14. Conic optimization

- conic linear program
- examples
- modeling
- duality
Generalized (conic) inequalities

**conic inequality:** a constraint $x \in K$ where $K$ is a convex cone in $\mathbb{R}^m$

we will require that the cone $K$ is **proper:**

- closed
- pointed: $K \cap (-K) = \{0\}$
- with nonempty interior: $\text{int} K \neq \emptyset$; equivalently, $K + (-K) = \mathbb{R}^m$

**notation** (for proper $K$)

\[
\begin{align*}
x \succeq_K y & \iff x - y \in K \\
x \succ_K y & \iff x - y \in \text{int} K
\end{align*}
\]
Inequality notation

we will use a different convention than in EE236B

**vector inequalities:** for \( x, y \in \mathbb{R}^m \)

- \( x > y, \ x \geq y \) denote componentwise inequality
- \( x \succ y, \ x \succeq y \) denote conic inequality for generic (unspecified) cone \( K \)
- \( x \succ_{K} y, \ x \succeq_{K} y \) denote conic inequality for specific \( K \)

**matrix inequality:** for \( X, Y \in \mathbb{S}^p \)

\( X \succ Y, \ X \succeq Y \) mean \( X - Y \) is positive (semi-)definite
Properties of conic inequalities

preserved by nonnegative linear combinations: if \( x \preceq y \) and \( u \preceq v \), then

\[
\alpha x + \beta u \preceq \alpha y + \beta v \quad \forall \alpha, \beta \geq 0
\]

define a \textbf{partial order} of vectors

- \( x \preceq x \)
- \( x \preceq y \preceq z \) implies \( x \preceq z \)
- \( x \preceq y \) and \( y \preceq x \) imply \( y = x \)

in general, not a \textit{total} order (requires that \( x \preceq y \) or \( y \preceq x \) for all \( x, y \))
Conic linear program

\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \preceq b \\
               & \quad Fx = g
\end{align*}

- $A \in \mathbb{R}^{m \times n}$, $F \in \mathbb{R}^{p \times n}$; without loss of generality, can assume

$$\text{rank}(F) = p, \quad \text{rank}\left(\begin{bmatrix} A \\ F \end{bmatrix}\right) = n$$

- $K$ is a proper cone in $\mathbb{R}^m$

- for $K = \mathbb{R}_+^m$, reduces to regular linear program (LP)

- by defining $K = K_1 \times \cdots \times K_r$, can represent multiple conic inequalities

$$A_1 x \preceq_{K_1} b_1, \quad \ldots, \quad A_r x \preceq_{K_r} b_r$$
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Norm cones

\[ K = \{(x, y) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|x\| \leq y\} \]

for the Euclidean norm this is the second-order cone (notation: \(Q^m\))
Second-order cone program

minimize \quad c^T x \\
subject to \quad \|B_{k0}x + d_{k0}\|_2 \leq B_{k1}x + d_{k1}, \quad k = 1, \ldots, r

**conic LP formulation:** express constraints as \( Ax \preceq_K b \)

\[
K = Q^{m_1} \times \cdots \times Q^{m_r}, \quad A = \begin{bmatrix} -B_{10} \\ -B_{11} \\ \vdots \\ -B_{r0} \\ -B_{r1} \end{bmatrix}, \quad b = \begin{bmatrix} d_{10} \\ d_{11} \\ \vdots \\ d_{r0} \\ d_{r1} \end{bmatrix}
\]

(assuming \( B_{k0}, d_{k0} \) have \( m_k - 1 \) rows)
Vector notation for symmetric matrices

- vectorized symmetric matrix: for $U \in S^p$

$$\text{vec}(U) = \sqrt{2} \left( \frac{U_{11}}{\sqrt{2}}, U_{21}, \ldots, \frac{U_{p1}}{\sqrt{2}}, U_{32}, \ldots, \frac{U_{p2}}{\sqrt{2}}, \ldots, \frac{U_{pp}}{\sqrt{2}} \right)$$

- inverse operation: for $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ with $n = p(p + 1)/2$

$$\text{mat}(u) = \frac{1}{\sqrt{2}} \begin{bmatrix}
\sqrt{2}u_1 & u_2 & \cdots & u_p \\
 u_2 & \sqrt{2}u_{p+1} & \cdots & u_{2p-1} \\
 \vdots & \vdots & & \vdots \\
 u_p & u_{2p-1} & \cdots & \sqrt{2}u_{p(p+1)/2}
\end{bmatrix}$$

Coefficients $\sqrt{2}$ are added so that standard inner products are preserved:

$$\text{tr}(UV) = \text{vec}(U)^T \text{vec}(V), \quad u^Tv = \text{tr}(\text{mat}(u) \text{mat}(v))$$
Positive semidefinite cone

\[ S^p = \{ \text{vec}(X) \mid X \in S^p_+ \} = \{ x \in \mathbb{R}^{p(p+1)/2} \mid \text{mat}(x) \succeq 0 \} \]

\[ S^2 = \left\{ (x, y, z) \mid \begin{bmatrix} x & y/\sqrt{2} \\ y/\sqrt{2} & z \end{bmatrix} \succeq 0 \right\} \]
Semidefinite program

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x_1 A_{11} + x_2 A_{12} + \cdots + x_n A_{1n} \preceq B_1 \\
& \quad \cdots \\
& \quad x_1 A_{r1} + x_2 A_{r2} + \cdots + x_n A_{rn} \preceq B_r
\end{align*}
\]

with \( A_{ij}, B_i \in S^{p_i} \)

conic LP formulation

\[
K = S^{p_1} \times S^{p_2} \times \cdots \times S^{p_r}
\]

\[
A = \begin{bmatrix}
\text{vec}(A_{11}) & \text{vec}(A_{12}) & \cdots & \text{vec}(A_{1n}) \\
\text{vec}(A_{21}) & \text{vec}(A_{22}) & \cdots & \text{vec}(A_{2n}) \\
\vdots & \vdots & \ddots & \vdots \\
\text{vec}(A_{r1}) & \text{vec}(A_{r2}) & \cdots & \text{vec}(A_{rn})
\end{bmatrix}, \quad b = \begin{bmatrix}
\text{vec}(B_1) \\
\text{vec}(B_2) \\
\vdots \\
\text{vec}(B_r)
\end{bmatrix}
\]
Exponential cone

the epigraph of the perspective of $\exp x$ is a non-proper cone

$$K = \left\{ (x, y, z) \in \mathbb{R}^3 \mid ye^{x/y} \leq z, \ y > 0 \right\}$$

the exponential cone is $K_{\exp} = \text{cl} K = K \cup \{(x, 0, z) \mid x \leq 0, z \geq 0\}$
Geometric program

minimize \( c^T x \)

subject to \( \log \sum_{k=1}^{n_i} \exp(a_{ik}^T x + b_{ik}) \leq 0, \quad i = 1, \ldots, r \)

conic LP formulation

minimize \( c^T x \)

subject to \[
\begin{bmatrix}
  a_{ik}^T x + b_{ik} \\
  1 \\
  z_{ik}
\end{bmatrix} \in K_{\exp}, \quad k = 1, \ldots, n_i, \quad i = 1, \ldots, r
\]

\( \sum_{k=1}^{n_i} z_{ik} \leq 1, \quad i = 1, \ldots, m \)
**Power cone**

**definition:** for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) > 0$, $\sum_{i=1}^{m} \alpha_i = 1$

$$K_\alpha = \left\{ (x, y) \in \mathbb{R}_+^m \times \mathbb{R} \mid |y| \leq x_1^{\alpha_1} \cdots x_m^{\alpha_m} \right\}$$

**examples** for $m = 2$

$\alpha = (\frac{1}{2}, \frac{1}{2})$  
$\alpha = (\frac{2}{3}, \frac{1}{3})$  
$\alpha = (\frac{3}{4}, \frac{1}{4})$
Cones constructed from convex sets

inverse image of convex set under perspective

\[ K = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y > 0, \ x/y \in C\} \]

- \( K \cup \{(0, 0)\} \) is a convex cone if \( C \) is a convex set
- \( \text{cl} \ K \) is proper if \( C \) has nonempty interior, does not contain straight lines

consequence

any convex constraint \( x \in C \) can be represented as a conic inequality

\[ x \in C \iff (x, 1) \in K \]

(with minor modifications to make \( K \) proper)
Cones constructed from functions

epigraph of perspective of convex function

\[ K = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid y > 0, \ yf(x/y) \leq z\} \]

- \( K \cup \{(0, 0, 0)\} \) is a convex cone if \( f \) is convex
- \( \text{cl} \ K \) is proper if \( \text{int} \ \text{dom} \ f \neq \emptyset \), \( \text{epi} \ f \) does not contain straight lines

consequence

can represent any convex constraint \( f(x) \leq t \) as a conic linear inequality

\[ f(x) \leq t \iff (x, 1, t) \in K \]

(with minor modifications to make \( K \) proper)
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Modeling software

modeling packages for convex optimization

• CVX, YALMIP (MATLAB)
• CVXPY, PICOS (Python)
• MOSEK Fusion (different platforms)

assist in formulating convex problems by automating two tasks:

• verifying convexity from convex calculus rules
• transforming problem in input format required by standard solvers

related packages

general-purpose optimization modeling: AMPL, GAMS
Modeling and conic optimization

**convex modeling systems**

- convert problems stated in standard mathematical notation to conic LPs
- in principle, any convex problem can be represented as a conic LP
- in practice, a small set of primitive cones is used ($\mathbb{R}_+^n, Q^p, S^p$)
- choice of cones is limited by available algorithms and solvers (see later)

modeling systems implement set of rules for expressing constraints

$$f(x) \leq t$$

as conic inequalities for the implemented cones
Examples of second-order cone representable functions

• convex quadratic

\[ f(x) = x^T P x + q^T x + r \quad (P \succeq 0) \]

• quadratic-over-linear function

\[ f(x, y) = \frac{x^T x}{y} \quad \text{with dom } f = \mathbb{R}^n \times \mathbb{R}_+ \quad (\text{assume } 0/0 = 0) \]

• convex powers with rational exponent

\[ f(x) = |x|^\alpha, \quad f(x) = \begin{cases} 
  x^\beta & x > 0 \\
  +\infty & x \leq 0
\end{cases} \]

for rational \( \alpha \geq 1 \) and \( \beta \leq 0 \)

• \( p \)-norm \( f(x) = \|x\|_p \) for rational \( p \geq 1 \)
Examples of SD cone representable functions

- matrix-fractional function
  \[ f(X, y) = y^T X^{-1} y \quad \text{with} \quad \text{dom} \, f = \{(X, y) \in S^n_+ \times \mathbb{R}^n \mid y \in \mathcal{R}(X)\} \]

- maximum eigenvalue of symmetric matrix

- maximum singular value \( f(X) = \|X\|_2 = \sigma_1(X) \)
  \[ \|X\|_2 \leq t \iff \begin{bmatrix} tI & X \\ X^T & tI \end{bmatrix} \succeq 0 \]

- nuclear norm \( f(X) = \|X\|_* = \sum_i \sigma_i(X) \)
  \[ \|X\|_* \leq t \iff \exists U, V : \begin{bmatrix} U & X \\ X^T & V \end{bmatrix} \succeq 0, \quad \frac{1}{2}(\text{tr } U + \text{tr } V) \leq t \]
Functions representable with exponential and power cone

exponential cone

- exponential and logarithm
- entropy $f(x) = x \log x$

power cone

- increasing power of absolute value: $f(x) = |x|^p$ with $p \geq 1$
- decreasing power: $f(x) = x^q$ with $q \leq 0$ and domain $\mathbb{R}_{++}$
- $p$-norm: $f(x) = \|x\|_p$ with $p \geq 1$
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Dual cone

\[ K^* = \{ y \mid x^T y \geq 0 \text{ for all } x \in K \} \]

**Properties** (if \( K \) is a proper cone)

- \( K^* \) is a proper cone
- \( (K^*)^* = K \)
- \( \text{int } K^* = \{ y \mid x^T y > 0 \text{ for all } x \in K, x \neq 0 \} \)

**Dual inequality:** \( x \succeq^* y \) means \( x \succeq_{K^*} y \) for generic proper cone \( K \)

**Note:** dual cone depends on choice of inner product
Examples

- $\mathbb{R}_+^p$, $\mathbb{Q}_+^p$, $\mathbb{S}_+^p$ are self-dual: $K = K^*$

- dual of norm cone is norm cone for dual norm

- dual of exponential cone

$$K^*_{\text{exp}} = \{(u, v, w) \in \mathbb{R}_- \times \mathbb{R} \times \mathbb{R}^+ | -u \log(-u/w) + u - v \leq 0\}$$

(with $0 \log(0/w) = 0$ if $w \geq 0$)

- dual of power cone is

$$K^*_{\alpha} = \{(u, v) \in \mathbb{R}_+^m \times \mathbb{R} | |v| \leq (u_1/\alpha_1)^{\alpha_1} \cdots (u_m/\alpha_m)^{\alpha_m}\}$$
Primal and dual conic LP

**Primal** (optimal value $p^*$)

minimize $c^T x$
subject to $Ax \leq b$

**Dual** (optimal value $d^*$)

maximize $-b^T z$
subject to $A^T z + c = 0$
\[ z \succeq 0 \]

**Weak duality:** $p^* \geq d^*$ (without exception)
**Strong duality**

**main theorem:** \( p^* = d^* \) if primal or dual problem is strictly feasible

**other implications of strict feasibility**

- if primal is strictly feasible, then dual optimum is attained (if \( d^* \) is finite)
- if dual is strictly feasible then primal optimum is attained (if \( p^* \) is finite)

**compare with linear programming duality** \((K = \mathbb{R}^m_+)\)

- for an LP, only exception to strong duality is \( p^* = +\infty, \ d^* = -\infty \)
- strong duality holds if primal or dual is feasible
- if optimal value is finite then it is attained (in primal and dual)
Example with finite nonzero duality gap

primal problem

\[
\begin{align*}
\text{minimize} & \quad x_1 \\
\text{subject to} & \quad \begin{bmatrix} 0 & x_1 \\ x_1 & x_2 \end{bmatrix} \succeq 0 \\
               & \quad x_1 \geq -1
\end{align*}
\]

optimal value \( p^* = 0 \)

dual problem

\[
\begin{align*}
\text{maximize} & \quad -z \\
\text{subject to} & \quad \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix} \succeq 0, \quad z \geq 0 \\
                  & \quad 2Z_{12} + z = 1, \quad Z_{22} = 0
\end{align*}
\]

optimal value \( d^* = -1 \)
Optimality conditions

if strong duality holds, then $x$ and $z$ are optimal if and only if

$$\begin{bmatrix} 0 \\ s \end{bmatrix} = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} c \\ b \end{bmatrix}$$

(1)

$s \succeq 0, \quad z \succeq^* 0, \quad z^T s = 0$

**primal feasibility:** block 2 of (1) and $s \succeq 0$

**dual feasibility:** block 1 of (1) and $z \succeq^* 0$

**zero duality gap:** inner product of $(x, z)$ and $(0, s)$ gives

$$z^T s = c^T x + b^T z$$
Strong theorems of alternative

strict primal feasibility

exactly one of the following two systems is solvable

1. \( Ax \prec b \)
2. \( A^T z = 0, \ z \neq 0, \ z \succeq_* 0, \ b^T z \leq 0 \)

strict dual feasibility

if \( c \in \mathcal{R}(A^T) \), exactly one of the following two systems is solvable

1. \( Ax \preceq_K 0, \ Ax \neq 0, \ c^T x \leq 0 \)
2. \( A^T z + c = 0, \ z \succ_{K^*} 0 \)

solution of one system is a certificate of infeasibility of the other system
Weak theorems of alternative

primal feasibility

at most one of the following two systems is solvable

1. $Ax \preceq b$
2. $A^T z = 0, \quad z \succeq_0 *, \quad b^T z < 0$

dual feasibility

at most one of the following two systems is solvable

1. $Ax \preceq 0, \quad c^T x < 0$
2. $A^T z + c = 0, \quad z \succeq_* 0$

these are strong alternatives if a constraint qualification holds
Self-dual embeddings

idea

embed primal, dual conic LPs into a single (self-dual) conic LP, so that:

• embedded problem is feasible, with known feasible points

• from the solution of embedded problem can extract primal and dual solutions of original problem, or certificates of primal or dual infeasibility

purpose: a feasible algorithm applied to the embedded problem

• can detect infeasibility in original problem

• does not require a phase I to find initial feasible points

used by some interior-point solvers
Basic self-dual embedding

minimize \( 0 \)

subject to

\[
\begin{bmatrix}
0 \\
s \\
\kappa
\end{bmatrix} =
\begin{bmatrix}
0 & A^T & c \\
-A & 0 & b \\
-c^T & -b^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
\tau
\end{bmatrix}
\]

\( s \succeq 0, \quad \kappa \geq 0, \quad z \succeq 0, \quad \tau \geq 0 \)

- a self-dual conic LP
- has a trivial solution (all variables zero)
- \( z^T s + \tau \kappa = 0 \) for all feasible points (follows from equality constraint)
- hence, problem is not strictly feasible
Optimality condition for embedded problem

\[
\begin{bmatrix}
0 \\
s \\
\kappa
\end{bmatrix} = \begin{bmatrix}
0 & A^T & c \\
-A & 0 & b \\
-c^T & -b^T & 0
\end{bmatrix} \begin{bmatrix}
x \\
z \\
\tau
\end{bmatrix}
\]

\[s \succeq 0, \quad \kappa \geq 0, \quad z \succeq 0, \quad \tau \geq 0\]

\[z^T s + \tau \kappa = 0\]

- follows from self-dual property
- a (mixed) linear complementarity problem
Classification of nonzero solution

let $s, \kappa, x, z, \tau$ be a nonzero solution of the self-dual embedding

case 1: $\tau > 0$, $\kappa = 0$

$$\hat{s} = s/\tau, \quad \hat{x} = x/\tau, \quad \hat{z} = z/\tau$$

are primal and dual solutions of the conic LPs, i.e., satisfy

$$\begin{bmatrix} 0 \\ \hat{s} \end{bmatrix} = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} + \begin{bmatrix} c \\ b \end{bmatrix}$$

$$\hat{s} \succeq 0, \quad \hat{z} \preceq_{\ast} 0, \quad \hat{s}^T \hat{z} = 0$$
case 2: $\tau = 0$, $\kappa > 0$; this implies $c^T x + b^T z < 0$

- if $b^T z < 0$, then $\hat{z} = z / (-b^T z)$ is a proof of primal infeasibility:

$$A^T \hat{z} = 0, \quad b^T \hat{z} = -1, \quad \hat{z} \succeq 0$$

- if $c^T x < 0$, then $\hat{x} = x / (-c^T x)$ is a proof of dual infeasibility:

$$A \hat{x} \preceq 0, \quad c^T \hat{x} = -1$$

case 3: $\tau = \kappa = 0$; no conclusion can be made about the original problem
Extended self-dual embedding

minimize $\theta$

subject to

$$
\begin{bmatrix}
0 \\
 s \\
 \kappa \\
0
\end{bmatrix} = \begin{bmatrix}
0 & A^T & c & q_x \\
-A & 0 & b & q_z \\
-c^T & -b^T & 0 & q_\tau \\
-q_x^T & -q_z^T & -q_\tau & 0
\end{bmatrix} \begin{bmatrix}
x \\
z \\
\tau \\
\theta
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
$$

$s \succeq 0, \quad \kappa \geq 0, \quad z \succeq_\ast 0, \quad \tau \geq 0$

- $q_x, q_z, q_\tau$ chosen so that

$$(s, \kappa, x, z, \tau, \theta) = (s_0, 1, x_0, z_0, 1, z_0^T s_0 + 1)$$

is feasible, for some given $s_0 > 0, x_0, z_0 \succeq_\ast 0$

- a self-dual conic LP
Optimality condition

\[
\begin{bmatrix}
0 \\
\mathbf{s} \\
\kappa \\
0
\end{bmatrix} = \begin{bmatrix}
0 & A^T & c & q_x \\
-A & 0 & b & q_z \\
-c^T & -b^T & 0 & q_\tau \\
-q_x^T & -q_z^T & -q_\tau & 0
\end{bmatrix} \begin{bmatrix}
x \\
z \\
\tau \\
\theta
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

\[s \succeq 0, \quad \kappa \geq 0, \quad z \succeq* 0, \quad \tau \geq 0\]

\[z^T s + \tau \kappa = 0\]

- follows from self-dual property
- a (mixed) linear complementarity problem
Properties of extended self-dual embedding

• problem is strictly feasible by construction

• if \( s, \kappa, x, z, \tau, \theta \) satisfy equality constraint, then

\[
\theta = s^T z + \kappa \tau
\]

(take inner product with \((x, z, \tau, \theta)\) of each side of the equality)

• at optimum, \( \theta = 0 \) and problem reduces to the embedding on p.14-30

• classification of p.14-32 also applies to solutions of extended embedding
Reference