Ellipsoid method

- ellipsoid method
- convergence proof
- inequality constraints

Ellipsoid method

history

- developed by Shor, Nemirovski, Yudin in 1970s
- used in 1979 by Khachian to show polynomial solvability of LPs

properties

- each step requires cutting-plane or subgradient evaluation
- modest storage ($O(n^2)$)
- modest computation per step ($O(n^2)$), via analytical formula
- extremely simple to implement
- efficient in theory
- slow but steady in practice; rarely used

Motivation

drawbacks of cutting-plane methods

- serious computation needed to find next query point

  typically, $O(n^2m)$ for analytic centering in ACCPM, with m inequalities

- localization polyhedron grows in complexity as algorithm progresses

  (with pruning, can keep m proportional to n, $e.g.$, $m = 4n$)

ellipsoid method addresses both issues, but retains theoretical efficiency

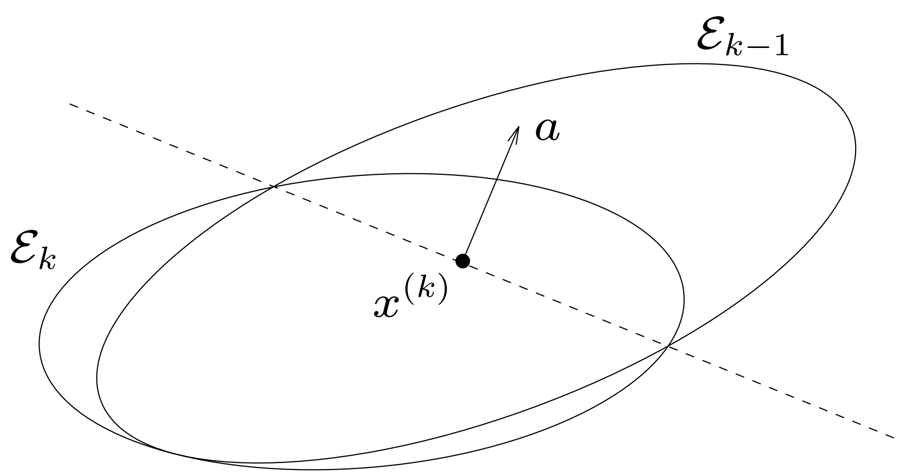
Ellipsoid algorithm for minimizing convex function

idea: localize $x^\star$ in an *ellipsoid* instead of a *polyhedron*

given an initial ellipsoid $\mathcal{E}_0$ known to contain optimal set

repeat for $k = 1, 2, \ldots$

1. query oracle to get a neutral cut $a^T z \leq b$ at $x^{(k))}$, the center of $\mathcal{E}_{k-1}$
2. set $\mathcal{E}_k :=$ minimum volume ellipsoid covering $\mathcal{E}_{k-1} \cap \{z \mid a^T z \leq b\}$



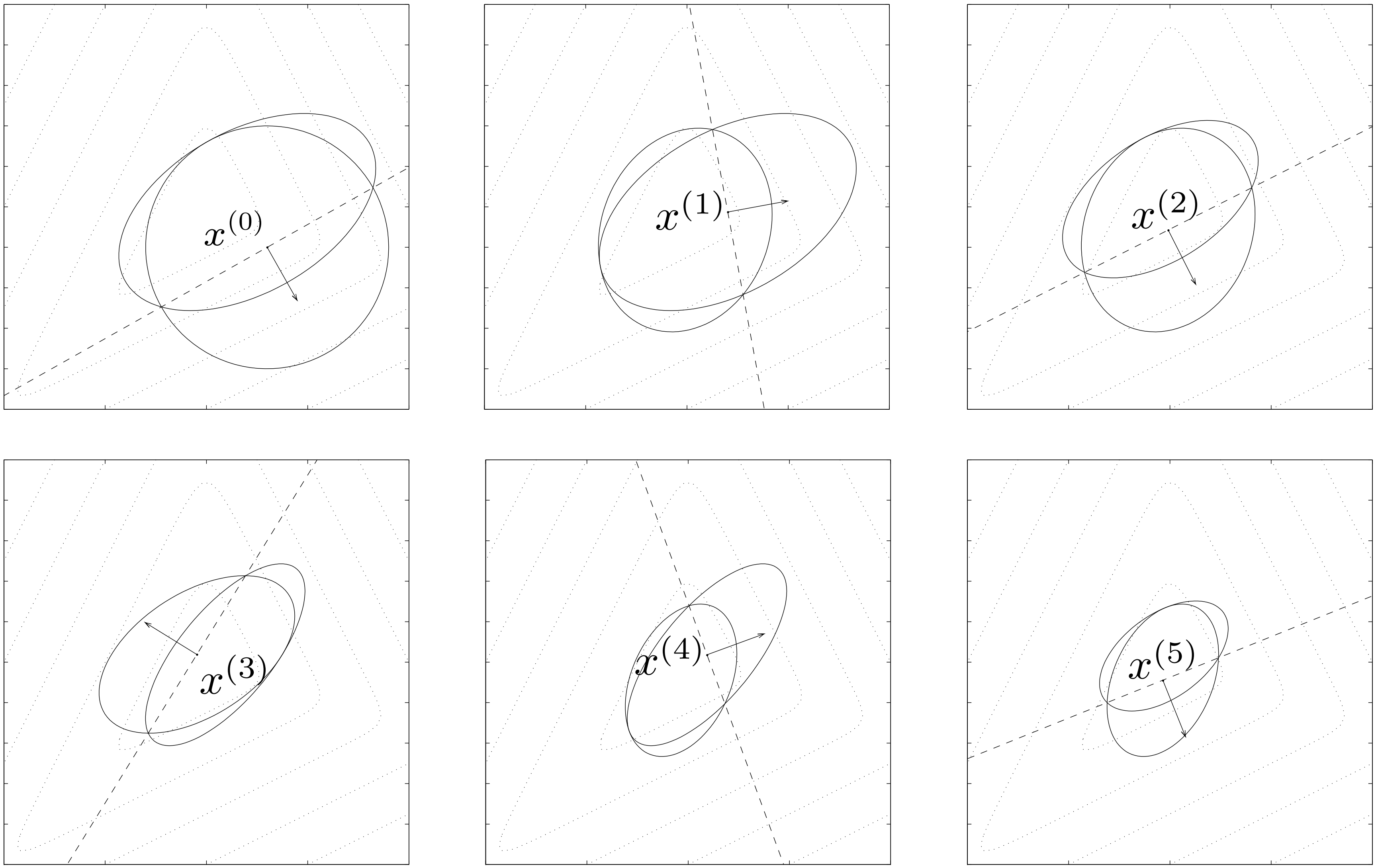
differences with cutting-plane methods

- localization set doesn't grow more complicated
- generating query point is trivial
- but, we add unnecessary points in step 2

interpretation

- reduces to bisection for $n = 1$
- can be viewed as an implementable version of the center-of-gravity cutting-plane method

Example

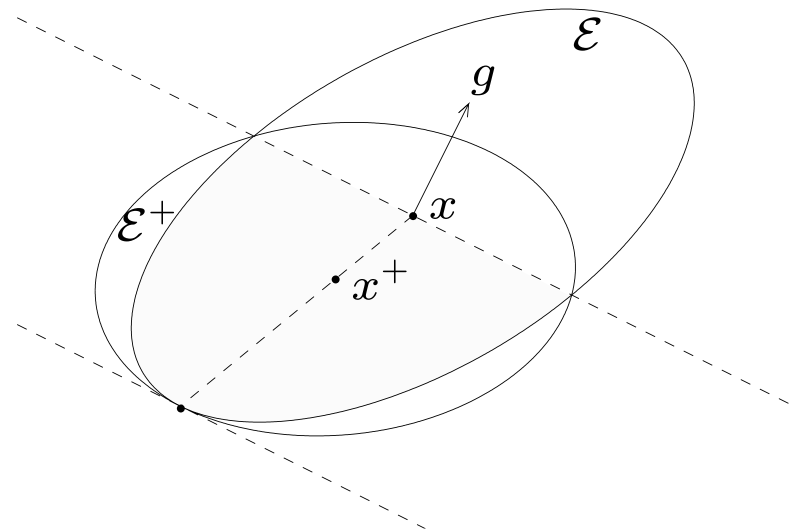


Updating the ellipsoid

$$\mathcal{E} = \left\{z \mid (z-x)^T P^{-1}(z-x) \leq 1\right\}$$

$\mathcal{E}^+$ is min. volume ellipsoid covering

$$\mathcal{E} \cap \left\{z \mid g^T(z-x) \leq 0\right\}$$



update formula (for $n > 1$): $\mathcal{E}^+ = \left\{z \mid (z-x^+)^T (P^+)^{-1}(z-x^+) \leq 1\right\}$,

$$x^+ = x - \frac{1}{n+1} P\tilde{g}, \qquad P^+ = \frac{n^2}{n^2-1}\left(P - \frac{2}{n+1} P\tilde{g}\tilde{g}^T P\right)$$

where $\tilde{g} = (1/\sqrt{g^T P g})g$

Simple stopping criterion

for unconstrained problem of minimizing $f(x)$

lower bound on optimal value

$$\begin{aligned} f(x^\star) &\geq f(x^{(k)}) + g^{(k)T}(x^\star - x^{(k)}) \\ &\geq f(x^{(k)}) + \inf_{z \in \mathcal{E}_{k-1}} g^{(k)T}(z - x^{(k)}) \\ &= f(x^{(k)}) - \sqrt{g^{(k)T} P^{(k-1)} g^{(k)}} \end{aligned}$$

second inequality holds since $x^\star \in \mathcal{E}_{k-1}$

simple stopping criterion to guarantee $f(x^{(k)}) - f(x^\star) \leq \epsilon$:

$$\sqrt{g^{(k)T} P^{(k-1)} g^{(k)}} \leq \epsilon$$

Basic ellipsoid algorithm

ellipsoid described as

$$\mathcal{E}(x, P) = \{z \mid (z - x)^T P^{-1}(z - x) \le 1\}$$

given ellipsoid $\mathcal{E}(x, P)$ containing $x^\star$, accuracy $\epsilon > 0$

repeat

1. evaluate $g \in \partial f(x)$
2. if $\sqrt{g^T P g} \le \epsilon$, return x; else, update ellipsoid

$$x := x - \frac{1}{n+1} P\tilde{g}, \qquad P := \frac{n^2}{n^2-1}\left(P - \frac{2}{n+1} P\tilde{g}\tilde{g}^T P\right)$$

where $\tilde{g} = (1/\sqrt{g^T P g})g$

Interpretation

- change coordinates
$$\tilde{z} = P^{-1/2} z$$
so uncertainty is isotropic (same in all directions), $i.e.$, $\mathcal{E}$ is unit ball

- take subgradient step with fixed length $1/(n+1)$

Shor calls ellipsoid method 'gradient method with space dilation in direction of gradient'

Improvements

- keep track of best upper and lower bounds:

$$
\begin{aligned}
f_{\text{best}}^{(k)} &= \min_{i=1,\ldots,k} f(x^{(i)}) \\
l_{\text{best}}^{(k)} &= \max_{i=1,\ldots,k} \left(f(x^{(i)}) - \sqrt{g^{(i)T} P^{(i-1)} g^{(i)}} \right)
\end{aligned}
$$

  stop when $f_{\text{best}}^{(k)} - l_{\text{best}}^{(k)} \leq \epsilon$

- propagate Cholesky factor of P (improves numerical stability)

Proof of convergence

assumptions: we consider the unconstrained problem

$$\text{minimize} \quad f(x)$$

- f is Lipschitz: $|f(y) - f(x)| \leq G\|y - x\|_2$
- $\{x \mid f(x) \leq f^\star + \epsilon\} \subseteq \mathcal{E}_0$
- $\mathcal{E}_0$ is ball with radius R

reduction of volume: can show that

$$\mathbf{vol}\,\mathcal{E}_{k+1} < e^{-\frac{1}{2n}}\,\mathbf{vol}\,\mathcal{E}_k$$

(reduction factor degrades rapidly with n, compared to CG or MVE cutting-plane methods)

proof. suppose $f(x^{(i)}) > f^\star + \epsilon$, $i = 1, \ldots, k$

- at iteration i we only discard points with $f(z) \geq f(x^{(i)})$; therefore

$$\{z \mid f(z) \leq f^\star + \epsilon\} \subseteq \mathcal{E}_k$$

- from Lipschitz condition, $\|z - x^\star\|_2 \leq \epsilon/G$ implies $f(z) \leq f^\star + \epsilon$; hence

$$B = \{z \mid \|z - x^\star\|_2 \leq \epsilon/G\} \subseteq \mathcal{E}_k$$

- therefore $\mathbf{vol}\, B \leq \mathbf{vol}\, \mathcal{E}_k$, so

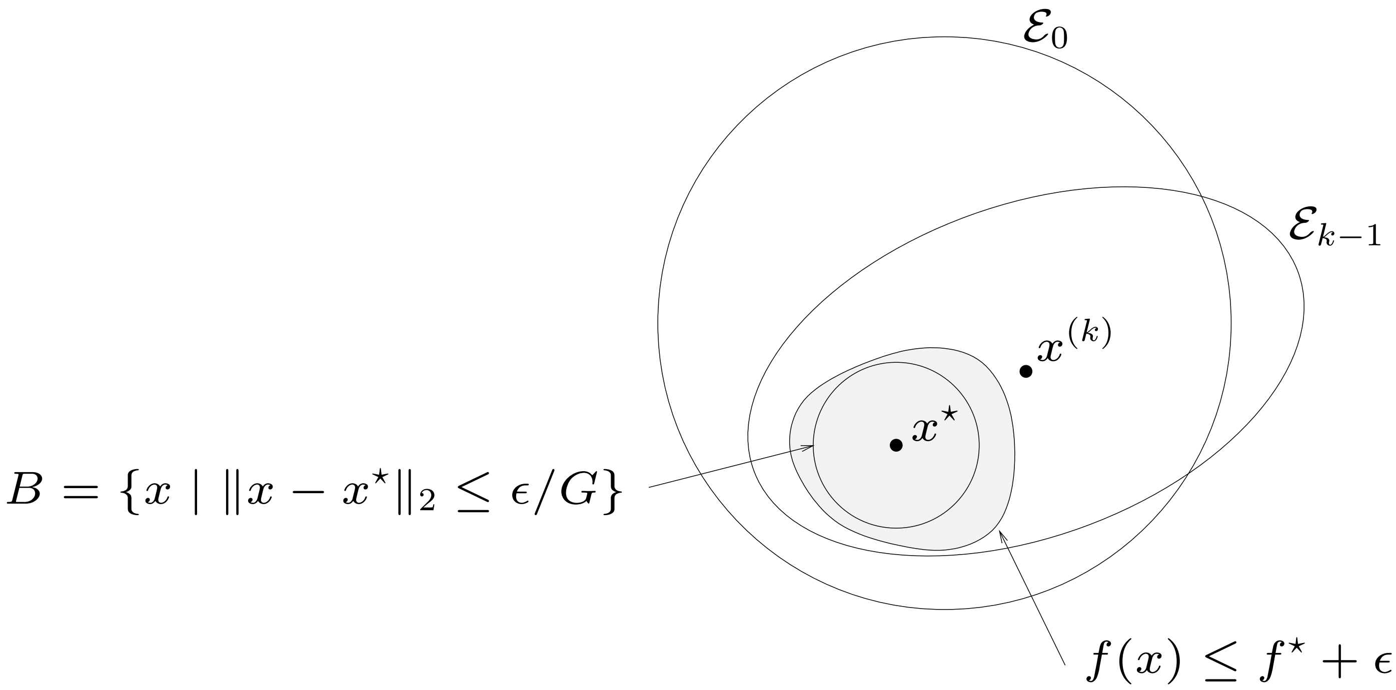
$$\alpha_n(\epsilon/G)^n \leq e^{-\frac{k}{2n}}\, \mathbf{vol}\, \mathcal{E}_0 = e^{-\frac{k}{2n}} \alpha_n R^n$$

  (α_n is volume of unit ball in $\mathbf{R}^n$)

- this gives

$$k \leq 2n^2 \log(RG/\epsilon)$$

geometrical illustration



conclusion: for $k > 2n^2 \log(RG/\epsilon)$,

$$f_{\text{best}}^{(k)} \leq f^\star + \epsilon$$

Interpretation of complexity

- since $x^\star \in \mathcal{E}_0 = \{x \mid \|x - x^{(1)}\|_2 \leq R\}$, our prior knowledge of $f^\star$ is

$$f(x^{(1)}) - GR \leq f^\star \leq f(x^{(1)})$$

  our prior uncertainty in $f^\star$ is GR

- after k iterations our knowledge of $f^\star$ is

$$f_{\rm best}^{(k)} - \epsilon \leq f^\star \leq f_{\rm best}^{(k)}$$

  posterior uncertainty in $f^\star$ is $\leq \epsilon$

- iterations required:

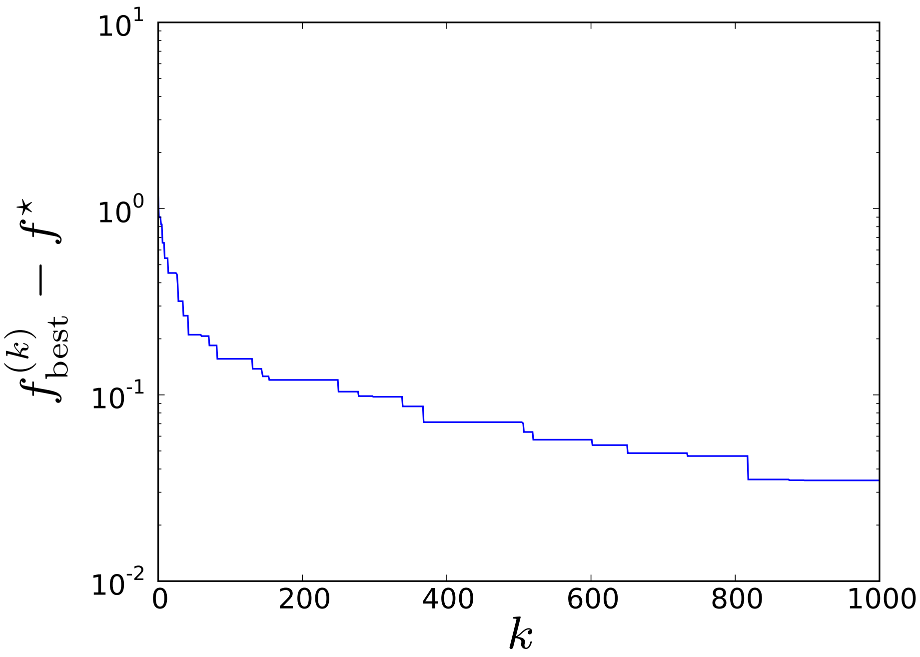
$$2n^2 \log \frac{RG}{\epsilon} = \frac{2n^2 \log_2(RG/\epsilon)}{\log_2 e} = 1.39\, n^2 \log_2 \frac{\text{prior uncertainty}}{\text{posterior uncertainty}}$$

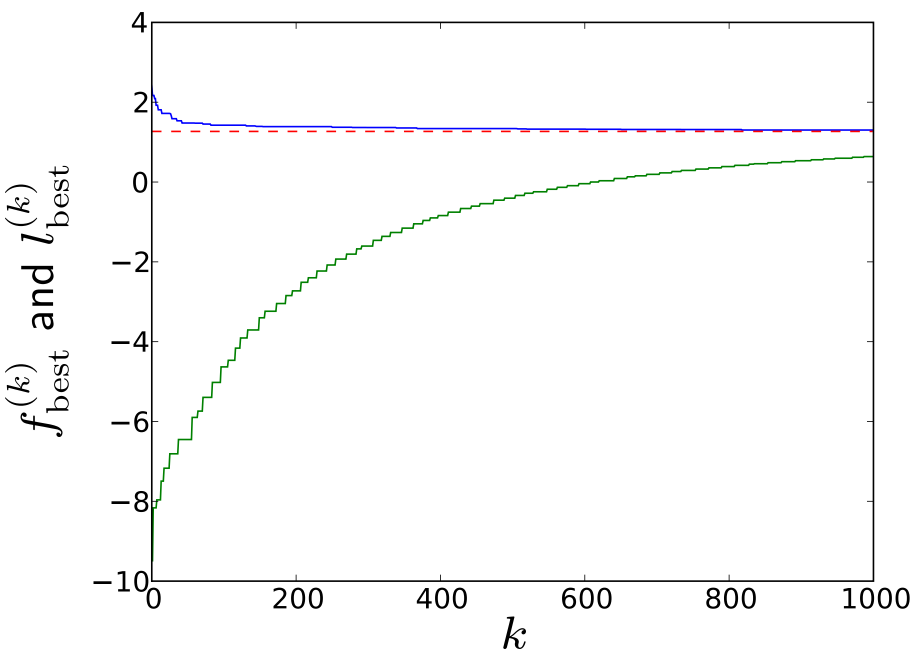
  efficiency: $1/(1.39n^2) = 0.72/n^2$ bits per gradient evaluation

Example

$$\text{minimize} \quad \max_{i=1,\ldots,m} (a_i^T x + b_i)$$

$m = 100$, $n = 20$, $\|x^\star\|_2 \approx 1.0$, start with $\mathcal{E} = \{x \mid \|x\|_2 \leq 10\}$





Deep cut ellipsoid method

minimum volume ellipsoid containing ellipsoid intersected with halfspace

$$\mathcal{E} \cap \left\{z \mid g^T(z-x)+h \leq 0\right\}$$

with $h \geq 0$, is given by

$$\begin{aligned} x^+ &= x - \frac{1+\alpha n}{n+1} P\tilde{g} \\ P^+ &= \frac{n^2(1-\alpha^2)}{n^2-1}\left(P - \frac{2(1+\alpha n)}{(n+1)(1+\alpha)} P\tilde{g}\tilde{g}^T P\right) \end{aligned}$$

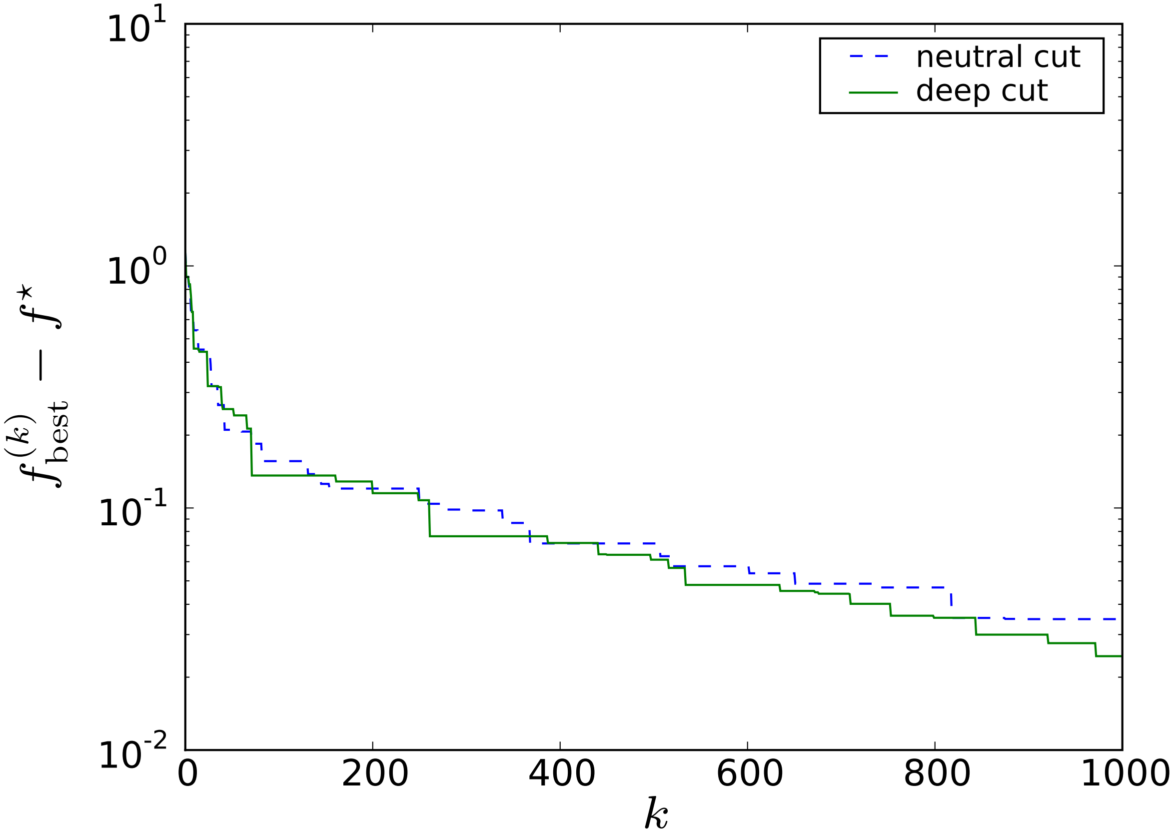
where

$$\tilde{g} = \frac{g}{\sqrt{g^T P g}}, \qquad \alpha = \frac{h}{\sqrt{g^T P g}}$$

(if $\alpha > 1$, intersection is empty)

Ellipsoid method with deep objective cuts

same example as on page 16



Inequality constrained problems

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m
\end{array}
$$

- if $x^{(k)}$ is feasible, update ellipsoid with objective cut

$$g_0^T z \leq g_0^T x^{(k)} - f_0(x^{(k)}) + f_{\rm best}^{(k)}, \qquad g_0 \in \partial f_0(x^{(k)})$$

  $f_{\rm best}^{(k)}$ is best objective value of feasible iterates so far

- if $x^{(k)}$ is infeasible (say, $f_j(x^{(k)}) > 0$), use feasibility cut

$$g_j^T z \leq g_j^T x^{(k)} - f_j(x^{(k)}), \qquad g_j \in \partial f_j(x^{(k)})$$

Stopping criterion

- if $x^{(k)}$ is feasible, we have lower bound on $p^\star$ as before:

$$p^\star \geq f_0(x^{(k)}) - \sqrt{g_0^{(k)T} P^{(k-1)} g_0^{(k)}}$$

- if $x^{(k)}$ is infeasible, we have for all $x \in \mathcal{E}_{k-1}$

$$\begin{aligned} f_j(x) &\geq f_j(x^{(k)}) + g_j^{(k)T}(x - x^{(k)}) \\ &\geq f_j(x^{(k)}) + \inf_{z \in \mathcal{E}_{k-1}} g^{(k)T}(z - x^{(k)}) \\ &= f_j(x^{(k)}) - \sqrt{g_j^{(k)T} P^{(k-1)} g_j^{(k)}} \end{aligned}$$

hence, problem is infeasible if for some j,

$$f_j(x^{(k)}) - \sqrt{g_j^{(k)T} P^{(k-1)} g_j^{(k)}} > 0$$

stopping criteria: terminate algorithm when

- $x^{(k)}$ is known to be ϵ-suboptimal:

$$x^{(k)} \text{ is feasible and } \sqrt{g_0^{(k)T} P^{(k-1)} g_0^{(k)}} \leq \epsilon$$

- or problem is shown to be infeasible:

$$f_j(x^{(k)}) - \sqrt{g_j^{(k)T} P^{(k-1)} g_j^{(k)}} > 0 \text{ for some } j$$