14. Generalized proximal gradient method

- proximal gradient method with Bregman distance
- accelerated proximal gradient method
Generalized proximal gradient method

- we extend the proximal gradient method of lecture 4 to Bregman distances
- the method applies to convex optimization problems with differentiable term $g$:

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

Algorithm: start at $x_0 \in \text{dom } f \cap \text{int}(\text{dom } \phi)$ and repeat

$$x_{k+1} = \arg\min_x \left( g(x_k) + \nabla g(x_k)^T (x - x_k) + h(x) + \frac{1}{t_k}d(x, x_k) \right)$$

$$= \text{prox}_{t_k h}(x_k, t_k \nabla g(x_k))$$

$t_k$ is a positive step size, fixed or selected by line search
Assumptions

\[
\text{minimize } f(x) = g(x) + h(x)
\]

- \( h \) is convex and \( \text{prox}_t^d \) is well defined: for every \( x \in \text{int} (\text{dom} \phi) \) and every \( a \),

\[
\text{minimize } h(u) + a^T u + \frac{1}{t} d(u, x)
\]

has a unique solution \( \text{prox}_t^d(x, ta) \in \text{int} (\text{dom} \phi) \)

- \( g \) is convex and differentiable with \( \text{dom} \phi \subseteq \text{dom} g \)

- the function \( L\phi - g \) is convex, for some \( L > 0 \); equivalently,

\[
\begin{aligned}
g(x) &\leq g(y) + \nabla g(y)^T (x - y) + Ld(x, y) \\
\text{for all } (x, y) &\in \text{dom} d
\end{aligned}
\]

(1)

this is sometimes called relative smoothness

- the optimal value \( f^* \) is finite and attained at \( x^* \in \text{dom} \phi \)
Consequence of relative smoothness

- the following inequality holds if $0 < t_k \leq 1/L$:

$$g(x_{k+1}) \leq g(x_k) + \nabla g(x_k)^T (x_{k+1} - x_k) + \frac{1}{t_k} d(x_{k+1}, x_k) \quad (2)$$

- if this inequality holds, then for all $x \in \text{dom } f \cap \text{dom } \phi$,

$$f(x_{k+1}) \leq g(x_k) + \nabla g(x_k)^T (x_{k+1} - x_k) + h(x_{k+1}) + \frac{1}{t_k} d(x_{k+1}, x_k)$$

$$\leq g(x_k) + \nabla g(x_k)^T (x - x_k) + h(x) + \frac{1}{t_k} (d(x, x_k) - d(x, x_{k+1}))$$

$$\leq f(x) + \frac{1}{t_k} (d(x, x_k) - d(x, x_{k+1})) \quad (3)$$

2nd line is optimality condition for $\text{prox}_{t_k h}$ on p.13.21; 3rd line is convexity of $g$
Descent properties

• substituting $x = x_k$ in (3) shows that

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{t_k}d(x_k, x_{k+1})$$

$$\leq f(x_k)$$

strict inequality holds if $x_k \neq x_{k+1}$ and the kernel $\phi$ is strictly convex

• substituting $x = x^*$ in (3) shows that

$$d(x^*, x_{k+1}) - d(x^*, x_k) \leq t_k(f^* - f(x_{k+1}))$$

$$\leq 0 \quad (4)$$

Generalized proximal gradient method
Convergence of function values

suppose (2) holds at every iteration

\[
\left( \sum_{i=0}^{k-1} t_i (f(x_k) - f^*) \right) \leq \sum_{i=1}^{k} t_{i-1} (f(x_i) - f^*)
\]

\[
\leq \sum_{i=1}^{k} (d(x^*, x_{i-1}) - d(x^*, x_i))
\]

\[
= d(x^*, x_0) - d(x^*, x_k)
\]

\[
\leq d(x^*, x_0)
\]

• first inequality holds because function values \( f(x_i) \) are non-increasing

• second inequality is (4)

this shows that

\[
f(x_k) - f^* \leq \frac{d(x^*, x_0)}{\sum_{i=0}^{k-1} t_i}
\]
Step size selection

**Fixed step size:** for \( t_i = 1/L \), the upper bound on the previous page is

\[
f(x_k) - f^* \leq \frac{Ld(x^*, x_0)}{k}
\]

**Line search:** start at \( t_k = \hat{t} \) and backtrack \( (t_k := \beta t_k, \text{ with } \beta \in (0, 1)) \) until (2) holds

- since (2) holds for \( t_k \leq 1/L \), the selected step size satisfies

\[
t_k \geq t_{\text{min}} = \min\{\hat{t}, \beta/L\}
\]

- the upper bound on the previous page implies that

\[
f(x_k) - f^* \leq \frac{d(x^*, x_0)}{kt_{\text{min}}}
\]
Outline

- proximal gradient method with Bregman distance
- accelerated proximal gradient method
we discuss a Bregman distance variant of FISTA (p. 7.8) for the problem on p. 14.2

Algorithm: start at $x_0 = v_0 \in \text{dom } f \cap \text{int(dom } \phi\text{)}$, and repeat for $k = 0, 1, \ldots$:

\[
y_{k+1} = x_k + \theta_k (v_k - x_k)
\]

\[
v_{k+1} = \arg\min_v (h(v) + \nabla g(y_{k+1})^T v + \frac{1}{\tau_k} d(v, v_k))
\]

\[
x_{k+1} = x_k + \theta_k (v_{k+1} - x_k)
\]

• step 2 can be written as $v_{k+1} = \text{prox}_{\frac{d}{\tau_k} h} (v_k, \tau_k \nabla g(y_{k+1}))$

• choice of parameters $\theta_k \in (0, 1]$, $\tau_k > 0$ will be discussed on page 14.16

• known as the improved interior gradient algorithm (Auslender & Teboulle, 2006)

• Bregman extension of a gradient projection method by Nesterov (1988)
Feasibility of the iterates

step 2 requires that $\nabla g(y_{k+1})$ exists and that $v_k \in \text{int}(\text{dom } \phi)$

\[
y_{k+1} = \theta_k v_k + (1 - \theta_k)x_k
\]

\[
v_{k+1} = \arg\min_v (h(v) + \nabla g(y_{k+1})^Tv + \frac{1}{\tau_k}d(v, v_k))
\]

\[
x_{k+1} = \theta_k v_{k+1} + (1 - \theta_k)x_k
\]

suppose $x_0 = v_0 \in \text{dom } f \cap \text{int}(\text{dom } \phi)$ and $\text{dom } \phi \subseteq \text{dom } g$

- step 1: $y_{k+1}$ is a convex combination of $v_k$ and $x_k$
- step 2: $v_{k+1} \in \text{dom } h \cap \text{int}(\text{dom } \phi)$, by assumption that $\text{prox}^{d}_{\tau_k h}$ is well defined
- step 3: $x_{k+1}$ is a convex combination of $v_{k+1}$ and $x_k$

hence, the sequences $y_k, v_k, x_k$ remain in $\text{dom } f \cap \text{int}(\text{dom } \phi)$
Quadratic kernel

for the quadratic distance $d(x, y) = \frac{1}{2} \| x - y \|_2^2$ the algorithm can be written as

$$
\begin{align*}
    y_{k+1} &= x_k + \theta_k (v_k - x_k) \\
    v_{k+1} &= \text{prox}_{\tau_k h}(v_k - \tau_k \nabla g(y_{k+1})) \\
    x_{k+1} &= x_k + \theta_k (v_{k+1} - x_k)
\end{align*}
$$

• compare with FISTA (page 7.8): same $y$-update, different $x$, $v$-updates

$$
\begin{align*}
    y_{k+1} &= x_k + \theta_k (v_k - x_k) \\
    x_{k+1} &= \text{prox}_{\tau_k h}(y_{k+1} - t_k \nabla g(y_{k+1})) \\
    v_{k+1} &= x_k + \frac{1}{\theta_k} (x_{k+1} - x_k)
\end{align*}
$$

• if $h = 0$ and $t_k = \theta_k \tau_k$, the two methods are equivalent

• if $h \neq 0$, points $v_k$, $y_k$ in FISTA may be outside $\text{dom} \ h$ (in contrast to 1st method)
Assumptions

\[ \text{minimize } f(x) = g(x) + h(x) \]

we make the same assumptions as on page 14.3 with one difference

- \( \nabla g \) is \( L \)-Lipschitz continuous for some norm \( \| \cdot \| \):

\[
g(x) \leq g(y) + \nabla g(y)^T (x - y) + \frac{L}{2} \| x - y \|^2 \quad \text{for all } x, y \in \text{dom } g
\]

- the Bregman kernel \( \phi \) is 1-strongly convex with respect to the same norm:

\[
d(x, y) \geq \frac{1}{2} \| x - y \|^2 \quad \text{for all } (x, y) \in \text{dom } d
\]

these two assumptions replace the relative smoothness assumption on page 14.3:

\[
g(x) \leq g(y) + \nabla g(y)^T (x - y) + Ld(x, y)
\]
Consequence of Lipschitz continuity of gradient

- the following inequality holds if $0 < \tau_k \leq 1/(L\theta_k)$:

$$g(x_{k+1}) \leq (1 - \theta_k)g(x_k) + \theta_k \left( g(y_{k+1}) + \nabla g(y_{k+1})^T(v_{k+1} - y_{k+1}) + \frac{1}{\tau_k}d(v_{k+1}, v_k) \right)$$

- if this inequality holds, then for all $x \in \text{dom } f \cap \text{dom } \phi$,

$$\frac{\tau_k}{\theta_k} (f(x_{k+1}) - f(x)) + d(x, v_{k+1}) \leq \frac{\tau_k(1 - \theta_k)}{\theta_k} (f(x_k) - f(x)) + d(x, v_k)$$

(proofs on next pages)
Proof: we show that the inequality (5) holds for \( \tau_k = 1/(L\theta_k) \)

- we use notation \( x^+ = x_{k+1}, \ x = x_k, \ v^+ = v_{k+1}, \ v = v_k, \ y = y_{k+1}, \ \theta = \theta_k \)
- from the Lipschitz continuity of \( \nabla g \):

  \[
g(x^+) \leq g(y) + \nabla g(y)^T(x^+ - y) + \frac{L}{2} ||x^+ - y||^2
  \]

- from steps 1 and 2 in the algorithm, \( \theta(v^+ - v) = x^+ - y \):

  \[
g(x^+) \leq g(y) + \nabla g(y)^T(x^+ - y) + \frac{L\theta^2}{2} ||v^+ - v||^2
  \]

- from strong convexity of the Bregman kernel:

  \[
g(x^+) \leq g(y) + \nabla g(y)^T(x^+ - y) + L\theta^2 d(v^+, v)
  \]

- from step 3 in the algorithm, \( x^+ = (1 - \theta)x + \theta v^+ \):

  \[
g(x^+) = g(y) + (1 - \theta)\nabla g(y)^T(x - y) + \theta \nabla g(y)^T(v^+ - y) + L\theta^2 d(v^+, v)
  \]

- inequality (5) now follows from \( g(y) + \nabla g(y)^T(x - y) \leq g(x) \) (convexity of \( g \))
Proof: we show that (5) implies that (6) holds for all \( x \in \text{dom } f \cap \text{dom } \phi \)

- the optimality condition for the \text{prox} evaluation in step 2 of the algorithm is

\[
h(v_{k+1}) \leq h(x) + \nabla g(y_{k+1})^T (x - v_{k+1}) + \frac{1}{\tau_k} (d(x, v_k) - d(x, v_{k+1}) - d(v_{k+1}, v_k))
\]

- from Jensen’s inequality and \( x_{k+1} = (1 - \theta_k)x_k + \theta_k v_{k+1} \):

\[
h(x_{k+1}) \leq (1 - \theta_k)h(x_k)
\]

\[
+ \theta_k \left( h(x) + \nabla g(y_{k+1})^T (x - v_{k+1}) + \frac{1}{\tau_k} (d(x, v_k) - d(x, v_{k+1}) - d(v_{k+1}, v_k)) \right)
\]

- combine this with (5):

\[
f(x_{k+1}) \leq (1 - \theta_k)f(x_k)
\]

\[
+ \theta_k \left( f(x) + g(y_{k+1}) + \nabla g(y_{k+1})^T (x - y_{k+1}) + \frac{1}{\tau_k}(d(x, v_k) - d(x, v_{k+1})) \right)
\]

- from convexity of \( g \):

\[
f(x_{k+1}) \leq (1 - \theta_k)f(x_k) + \theta_k \left( f(x) + \frac{1}{\tau_k}(d(x, v_k) - d(x, v_{k+1})) \right)
\]
Parameter selection

- the parameters $\theta_k \in (0, 1]$, $\tau_k > 0$ will be chosen to satisfy (5) and

$$\theta_0 = 1, \quad \frac{\tau_k(1 - \theta_k)}{\theta_k} \leq \frac{\tau_{k-1}}{\theta_{k-1}} \quad \text{for } k \geq 1 \quad (7)$$

- this allows us to combine the inequalities (6) at $x = x^*$ recursively to obtain

$$\frac{\tau_{k-1}}{\theta_{k-1}}(f(x_k) - f(x^*)) + d(x^*, v_k) \leq \frac{\tau_0}{\theta_0}(f(x_1) - f(x^*)) + d(x^*, v_1)$$

$$\leq \frac{\tau_0(1 - \theta_0)}{\theta_0}(f(x_0) - f(x^*)) + d(x^*, v_0)$$

$$= d(x^*, x_0)$$

hence,

$$f(x_k) - f^* \leq \frac{\theta_{k-1}}{\tau_{k-1}}d(x^*, x_0) \quad (8)$$
Fixed step size

if $L$ is known, we choose $\tau_k = 1/(L\theta_k)$ and $\theta_k$ that satisfies

$$\theta_0 = 1, \quad \frac{\theta_k^2}{1 - \theta_k} \geq \theta_{k-1}^2 \quad \text{for } k \geq 1$$

- a simple choice is $\theta_k = 2/(k + 2)$

- alternatively, find the smallest allowable $\theta_k$ by solving $\theta_k^2/(1 - \theta_k) = \theta_{k-1}^2$:

$$\theta_0 = 1, \quad \theta_k = \frac{-\theta_{k-1}^2 + \sqrt{\theta_{k-1}^4 + 4\theta_{k-1}^2}}{2}, \quad k \geq 1$$

with these choices the bound (8) implies $1/k^2$ convergence:

$$f(x_k) - f^* \leq \frac{4L}{(k + 1)^2}d(x^*, x_0)$$
Variable step size

if $L$ is unknown, we take $\tau_k = t_k/\theta_k$, where $t_k$ is estimate of $1/L$, and solve $\theta_k$ from

$$\theta_0 = 1, \quad \frac{t_k(1 - \theta_k)}{\theta_k^2} = \frac{t_{k-1}}{\theta_{k-1}^2} \quad \text{for } k \geq 1$$

- to find $t_k$, we start at $t_k = \hat{t}_k$ and backtrack ($t_k := \beta t_k$) until (5) holds
- for each tentative $t_k$, we need to recompute $y_{k+1}, v_{k+1}, x_{k+1}$ to evaluate (5)
- since (5) holds for $\tau_k \leq 1/(L\theta_k)$, the selected $t_k$ satisfies $t_k \geq \min \{\hat{t}_k, \beta/L\}$
- it was shown in lecture 7, equation (3), that

$$\frac{\theta_{k-1}^2}{t_{k-1}} = \frac{1}{t_0} \prod_{i=1}^{k-1} (1 - \theta_i) \leq \frac{4}{(2\sqrt{t_0} + \sum_{i=1}^{k-1} \sqrt{t_i})^2}$$

- if $t_{\text{min}} = \min \{\min_i \hat{t}_i, \beta/L\} > 0$, the bound (8) shows $1/k^2$ convergence:

$$f(x_k) - f^* \leq \frac{4/t_{\text{min}}}{(k + 1)^2} d(x^*, x_0)$$

Generalized proximal gradient method
Example

**Primal problem** (variable \( x \in \mathbb{R}^n \))

\[
\text{minimize } \quad f(x) + \lambda_{\text{max}}(A(x) + B)
\]

- \( f \) is strongly convex
- \( A \) maps \( n \)-vector \( x \) to \( m \times m \) symmetric matrix \( A(x) = x_1A_1 + \cdots + x_nA_n \)
- coefficient matrices \( A_1, \ldots, A_n, B \) are symmetric \( m \times m \) matrices

**Dual problem** (variable \( X \in S^m \))

\[
\text{maximize } \quad \text{tr}(BX) - f^*(-A^\text{adj}(X))
\]
\[
\text{subject to } \quad \text{tr}(X) = 1, \quad X \geq 0
\]

\( A^\text{adj} \) maps symmetric matrix \( X \) to \( n \)-vector \( A^\text{adj}(X) = (\text{tr}(A_1X), \ldots, \text{tr}(A_nX)) \)
Bregman proximal mapping

we’ll apply the generalized proximal gradient method to the dual problem

- kernel is matrix entropy (p.13.11): \( \phi(X) = \text{tr}(X \log X) \) with \( \text{dom} \phi = S_+^m \),

\[
d(X, Y) = \text{tr}(X \log X - X \log Y - X + Y)
\]

- proximal mapping of indicator \( \delta_C \) of the set \( C = \{X \succeq 0 \mid \text{tr}(X) = 1\} \) is

\[
\arg\min_{\text{tr}(X)=1, X \succeq 0} (\text{tr}(AX) + d(X, Y)) = \frac{\exp(-A + \log Y)}{\text{tr}(\exp(-A + \log Y))}
\]

exponential and logarithm of symmetric matrix are defined as

\[
\log U = \sum_i (\log \lambda_i) q_i q_i^T, \quad \exp U = \sum_i (\exp \lambda_i) q_i q_i^T
\]

if \( U \) has eigenvalue decomposition \( U = \sum_i \lambda_i q_i q_i^T \)
Example

minimize \[ \frac{1}{2} \| x \|^2 + \lambda_{\max}(\mathcal{A}(x) + B) \]
maximize \[ \text{tr}(BX) - \frac{1}{2} \| \mathcal{A}^{\text{adj}}(X) \|^2 \]
subject to \[ \text{tr}(X) = 1, \quad X \succeq 0 \]

- randomly generated data with \( m = 200, \ n = 100 \)
- basic and accelerated method, with the same, fixed step size
References