18. Generalized proximal gradient method

- proximal gradient method with Bregman distance
- accelerated proximal gradient method

Generalized proximal gradient method

- we extend the proximal gradient method of lecture 4 to Bregman distances
- the method applies to convex optimization problems with differentiable term g:

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

Algorithm: start at $x_0 \in \mathbf{dom}\, f \cap \mathbf{int}(\mathbf{dom}\, \phi)$ and repeat

$$\begin{aligned} x_{k+1} &= \underset{x}{\operatorname{argmin}} \left(g(x_k) + \nabla g(x_k)^T (x - x_k) + h(x) + \frac{1}{t_k} d(x, x_k) \right) \\ &= \operatorname{prox}^d_{t_k h}(x_k,\ t_k \nabla g(x_k)) \end{aligned}$$

t_k is a positive step size, fixed or selected by line search

Assumptions

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

- h is convex and prox^d_{th} is well defined: for every $x \in \text{int}\,(\text{dom}\,\phi)$ and every a,

$$\text{minimize} \quad h(u) + a^T u + \frac{1}{t} d(u, x)$$

  has a unique solution $\text{prox}^d_{th}(x, ta) \in \text{int}\,(\text{dom}\,\phi)$

- g is convex and differentiable with $\text{dom}\,\phi \subseteq \text{dom}\,g$

- the function $L\phi - g$ is convex, for some $L > 0$; equivalently,

$$g(x) \le g(y) + \nabla g(y)^T (x - y) + L d(x, y) \quad \text{for all } (x, y) \in \text{dom}\,d \qquad (1)$$

  this is sometimes called *relative smoothness*

- the optimal value $f^\star$ is finite and attained at $x^\star \in \text{dom}\,\phi$

Consequence of relative smoothness

- the following inequality holds if $0 < t_k \leq 1/L$:

$$g(x_{k+1}) \leq g(x_k) + \nabla g(x_k)^T (x_{k+1} - x_k) + \frac{1}{t_k} d(x_{k+1}, x_k) \qquad (2)$$

- if this inequality holds, then for all $x \in \mathbf{dom}\, f \cap \mathbf{dom}\, \phi$,

$$\begin{aligned} f(x_{k+1}) &\leq g(x_k) + \nabla g(x_k)^T (x_{k+1} - x_k) + h(x_{k+1}) + \frac{1}{t_k} d(x_{k+1}, x_k) \\ &\leq g(x_k) + \nabla g(x_k)^T (x - x_k) + h(x) + \frac{1}{t_k} (d(x, x_k) - d(x, x_{k+1})) \\ &\leq f(x) + \frac{1}{t_k} (d(x, x_k) - d(x, x_{k+1})) \qquad (3) \end{aligned}$$

2nd line is optimality condition for $\mathrm{prox}^d_{t_k h}$ on p.17.21; 3rd line is convexity of g

Descent properties

- substituting $x = x_k$ in (3) shows that

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) - \frac{1}{t_k} d(x_k, x_{k+1}) \\ &\leq f(x_k) \end{aligned}$$

  strict inequality holds if $x_k \neq x_{k+1}$ and the kernel ϕ is strictly convex

- substituting $x = x^\star$ in (3) shows that

$$\begin{aligned} d(x^\star, x_{k+1}) - d(x^\star, x_k) &\leq t_k(f^\star - f(x_{k+1})) \\ &\leq 0 \end{aligned} \tag{4}$$

Convergence of function values

suppose (2) holds at every iteration

$$
\begin{aligned}
(\sum_{i=0}^{k-1} t_i)(f(x_k) - f^\star) &\leq \sum_{i=1}^{k} t_{i-1}(f(x_i) - f^\star) \\
&\leq \sum_{i=1}^{k} \big(d(x^\star, x_{i-1}) - d(x^\star, x_i)\big) \\
&= d(x^\star, x_0) - d(x^\star, x_k) \\
&\leq d(x^\star, x_0)
\end{aligned}
$$

- first inequality holds because function values $f(x_i)$ are non-increasing
- second inequality is (4)

this shows that

$$
f(x_k) - f^\star \leq \frac{d(x^\star, x_0)}{\sum_{i=0}^{k-1} t_i}
$$

Step size selection

Fixed step size: for $t_i = 1/L$, the upper bound on the previous page is

$$f(x_k) - f^\star \leq \frac{L d(x^\star, x_0)}{k}$$

Line search: start at $t_k = \hat{t}$, backtrack ($t_k := \beta t_k$, with $\beta \in (0, 1)$) until (2) holds

- since (2) holds for $t_k \leq 1/L$, the selected step size satisfies

$$t_k \geq t_{\min} = \min\{\hat{t}, \beta/L\}$$

- the upper bound on the previous page implies that

$$f(x_k) - f^\star \leq \frac{d(x^\star, x_0)}{k t_{\min}}$$

Outline

- proximal gradient method with Bregman distance
- **accelerated proximal gradient method**

Accelerated proximal gradient method

we discuss a Bregman distance variant of FISTA (p. 7.8) for the problem on p. 18.2

Algorithm: start at $x_0 = v_0 \in \operatorname{dom} f \cap \operatorname{int}(\operatorname{dom} \phi)$, and repeat for $k = 0, 1, \ldots$:

$$\begin{aligned} y_{k+1} &= x_k + \theta_k (v_k - x_k) \\ v_{k+1} &= \underset{v}{\operatorname{argmin}} \left(h(v) + \nabla g(y_{k+1})^T v + \frac{1}{\tau_k} d(v, v_k) \right) \\ x_{k+1} &= x_k + \theta_k (v_{k+1} - x_k) \end{aligned}$$

- step 2 can be written as $v_{k+1} = \operatorname{prox}^d_{\tau_k h}(v_k, \tau_k \nabla g(y_{k+1}))$
- choice of parameters $\theta_k \in (0, 1]$, $\tau_k > 0$ will be discussed on page 18.16
- known as the *improved interior gradient algorithm* (Auslender & Teboulle, 2006)
- Bregman extension of a gradient projection method by Nesterov (1988)

Feasibility of the iterates

step 2 requires that $\nabla g(y_{k+1})$ exists and that $v_k \in \mathrm{int}(\mathrm{dom}\,\phi)$

$$\begin{aligned} y_{k+1} &= \theta_k v_k + (1-\theta_k)x_k \\ v_{k+1} &= \operatorname*{argmin}_v \left(h(v) + \nabla g(y_{k+1})^T v + \frac{1}{\tau_k} d(v, v_k)\right) \\ x_{k+1} &= \theta_k v_{k+1} + (1-\theta_k)x_k \end{aligned}$$

suppose $x_0 = v_0 \in \mathrm{dom}\, f \cap \mathrm{int}(\mathrm{dom}\,\phi)$ and $\mathrm{dom}\,\phi \subseteq \mathrm{dom}\, g$

- step 1: y_{k+1} is a convex combination of v_k and x_k
- step 2: $v_{k+1} \in \mathrm{dom}\, h \cap \mathrm{int}(\mathrm{dom}\,\phi)$, by assumption that $\mathrm{prox}^d_{\tau_k h}$ is well defined
- step 3: x_{k+1} is a convex combination of v_{k+1} and x_k

hence, the sequences y_k, v_k, x_k remain in $\mathrm{dom}\, f \cap \mathrm{int}(\mathrm{dom}\,\phi)$

Quadratic kernel

for the quadratic distance $d(x, y) = \frac{1}{2}\|x - y\|_2^2$ the algorithm can be written as

$$y_{k+1} = x_k + \theta_k(v_k - x_k) \tag{5a}$$

$$v_{k+1} = \text{prox}_{\tau_k h}(v_k - \tau_k \nabla g(y_{k+1})) \tag{5b}$$

$$x_{k+1} = x_k + \theta_k(v_{k+1} - x_k) \tag{5c}$$

- compare with FISTA (page 7.8): same y-update, different x-, v-updates

$$y_{k+1} = x_k + \theta_k(v_k - x_k) \tag{6a}$$

$$x_{k+1} = \text{prox}_{t_k h}(y_{k+1} - t_k \nabla g(y_{k+1})) \tag{6b}$$

$$v_{k+1} = x_k + \frac{1}{\theta_k}(x_{k+1} - x_k) \tag{6c}$$

- if $h = 0$ and $t_k = \theta_k \tau_k$, the two methods are equivalent
- if $h \neq 0$, points v_k, y_k in (6) may be outside $\text{dom}\, h$ (in contrast to method (5))

Assumptions

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

we make the same assumptions as on page 18.3 with one difference

- ∇g is L-Lipschitz continuous for some norm $\|\cdot\|$:

$$g(x) \leq g(y) + \nabla g(y)^T(x-y) + \frac{L}{2}\|x-y\|^2 \quad \text{for all } x, y \in \operatorname{dom} g$$

- the Bregman kernel ϕ is 1-strongly convex with respect to the same norm:

$$d(x,y) \geq \frac{1}{2}\|x-y\|^2 \quad \text{for all } (x,y) \in \operatorname{dom} d$$

these two assumptions replace the relative smoothness assumption on page 18.3:

$$g(x) \leq g(y) + \nabla g(y)^T(x-y) + L d(x,y)$$

Consequence of Lipschitz continuity of gradient

- the following inequality holds if $0 < \tau_k \leq 1/(L\theta_k)$:

$$\begin{aligned} g(x_{k+1}) &\leq (1-\theta_k)g(x_k) \\ &\quad + \theta_k \left(g(y_{k+1}) + \nabla g(y_{k+1})^T (v_{k+1} - y_{k+1}) + \frac{1}{\tau_k} d(v_{k+1}, v_k) \right) \end{aligned} \quad (7)$$

- if this inequality holds, then for all $x \in \mathbf{dom}\, f \cap \mathbf{dom}\, \phi$,

$$\begin{aligned} & \frac{\tau_k}{\theta_k}(f(x_{k+1}) - f(x)) + d(x, v_{k+1}) \\ & \quad \leq \quad \frac{\tau_k(1-\theta_k)}{\theta_k}(f(x_k) - f(x)) + d(x, v_k) \end{aligned} \quad (8)$$

(proofs on next pages)

Proof: we show that the inequality (7) holds for $\tau_k = 1/(L\theta_k)$

- we use notation $x^+ = x_{k+1}, \;\; x = x_k, \;\; v^+ = v_{k+1}, \;\; v = v_k, \;\; y = y_{k+1}, \;\; \theta = \theta_k$
- from the Lipschitz continuity of ∇g:

$$g(x^+) \leq g(y) + \nabla g(y)^T (x^+ - y) + \frac{L}{2}\|x^+ - y\|^2$$

- from steps 1 and 2 in the algorithm, $\theta(v^+ - v) = x^+ - y$:

$$g(x^+) \leq g(y) + \nabla g(y)^T (x^+ - y) + \frac{L\theta^2}{2}\|v^+ - v\|^2$$

- from strong convexity of the Bregman kernel:

$$g(x^+) \leq g(y) + \nabla g(y)^T (x^+ - y) + L\theta^2 d(v^+, v)$$

- from step 3 in the algorithm, $x^+ = (1-\theta)x + \theta v^+$:

$$g(x^+) \leq g(y) + (1-\theta)\nabla g(y)^T (x - y) + \theta \nabla g(y)^T (v^+ - y) + L\theta^2 d(v^+, v)$$

- inequality (7) now follows from $g(y) + \nabla g(y)^T (x - y) \leq g(x)$ (convexity of g)

Proof: we show that (7) implies that (8) holds for all $x \in \operatorname{dom} f \cap \operatorname{dom} \phi$

- the optimality condition for the prox evaluation in step 2 of the algorithm is

$$h(v_{k+1}) \leq h(x) + \nabla g(y_{k+1})^T (x - v_{k+1}) + \frac{1}{\tau_k} \left(d(x, v_k) - d(x, v_{k+1}) - d(v_{k+1}, v_k) \right)$$

- from Jensen's inequality and $x_{k+1} = (1 - \theta_k) x_k + \theta_k v_{k+1}$:

$$\begin{aligned} h(x_{k+1}) &\leq (1 - \theta_k) h(x_k) \\ &\quad + \theta_k \left(h(x) + \nabla g(y_{k+1})^T (x - v_{k+1}) + \frac{1}{\tau_k} \left(d(x, v_k) - d(x, v_{k+1}) - d(v_{k+1}, v_k) \right) \right) \end{aligned}$$

- combine this with (7):

$$\begin{aligned} f(x_{k+1}) &\leq (1 - \theta_k) f(x_k) \\ &\quad + \theta_k \left(h(x) + g(y_{k+1}) + \nabla g(y_{k+1})^T (x - y_{k+1}) + \frac{1}{\tau_k} \left(d(x, v_k) - d(x, v_{k+1}) \right) \right) \end{aligned}$$

- from convexity of g:

$$f(x_{k+1}) \leq (1 - \theta_k) f(x_k) + \theta_k \left(f(x) + \frac{1}{\tau_k} \left(d(x, v_k) - d(x, v_{k+1}) \right) \right)$$

Parameter selection

- the parameters $\theta_k \in (0,1]$, $\tau_k > 0$ will be chosen to satisfy (7) and

$$\theta_0 = 1, \qquad \frac{\tau_k(1-\theta_k)}{\theta_k} \le \frac{\tau_{k-1}}{\theta_{k-1}} \quad \text{for } k \ge 1 \tag{9}$$

- this allows us to combine the inequalities (8) at $x = x^\star$ recursively to obtain

$$\begin{aligned}
\frac{\tau_{k-1}}{\theta_{k-1}}(f(x_k) - f(x^\star)) + d(x^\star, v_k) &\le \frac{\tau_0}{\theta_0}(f(x_1) - f(x^\star)) + d(x^\star, v_1) \\
&\le \frac{\tau_0(1-\theta_0)}{\theta_0}(f(x_0) - f(x^\star)) + d(x^\star, v_0) \\
&= d(x^\star, x_0))
\end{aligned}$$

hence,

$$f(x_k) - f^\star \le \frac{\theta_{k-1}}{\tau_{k-1}} d(x^\star, x_0) \tag{10}$$

Fixed step size

if L is known, we choose $\tau_k = 1/(L\theta_k)$ and θ_k that satisfies

$$\theta_0 = 1, \qquad \frac{\theta_k^2}{1-\theta_k} \geq \theta_{k-1}^2 \quad \text{for } k \geq 1$$

- a simple choice is $\theta_k = 2/(k+2)$
- alternatively, find the smallest allowable θ_k by solving $\theta_k^2/(1-\theta_k) = \theta_{k-1}^2$:

$$\theta_0 = 1, \qquad \theta_k = \frac{-\theta_{k-1}^2 + \sqrt{\theta_{k-1}^4 + 4\theta_{k-1}^2}}{2}, \quad k \geq 1$$

with these choices the bound (10) implies $1/k^2$ convergence:

$$f(x_k) - f^\star \leq \frac{4L}{(k+1)^2} d(x^\star, x_0)$$

Variable step size

if L is unknown, we take $\tau_k = t_k/\theta_k$, where t_k is estimate of $1/L$, and solve θ_k from

$$\theta_0 = 1, \qquad \frac{t_k(1-\theta_k)}{\theta_k^2} = \frac{t_{k-1}}{\theta_{k-1}^2} \quad \text{for } k \geq 1$$

- to find t_k, we start at $t_k = \hat{t}_k$ and backtrack ($t_k := \beta t_k$) until (7) holds
- for each tentative t_k, we need to recompute $y_{k+1}, v_{k+1}, x_{k+1}$ to evaluate (7)
- since (7) holds for $\tau_k \leq 1/(L\theta_k)$, the selected t_k satisfies $t_k \geq \min\{\hat{t}_k, \beta/L\}$
- it was shown in lecture 7, equation (3), that

$$\frac{\theta_{k-1}^2}{t_{k-1}} = \frac{1}{t_0}\prod_{i=1}^{k-1}(1-\theta_i) \leq \frac{4}{(2\sqrt{t_0} + \sum_{i=1}^{k-1}\sqrt{t_i})^2}$$

- if $t_{\min} = \min\{\min_i \hat{t}_i, \beta/L\} > 0$, the bound (10) shows $1/k^2$ convergence:

$$f(x_k) - f^\star \leq \frac{4/t_{\min}}{(k+1)^2} d(x^\star, x_0)$$

Example

Primal problem (variable $x \in \mathbf{R}^n$)

$$\text{minimize} \quad f(x) + \lambda_{\max}(\mathcal{A}(x) + B)$$

- f is strongly convex
- $\mathcal{A}$ maps n-vector x to $m \times m$ symmetric matrix $\mathcal{A}(x) = x_1 A_1 + \cdots + x_n A_n$
- coefficient matrices $A_1, \ldots, A_n, B$ are symmetric $m \times m$ matrices

Dual problem (variable $X \in \mathbf{S}^m$)

$$\begin{array}{ll} \text{maximize} & \operatorname{tr}(BX) - f^*(-\mathcal{A}^{\text{adj}}(X)) \\ \text{subject to} & \operatorname{tr}(X) = 1 \\ & X \succeq 0 \end{array}$$

$\mathcal{A}^{\text{adj}}$ maps symmetric matrix X to n-vector $\mathcal{A}^{\text{adj}}(X) = (\operatorname{tr}(A_1 X), \ldots, \operatorname{tr}(A_n X))$

Bregman proximal mapping

we'll apply the generalized proximal gradient method to the dual problem

- kernel is matrix entropy (page 17.11): $\phi(X) = \mathrm{tr}(X \log X)$ with $\mathrm{dom}\,\phi = \mathbf{S}_+^m$,

$$d(X, Y) = \mathrm{tr}(X \log X - X \log Y - X + Y)$$

- proximal mapping of indicator δ_H of the set $H = \{X \mid \mathrm{tr}(X) = 1\}$ is

$$\underset{\mathrm{tr}(X)=1}{\mathrm{argmin}}\, (\mathrm{tr}(AX) + d(X, Y)) = \frac{\exp(-A + \log Y)}{\mathrm{tr}(\exp(-A + \log Y))}$$

  exponential and logarithm of symmetric matrix are defined as

$$\log U = \sum_i (\log \lambda_i) q_i q_i^T, \qquad \exp U = \sum_i (\exp \lambda_i) q_i q_i^T$$

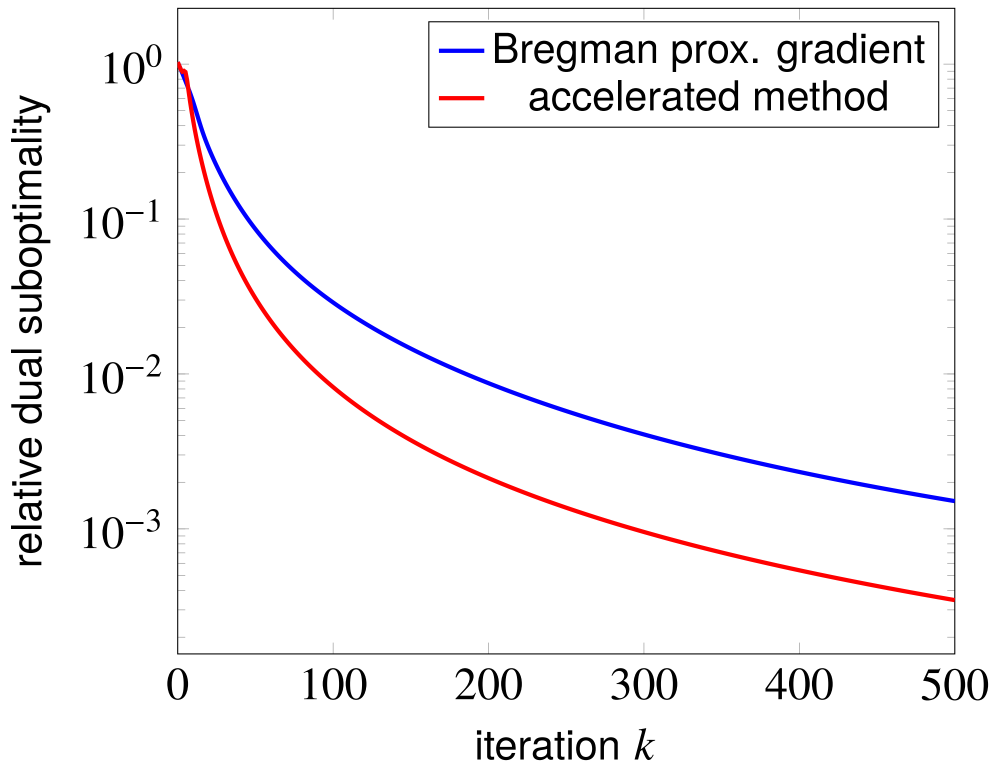
  where $U = \sum_i \lambda_i q_i q_i^T$ is eigendecomposition of U

Example

minimize $\quad \frac{1}{2}\|x\|_2^2 + \lambda_{\max}(\mathcal{A}(x) + B)$

maximize $\quad \mathrm{tr}(BX) - \frac{1}{2}\|\mathcal{A}^{\mathrm{adj}}(X)\|_2^2$
subject to $\quad \mathrm{tr}(X) = 1, \quad X \succeq 0$

- randomly generated data with $m = 200$, $n = 100$
- basic and accelerated method, with the same, fixed step size



References

- A. Auslender and M. Teboulle, *Interior gradient and proximal methods for convex and cone optimization*, SIAM J. Optim. (2006).
- P. Tseng, *On accelerated proximal gradient methods for convex-concave optimization* (2008). The algorithm on page 18.8 is Algorithm 1 in Tseng's paper.