14. Primal-dual proximal methods

- primal-dual optimality conditions
- monotone operators
- proximal point algorithm
- Chambolle-Pock algorithm
- Douglas-Rachford operator splitting

Primal and dual problem

$$\text{primal:} \quad \text{minimize} \quad f(x) + g(Ax)$$

$$\text{dual:} \quad \text{maximize} \quad -g^*(z) - f^*(-A^T z)$$

- f and g are closed convex functions
- dual problem is Lagrange dual of reformulated problem

$$\begin{array}{ll} \text{minimize} & f(x) + g(y) \\ \text{subject to} & Ax = y \end{array}$$

Optimality (KKT) conditions

- primal feasibility: $x \in \operatorname{dom} f$ and $Ax = y \in \operatorname{dom} g$
- (x, y) is a minimizer of the Lagrangian $f(x) + g(y) + z^T(Ax - y)$:

$$-A^T z \in \partial f(x), \qquad z \in \partial g(y) \quad \text{(equivalently, } y \in \partial g^*(z)\text{)}$$

Primal-dual optimality conditions

- the optimality conditions can be written symmetrically as

$$0 \in \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g^*(z) \end{bmatrix}$$

- second term on right-hand side denotes the product set

$$\partial f(x) \times \partial g^*(z) = \{(u, v) \mid u \in \partial f(x),\ v \in \partial g^*(z)\}$$

- solutions are saddle points of convex-concave function

$$f(x) - g^*(z) + z^T A x$$

in this lecture we assume that the optimality conditions are solvable
(a sufficient condition is that primal is solvable and Slater's condition holds)

Outline

- primal-dual optimality conditions
- **monotone operators**
- resolvent
- proximal point algorithm
- Chambolle-Pock algorithm
- Douglas-Rachford operator splitting

Multivalued (set-valued) operator

Definition: operator F maps vectors $x \in \mathbf{R}^n$ to sets $F(x) \subseteq \mathbf{R}^n$

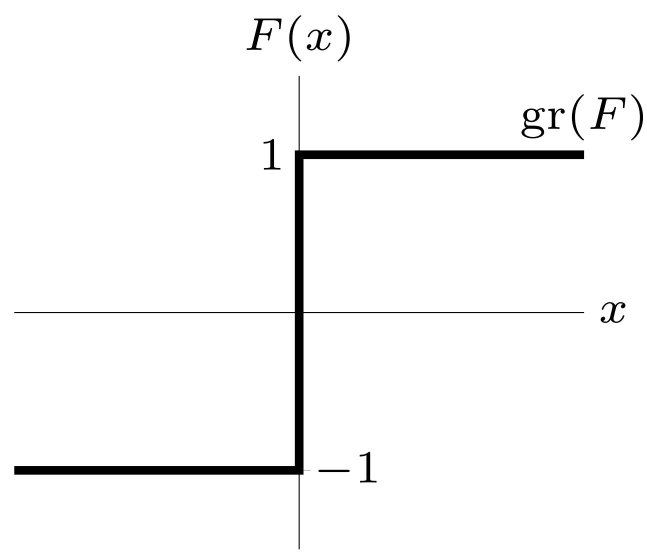
- the domain and graph of F are defined as

$$\begin{aligned} \operatorname{dom} F &= \{x \in \mathbf{R}^n \mid F(x) \neq \emptyset\} \\ \operatorname{gr}(F) &= \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n \mid x \in \operatorname{dom} F,\ y \in F(x)\} \end{aligned}$$

- if $F(x)$ is a singleton, we write $F(x) = y$ instead of $F(x) = \{y\}$

Example: sign operator

$$F(x) = \begin{cases} -1 & x < 0 \\ [-1, 1] & x = 0 \\ 1 & x > 0 \end{cases}$$



Transformations as operations on graph

Inverse: $F^{-1}(x) = \{y \mid x \in F(y)\}$

$$\mathrm{gr}(F^{-1}) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathrm{gr}(F)$$

Composition with scaling: $(\lambda F)(x) = \lambda F(x)$ and $(F\lambda)(x) = F(\lambda x)$

$$\mathrm{gr}(\lambda F) = \begin{bmatrix} I & 0 \\ 0 & \lambda I \end{bmatrix} \mathrm{gr}(F), \qquad \mathrm{gr}(F\lambda) = \begin{bmatrix} (1/\lambda)I & 0 \\ 0 & I \end{bmatrix} \mathrm{gr}(F)$$

Addition to identity: $(I + \lambda F)(x) = \{x + \lambda y \mid y \in F(x)\}$

$$\mathrm{gr}(I + \lambda F) = \begin{bmatrix} I & 0 \\ I & \lambda I \end{bmatrix} \mathrm{gr}(F)$$

note that these are all *linear* operations on the graph

Monotone operator

Definition: F is a monotone operator if

$$(y - \hat{y})^T (x - \hat{x}) \geq 0 \qquad \forall x, \hat{x} \in \operatorname{dom} F,\ y \in F(x),\ \hat{y} \in F(\hat{x})$$

in terms of the graph,

$$\begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix} \geq 0 \qquad \forall (x, y),\ (\hat{x}, \hat{y}) \in \operatorname{gr}(F)$$

Monotone inclusion problem: find $x \in F^{-1}(0)$, *i.e.*, solve

$$0 \in F(x)$$

includes many equilibrium/optimality conditions as special cases

Examples

we will encounter the following three types of monotone operators

- subdifferentials $\partial f(x)$ of convex functions f
- affine monotone operators: $F(x) = Cx + d$ is monotone if

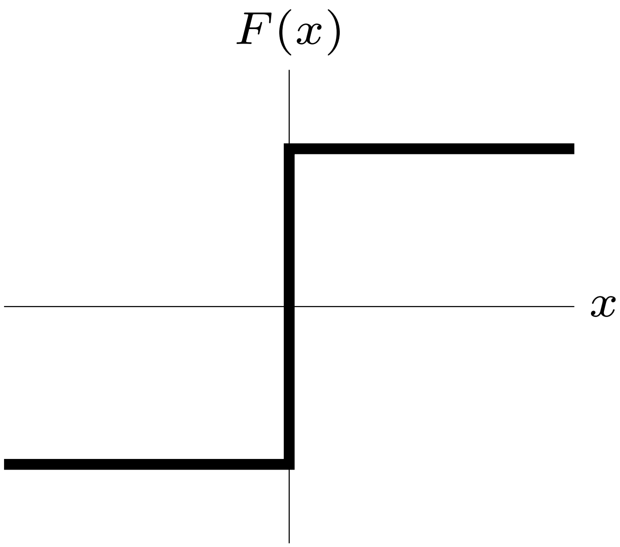
$$C + C^T \succeq 0$$

- sums of the above; in particular,

$$F(x,z) = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g^*(z) \end{bmatrix}$$

Maximal monotone operator

graph is not properly contained in the graph of another monotone operator



maximal monotone

monotone, but not maximal monotone

Conditions for maximal monotonicity

- the subdifferential of a closed convex function is maximal monotone
- affine monotone operators are maximal monotone
- (Minty) a monotone operator F is maximal monotone if and only if

$$\mathrm{im}(I+F) = \bigcup_{x \in \mathrm{dom}\, F} (x + F(x)) = \mathbf{R}^n$$

  i.e., for every $y \in \mathbf{R}^n$, there exists an x such that $y \in x + F(x)$

- sums $F + G$ of maximal monotone operators are not necessarily maximal

  (sufficient condition: $\mathrm{int}\,\mathrm{dom}\, F \cap \mathrm{dom}\, G \neq \emptyset$)

Coercivity (strong monotonicity)

F is **coercive** with parameter $\mu > 0$ if

$$(y - \hat{y})^T(x - \hat{x}) \geq \mu \|x - \hat{x}\|_2^2 \qquad \forall x, \hat{x} \in \operatorname{dom} F,\ y \in F(x),\ \hat{y} \in F(\hat{x})$$

- $F - \mu I$ is a monotone operator
- equivalently,

$$\begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix}^T \begin{bmatrix} -2\mu I & I \\ I & 0 \end{bmatrix} \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix} \geq 0 \qquad \forall (x, y),\ (\hat{x}, \hat{y}) \in \operatorname{gr}(F)$$

Examples

- subdifferential of strongly convex function
- affine operator $F(x) = Ax + b$ if $A + A^T \succ 0$ (with $\mu = \lambda_{\min}(A + A^T)/2$)

Co-coercivity

F is **co-coercive** with parameter $\gamma > 0$ if F^{-1} is coercive:

$$(F(x) - F(\hat{x}))^T (x - \hat{x}) \geq \gamma \|F(x) - F(\hat{x})\|_2^2 \qquad \forall x, \hat{x} \in \mathrm{dom}\, F$$

- equivalently,

$$\left[\begin{array}{c} x - \hat{x} \\ y - \hat{y} \end{array}\right]^T \left[\begin{array}{cc} 0 & I \\ I & -2\gamma I \end{array}\right] \left[\begin{array}{c} x - \hat{x} \\ y - \hat{y} \end{array}\right] \geq 0 \qquad \forall (x, y),\ (\hat{x}, \hat{y}) \in \mathrm{gr}(F)$$

- F is **firmly nonexpansive** if it is co-coercive with $\gamma = 1$

Example: affine operator $F(x) = Ax + b$ with

$$A + A^T \succeq 2\gamma A^T A \qquad \Longleftrightarrow \qquad \|2\gamma A - I\|_2 \leq 1$$

for symmetric positive definite A, this means $\lambda_{\max}(A) \leq 1/\gamma$

Lipschitz continuity

- F is **Lipschitz continuous** with parameter L if

$$\|F(x) - F(\hat{x})\|_2 \leq L\|x - \hat{x}\|_2 \qquad \forall x, \hat{x} \in \mathrm{dom}\, F$$

- F is **nonexpansive** if it is Lipschitz continuous with $L = 1$

Example: any affine $F(x) = Ax + b$ (parameter $L = \|A\|_2$)

Relation to co-coercivity

- co-coercivity implies Lipschitz continuity (with $L = 1/\gamma$)
- Lipschitz continuity does not imply co-coercivity (see homework 1)
- properties are equivalent for gradients of closed convex functions (page 1-15)

Outline

- primal-dual optimality conditions
- monotone operators
- **resolvent**
- proximal point algorithm
- Chambolle-Pock algorithm
- Douglas-Rachford operator splitting

Resolvent

the **resolvent** of an operator F is the operator

$$(I + \lambda F)^{-1} \qquad \text{(with } \lambda > 0\text{)}$$

- inverse denotes the operator inverse:

$$y \in (I + \lambda F)^{-1}(x) \qquad \Longleftrightarrow \qquad x - y \in \lambda F(y)$$

- graph of resolvent is a linear transformation of graph of F:

$$\mathrm{gr}((I + \lambda F)^{-1}) = \begin{bmatrix} I & \lambda I \\ I & 0 \end{bmatrix} \mathrm{gr}(F)$$

Examples

Subdifferential: resolvent is proximal mapping

$$(I + \lambda \partial f)^{-1}(x) = \mathrm{prox}_{\lambda f}(x)$$

follows from subgradient characteriation of $\mathrm{prox}_{\lambda f}$ (page 6-7)

$$y = \mathrm{prox}_{\lambda f}(x) \quad \Longleftrightarrow \quad x - y \in \lambda \partial f(y)$$

Monotone affine mapping: resolvent of $F(x) = Ax + b$ is

$$(I + \lambda F)^{-1}(x) = (I + \lambda A)^{-1}(x - \lambda b)$$

- inverse on right-hand side is standard matrix inverse
- $I + \lambda A$ is nonsingular for all $\lambda \geq 0$ because $A + A^T \succeq 0$

Monotonicity properties

- an operator is monotone if and only if its resolvent is firmly nonexpansive:

  this follows from the matrix identity

$$\lambda \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & I \\ \lambda I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -2I \end{bmatrix} \begin{bmatrix} I & \lambda I \\ I & 0 \end{bmatrix}$$

  and the expression of the graph of the resolvent on page 14-13

- a monotone operator F is *maximal* monotone if and only

$$\mathrm{dom}(I + \lambda F)^{-1} = \mathbf{R}^n$$

  follows from Minty's theorem on page 14-9

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Proximal point algorithm

Monotone inclusion problem: given maximal monotone F, find x such that

$$0 \in F(x)$$

this is equivalent to finding a fixed point of the resolvent $R_t = (I + tF)^{-1}$ of F:

$$x = R_t(x) \quad \Longleftrightarrow \quad x \in (I + tF)(x) \quad \Longleftrightarrow \quad 0 \in F(x)$$

Proximal-point algorithm: fixed point iteration

$$x^+ = R_t(x)$$

Proximal-point algorithm with relaxation (relaxation parameter $\rho \in (0, 2)$):

$$x^+ = x + \rho(R_t(x) - x)$$

Convergence

if $F^{-1}(0) \neq \emptyset$, proximal point algorithm converges

- with constant $t > 0$ and $\rho \in (0, 2)$
- with t_k, ρ_k varying and bounded away from their limits, *i.e.*,

$$t_k \geq t_{\min} > 0, \qquad 0 < \rho_{\min} \leq \rho_k \leq \rho_{\max} < 2 \qquad \text{for all } k$$

proof relies on firm nonexpansiveness of resolvent

Linear change of variables

make a change of variables $x = Ay$, with A nonsingular:

$$G(y) = A^T F(Ay)$$

- graph of G is

$$\mathrm{gr}(G) = \begin{bmatrix} A^{-1} & 0 \\ 0 & A^T \end{bmatrix} \mathrm{gr}(F)$$

- (maximal) monotonicity of G follows from (maximal) monotonicity of F and

$$\begin{bmatrix} A^{-1} & 0 \\ 0 & A^T \end{bmatrix}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & A^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

'Preconditioned' proximal point algorithm

$$y^{(k)} = (I + tG)^{-1}(y^{(k-1)})$$

- $y^{(k)}$ is the solution of the inclusion problem

$$\frac{1}{t}(y^{(k-1)} - y) \in A^T F(Ay)$$

- in the original coordinates $x = Ay$, this can be written as

$$\frac{1}{t}H(x^{(k-1)} - x) \in F(x)$$

  where $H = A^{-T}A^{-1}$ and $x^{(k-1)} = Ay^{(k-1)}$

- we obtain a generalized proximal point update, with $H \succ 0$ substituted for I:

$$x^{(k)} = (H + tF)^{-1}(Hx^{(k-1)})$$

the purpose is often to make the resolvents cheaper, not preconditioning

Proximal method of multipliers

the proximal point algorithm applied to

$$F(x,z) = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g^*(z) \end{bmatrix}$$

is known as the proximal method of multipliers

- basic iteration (without relaxation) is

$$(x^{(k)}, z^{(k)}) = (I + tF)^{-1}(x^{(k-1)}, z^{(k-1)})$$

- $(x^{(k)}, z^{(k)})$ is the solution of the monotone inclusion

$$0 \in \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g^*(z) \end{bmatrix} + \frac{1}{t} \begin{bmatrix} x - x^{(k-1)} \\ z - z^{(k-1)} \end{bmatrix}$$

Evaluation of the resolvent

- equivalent inclusion problem

$$0 \in \begin{bmatrix} 0 & 0 & A^T \\ 0 & 0 & -I \\ -A & I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g(y) \\ 0 \end{bmatrix} + \frac{1}{t} \begin{bmatrix} x - x^{(k-1)} \\ 0 \\ z - z^{(k-1)} \end{bmatrix}$$

- this is the optimality condition of the optimization problem (variables x, y)

$$\text{minimize} \quad f(x) + g(y) + \frac{t}{2}\|Ax - y + (1/t)z^{(k-1)}\|_2^2 + \frac{1}{2t}\|x - x^{(k-1)}\|_2^2$$

  (the augmented Lagrangian with an extra penalty term on x)

- from the minimizer $(\hat{x}, \hat{y})$, we make the update

$$x^{(k)} = \hat{x}, \qquad z^{(k)} = z^{(k-1)} + t(A\hat{x} - \hat{y})$$

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Chambolle-Pock algorithm

$$0 \in \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g^*(z) \end{bmatrix}$$

Algorithm

$$\begin{aligned} x^{(k)} &= \operatorname{prox}_{tf}(x^{(k-1)} - tA^T z^{(k-1)}) \\ z^{(k)} &= \operatorname{prox}_{sg^*}(z^{(k-1)} + sA(2x^{(k)} - x^{(k-1)})) \end{aligned}$$

- primal and dual step sizes t, s are positive and satisfy $st\|A\|_2^2 \leq 1$
- each iteration requires evaluations of proximal mappings of f and g^*
- also requires multiplications with A, A^T, but no solutions of linear equations
- for $A = I$, $s = t = 1$ this is the Douglas-Rachford algorithm (page 13-8)

Relation to proximal point algorithm

apply 'preconditioned' proximal point algorithm of page 14-19 with

$$H = \begin{bmatrix} I & -tA^T \\ -tA & (t/s)I \end{bmatrix}$$

- H is positive definite for $st\|A\|_2^2 < 1$
- $x^{(k)}$ and $z^{(k)}$ are the solution of

$$\frac{1}{t}\begin{bmatrix} I & -tA^T \\ -tA & (t/s)I \end{bmatrix}\begin{bmatrix} x^{(k-1)} - x \\ z^{(k-1)} - z \end{bmatrix} \in \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g^*(z) \end{bmatrix}$$

- this simplifies to

$$\begin{aligned} 0 &\in \partial f(x) + \frac{1}{t}(x - x^{(k-1)} + tA^T z^{(k-1)}) \\ 0 &\in \partial g^*(z) + \frac{1}{s}(z - z^{(k-1)} - sA(2x - x^{(k-1)})), \end{aligned}$$

  and writing the solution in terms of prox-operators gives the CP algorithm

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- **Douglas-Rachford operator splitting**

Operator splitting

given maximal monotone operators F and G, solve

$$0 \in F(x) + G(x)$$

Algorithm: start at any $y^{(0)}$ and repeat for $k = 1, 2, \ldots$

$$\begin{aligned} x^{(k)} &= (I + tF)^{-1}(y^{(k-1)}) \\ y^{(k)} &= y^{(k-1)} + (I + tG)^{-1}(2x^{(k)} - y^{(k-1)}) - x^{(k)} \end{aligned}$$

- for $F = \partial f$ and $G = \partial g$, this is the algorithm of page 13-2
- useful when resolvents of F and G are inexpensive, but not resolvent of sum
- converges under weak conditions (existence of solution)
- can add relaxation to y-update

Primal-dual optimality conditions

$$0 \in \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g^*(z) \end{bmatrix}$$

Simplest splitting

$$F(x,z) = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix}, \qquad G(x,z) = \begin{bmatrix} \partial f(x) \\ \partial g^*(z) \end{bmatrix}$$

- resolvent of F: reduces to linear equation with coefficient $I + t^2 A^T A$
- resolvent of G: apply prox-operators of f and g
- complexity per iteration is similar to primal or dual DR (p. 13-11 and p. 13-19)

Other splittings: exploit additive structure in A, f, g^* (see references)

References

Monotone operators and the proximal point algorithm

- H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces* (2011).
- R. T. Rockafellar, *Monotone operators and the proximal point algorithm,* SIAM J. Control and Opt. (1976).
- J. Eckstein and D. Bertekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators,* Mathematical Programming (1992).

  The convergence result on page 14-17 is Theorem 3 of this paper.

Chambolle-Pock algorithm and extensions

- A. Chambolle and T. Pock, *A first-order primal-dual algorithm for convex problems with applications to imaging,* Journal of Mathematical Imaging and Vision (2011).
- B. He and X. Yuan, *Convergence analysis of primal-dual algorithms for a saddle-point problem: from contraction perspective,* SIAM J. Imaging Sciences (2012).
- L. Condat, *A primal-dual splitting method for convex optimization involving Lipschitzian, proximable, and linear composite terms,* JOTA (2013).

  Includes a proof of convergence for $st\|A\|_2^2 = 1$. Also includes an extension to cost functions $f(x) + g(Ax) + h(x)$, with differentiable h.

Douglas-Rachford operator splitting

- P. L. Lions and B. Mercier, *Splitting algorithms for the sum of two nonlinear operators*, SIAM Journal on Numerical Analysis (1979).
- J. Eckstein and D. Bertekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators,* Mathematical Programming (1992).
- H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces* (2011).
- D. O'Connor and L. Vandenberghe, *Primal-dual decomposition by operator splitting and applications to image deblurring,* SIAM J. Imaging Sciences (2014).

  Includes examples of other ways to split the primal-dual optimality conditions on page 14-25.