(i)})\) stored in the vector \(b_o\) and \(f_{\text{best}}^{(i)}\) in \(c_o\)
**piecewise-linear relaxation:** the problem

\[
\begin{align*}
\text{minimize} & \quad \max(A_0 x - b_0) \\
\text{subject to} & \quad A_f x \preceq b_f
\end{align*}
\]

is a relaxation of the problem (1) \((\max(y) \text{ for vector } y \text{ means } \max_i y_i)\)

- \(f_0(x) \geq \max(A_0 x - b_0)\) for all \(x\) (by convexity)
- optimal set is contained in the polyhedron \(A_f x \preceq b_f\) (by construction)

**dual of PWL relaxation**

\[
\begin{align*}
\text{maximize} & \quad -b_0^T u - b_f^T v \\
\text{subject to} & \quad A_o^T u + A_f^T v = 0 \\
& \quad 1^T u = 1 \\
& \quad u \succeq 0, \quad v \succeq 0
\end{align*}
\]

dual feasible points give lower bounds on optimal value of (1)
dual feasible point from analytic centering

\( x^{(k)} \) is the analytic center of \( A_o x \leq b_o + c_o, \ A_f x \leq b_f \); hence

\[
A_o^T z_o + A_f^T z_f = 0,
\]

where

\[
z_o = \text{diag} \left( b_o + c_o - A_o x^{(k)} \right)^{-1} \mathbf{1}, \quad z_f = \text{diag} \left( b_f - A_f x^{(k)} \right)^{-1} \mathbf{1}
\]

- normalizing gives a dual feasible point for the PWL relaxation:

\[
u = \frac{1}{1^T z_o} z_o, \quad u = \frac{1}{1^T z_o} z_f
\]

- \( l^{(k)} = -b_o^T u - b_f^T v \) is a lower bound on optimal value of (1)

from \( x^{(k)} \) we get a readily computed lower bound
Stopping criterion

keep track of best point found, and best lower bound

- best function value so far

\[ f_{\text{best}}^{(k)} = \min_{i=1,\ldots,k} f_0(x^{(i)}) \]

- best lower bound so far

\[ l_{\text{best}}^{(k)} = \max_{i=1,\ldots,k} l^{(i)} \]

can stop when \( f_{\text{best}}^{(k)} - l_{\text{best}}^{(k)} \leq \epsilon \) to guarantee \( \epsilon \)-suboptimality
Example: piecewise linear minimization

\[
\begin{align*}
\text{minimize} & \quad \max_{i=1,\ldots,m} (a_i^T x + b_i) \\
\text{subject to} & \quad -1 \leq x \leq 1
\end{align*}
\]

\(n = 100\) variables, \(m = 200\) terms, \(f^* \approx 0.36\)
computed lower bound on optimal value

\[ f_{\text{best}}^{(k)} - l^{(k)} \] (dashed line) and \[ f_{\text{best}}^{(k)} - f^* \] (solid line)
ACCPM with constraint dropping

same problem; convergence with and without pruning (to $3n$ constraints)

![Graph showing convergence with and without pruning](image-url)

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References


  §8.2.3 gives a convergence proof of ACCPM (the bound on the number of iterations is $n^2$ times a function of $R/r$)

• S. Boyd, course notes for EE364b, *Convex Optimization II*